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Optimal output feedback for non-zero set point regulation: the discrete-time case

WASSIM M. HADDAD† and DENNIS S. BERNSTEIN‡

Optimal discrete-time static output feedback is considered for a non-zero set point problem with non-zero mean disturbances. The optimal control law consists of a closed-loop component for feeding back the measurements and a constant open-loop component which accounts for the non-zero set point and non-zero disturbance mean. An additional feature is the presence of state-, control- and measurement-dependent white noise. It is shown that in the absence of multiplicative disturbances, the closed-loop controller can be designed independently of the open-loop control.

Notation and definitions

$\mathbf{R}, \mathbf{R}^{r\times s}, \mathbf{R}^r, \mathbf{E}$	real numbers, $r \times s$ real matrices, $\mathbf{R}^{r\times 1}$, expectation
$I_n, (\ )^{\mathrm{T}}$	$n \times n$ identity, transpose
$\otimes$	Kronecker product
$\operatorname{tr} Z$	trace of square matrix Z
asymptotically stable matrix	matrix with eigenvalues in the open unit disk
n, m, l, p	positive integers
x	n-dimensional vector
u, y	m-, l-dimensional vectors
$A, A_i; B, B_i; C, C_i$	$n \times n$ matrices, $n \times m$ matrices, $l \times n$ matrices, $i = 1, \ldots, p$
L, K	$r \times n$ matrix, $m \times l$ matrix
δ, γ, α	r-, n-, m-dimensional vectors
k	discrete-time index $1, 2, \ldots$
$v_i(k)$	unit variance white noise, $i = 1, \ldots, p$
$w_1(k), w_2(k)$	n-dimensional, l-dimensional white noise processes
V_1, V_2	$n \times n$ covariance of w_1, $l \times l$ covariance of w_2; $V_1 \geqslant 0$, $V_2 \geqslant 0$
V_{12}	$n \times l$ cross-covariance of w_1, w_2
R_1, R_2	$r \times r$ and $m \times m$ state and control weightings; $R_1 \geqslant 0$, $R_2 \geqslant 0$
R_{12}	$r \times m$ cross weighting; $R_1 - R_{12} R_2^{-1} R_{12}^{\mathrm{T}} \geqslant 0$
$\tilde{A}, \tilde{A}_i$	$A + BKC$, $A_i + B_i KC + BKC_i$, $i = 1, \ldots, p$
$\mathbf{A}$	$I_n - \tilde{A}$
$\tilde{B}$	$B\alpha + \gamma$
$\tilde{B}_i$	$B_i \alpha$

$$\tilde{w} = w_1 + BKw_2 + \sum_{i=1}^{p} B_i K w_2$$

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$\tilde{V}$ $\quad V_1 + V_{12}K^{\mathrm{T}}B^{\mathrm{T}} + BKV_{12}^{\mathrm{T}} + BKV_2K^{\mathrm{T}}B^{\mathrm{T}} + \sum_{i=1}^{p} B_iKV_2K^{\mathrm{T}}B_i^{\mathrm{T}}$

$\tilde{R}$ $\quad L^{\mathrm{T}}R_1L + L^{\mathrm{T}}R_{12}KC + C^{\mathrm{T}}K^{\mathrm{T}}R_{12}^{\mathrm{T}}L + C^{\mathrm{T}}K^{\mathrm{T}}R_2KC + \sum_{i=1}^{p} C_i^{\mathrm{T}}K^{\mathrm{T}}R_2KC_i$

For arbitrary $m \in \mathbf{R}^n$ and $Q, P \in \mathbf{R}^{n\times n}$ define:

$$R_{2s} \triangleq R_2 + B^{\mathrm{T}}PB + \sum_{i=1}^{p} B_i^{\mathrm{T}}PB_i, \qquad V_{2s} \triangleq V_2 + CQC^{\mathrm{T}} + \sum_{i=1}^{p} C_i(Q + mm^{\mathrm{T}})C_i^{\mathrm{T}}$$

$$R_{2a} \triangleq R_2 + B^{\mathrm{T}}PB, \qquad V_{2a} \triangleq V_2 + CQC^{\mathrm{T}}$$

$$P_s \triangleq B^{\mathrm{T}}PA + R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}PA_i, \qquad Q_s \triangleq AQC^{\mathrm{T}} + V_{12} + \sum_{i=1}^{p} A_i(Q + mm^{\mathrm{T}})C_i^{\mathrm{T}}$$

$$P_a \triangleq B^{\mathrm{T}}PA + R_{12}^{\mathrm{T}}, \qquad Q_a \triangleq AQC^{\mathrm{T}} + V_{12}$$

$$P_{s1} \triangleq R_{12}^{\mathrm{T}} + \sum_{i=1}^{p} B_i^{\mathrm{T}}PA_i, \qquad Q_{s1} \triangleq V_{12} + \sum_{i=1}^{p} A_i(Q + mm^{\mathrm{T}})C_i^{\mathrm{T}}$$

1. Introduction

The quadratic performance criterion

$$J = \sum_{k=0}^{N} x^{\mathrm{T}}(k)R_1x(k) + u^{\mathrm{T}}(k)R_2u(k) \tag{1}$$

expresses the desire to minimize deviations of the state $x(k)$ of the system

$$x(k+1) = Ax(k) + Bu(k) + w(k) \tag{2}$$

from the regulation point $x = 0$. As is well known (Kwakernaak and Sivan, 1972, pp. 504–509), the non-zero set point criterion

$$J_{\bar{x}} = \sum_{k=0}^{N} [x(k) - \bar{x}]^{\mathrm{T}}R_1[x(k) - \bar{x}] + u^{\mathrm{T}}(k)R_2u(k) \tag{3}$$

presents no additional difficulty so long as $x(k)$ and $u(k)$ are replaced by $x(k) - \bar{x}$ and $u(k) - \bar{u}$, where $\bar{u}$ satisfies

$$\bar{x} = A\bar{x} + B\bar{u} \tag{4}$$

Closer inspection, however, reveals that this approach is suboptimal. Specifically, the offset $\bar{u}$ in the control may correspond to an unacceptably high level of control effort when $\bar{u}^{\mathrm{T}}R_2\bar{u}$ is large. Hence (3) overlooks design tradeoffs concerning the control effort required for maintaining the non-zero regulation point $\bar{x}$. Moreover, such an approach is impossible when $\bar{u}$ satisfying (4) does not exist.

A significant advance in extending the full-state-feedback LQR formulation to steady-state periodic tracking problems (and hence to the special case of non-zero set point regulation) was given by Artstein and Leizarowitz (1985). Bernstein and Haddad (1987 b) generalize the results of Artstein and Leizarowitz (1985) for the non-zero set point regulation problem to include noisy and non-noisy measurements, weighted and unweighted controls, correlated plant/measurement noise, cross weighting, non-zero mean disturbances, and state-, control- and measurement-dependent multiplicative white noise. They consider the steady-state performance criterion

$$J = \lim_{t\to\infty} \mathbf{E}[(Lx(t) - \delta)^{\mathrm{T}}R_1(Lx(t) - \delta) + 2(Lx(t) - \delta)^{\mathrm{T}}R_{12}u(t) + u^{\mathrm{T}}(t)R_2u(t)] \tag{5}$$

where δ is the non-zero regulation point. For full-state feedback with $R_{12}=0$ and $L=$ identity, Artstein and Leizarowitz (1985) show that for a constant offset control law

$$u(t) = Kx(t) + \alpha \tag{6}$$

K and α are given bv

$$K = -R_2^{-1} B^{\mathrm{T}} P \tag{7}$$

$$\alpha = -R_2^{-1} B^{\mathrm{T}} (A - \Sigma P)^{-\mathrm{T}} R_1 \delta \tag{8}$$

where P satisfies the Riccati equation

$$0 = A^{\mathrm{T}} P + PA + R_1 - P\Sigma P$$

with

$$\Sigma \triangleq B R_2^{-1} B^{\mathrm{T}}$$

Two features of the control law (6)–(8) are noteworthy. First, (6) consists of both a closed-loop feedback component $Kx(t)$ and an open-loop component α depending upon the regulation point δ. And, second (and more important), is the observation that the closed-loop control component is independent of the open-loop component. From a practical point of view this feature is quite useful since it implies that the feedback gain K can be determined without regard to the set point. Hence a change in the desired set point δ during on-line operation does not necessitate re-solving the Riccati equation in real time; only α requires updating. For a new value of δ, α can readily be recomputed on-line via the matrix multiplication operation (8). In the presence of multiplicative disturbances, however, the independence of the closed-loop component from the open-loop component is lost.

The purpose of the present paper is to provide a self-contained derivation of the optimality conditions for the non-zero set point problem in the discrete-time case. To obtain a realistic problem setting, we consider the case in which the full state is not available, but rather only noise-corrupted measurements of linear combinations of states. For greater design flexibility, we also allow the possibility for correlated plant and measurement noise. In addition, we consider the dual design feature, namely, cross weighting in the performance criterion. The presence of a non-zero constant plant disturbance in conjunction with zero-mean white plant disturbances, i.e. a non-zero mean disturbance, is also considered. Our results show that the presence of a non-zero constant disturbance component leads to an additional offset in the open-loop component of the control. Finally, in addition to the above generalizations we allow for the presence of multiplicative disturbances in the plant. The control law thus generalizes previous results involving state-, control- and measurement-dependent noise (Bernstein and Haddad 1987). As shown in Bernstein and Greeley (1986) and Haddad (1987), the multiplicative white noise model can be used for robustness with respect to plant parameter variations.

2. Non-zero set point regulation

2.1. *Non-zero set point problem*

Given the nth-order controlled system

$$x(k+1) = \left(A + \sum_{i=1}^{p} v_i(k) A_i\right) x(k) + \left(B + \sum_{i=1}^{p} v_i(k) B_i\right) u(k) + w_1(k) + \gamma \tag{9}$$

with measurements

$$y(k) = \left(C + \sum_{i=1}^{p} v_i(k)C_i\right)x(k) + w_2(k) \tag{10}$$

where $k = 1, 2, \ldots$, determine K and α such that the static output feedback controller

$$u(k) = Ky(k) + \alpha \tag{11}$$

minimizes the steady-state performance criterion

$$J(K, \alpha) \triangleq \lim_{k\to\infty} \mathbf{E}[(Lx(k) - \delta)^{\mathrm{T}} R_1 (Lx(k) - \delta) + 2(Lx(k) - \delta)^{\mathrm{T}} R_{12} u(k) + u^{\mathrm{T}}(k) R_2 u(k)] \tag{12}$$

Using the notation of § 1 the closed-loop system (9)–(11) can be written as

$$x(k+1) = \left(\tilde{A} + \sum_{i=1}^{p} v_i(k)\tilde{A}_i\right)x(k) + \tilde{B} + \sum_{i=1}^{p} v_i(k)\tilde{B}_i + \tilde{w}(k) \tag{13}$$

To analyse (13) define the second-moment and covariance matrices

$$\tilde{Q}(k) \triangleq \mathbf{E}[x(k)x^{\mathrm{T}}(k)], \quad Q(k) \triangleq \tilde{Q}(k) - m(k)m^{\mathrm{T}}(k)$$

where $m(k) \triangleq \mathbf{E}[x(k)]$. It follows from (13) that $\tilde{Q}(k)$, $Q(k)$ and $m(k)$ satisfy

$$\begin{aligned}\tilde{Q}(k+1) &= \tilde{A}\tilde{Q}(k)\tilde{A}^{\mathrm{T}} + \tilde{A}m(k)\tilde{B}^{\mathrm{T}} + \tilde{B}m^{\mathrm{T}}(k)\tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} \\ &\quad + \sum_{i=1}^{p}[\tilde{A}_i\tilde{Q}(k)\tilde{A}_i^{\mathrm{T}} + \tilde{B}_i m^{\mathrm{T}}(k)\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m(k)\tilde{B}_i^{\mathrm{T}} + \tilde{B}_i\tilde{B}_i^{\mathrm{T}}] + \tilde{V}\end{aligned} \tag{14}$$

$$\begin{aligned}Q(k+1) &= \tilde{A}Q(k)\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p}[\tilde{A}_i Q(k)\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m(k)m^{\mathrm{T}}(k)\tilde{A}_i^{\mathrm{T}} \\ &\quad + \tilde{B}_i m^{\mathrm{T}}(k)\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m(k)\tilde{B}_i^{\mathrm{T}}] + \tilde{V}\end{aligned} \tag{15}$$

$$m(k+1) = \tilde{A}m(k) + \tilde{B} \tag{16}$$

To consider the steady state, we restrict our consideration to the set of closed-loop second-moment stabilizing gains

$$\mathbf{S}_s \triangleq \left\{K : \tilde{A}\otimes\tilde{A} + \sum_{i=1}^{p} \tilde{A}_i\otimes\tilde{A}_i \text{ is asymptotically stable}\right\}$$

It follows from fundamental properties of Lyapunov equations that if $K \in \mathbf{S}_s$, then $\tilde{A}$ is also asymptotically stable. Hence, for $K \in \mathbf{S}_s$, $\tilde{Q} \triangleq \lim_{k\to\infty} \tilde{Q}(k)$, $Q \triangleq \lim_{k\to\infty} Q(k)$ and $m \triangleq \lim_{k\to\infty} m(k)$ exist and satisfy

$$\begin{aligned}\tilde{Q} &= \tilde{A}\tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{A}m\tilde{B}^{\mathrm{T}} + \tilde{B}m^{\mathrm{T}}\tilde{A}^{\mathrm{T}} + \tilde{B}\tilde{B}^{\mathrm{T}} \\ &\quad + \sum_{i=1}^{p}[\tilde{A}_i\tilde{Q}\tilde{A}_i^{\mathrm{T}} + \tilde{B}_i m^{\mathrm{T}}\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m\tilde{B}_i^{\mathrm{T}} + \tilde{B}_i\tilde{B}_i^{\mathrm{T}}] + \tilde{V}\end{aligned} \tag{17}$$

$$Q = \tilde{A}Q\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p}[\tilde{A}_i Q\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i mm^{\mathrm{T}}\tilde{A}_i^{\mathrm{T}} + \tilde{B}_i m^{\mathrm{T}}\tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m\tilde{B}_i^{\mathrm{T}} + \tilde{B}_i\tilde{B}_i^{\mathrm{T}}] + \tilde{V} \tag{18}$$

$$m = \mathbf{A}^{-1}\tilde{B} \tag{19}$$

Note that since $\tilde{A}$ is asymptotically stable, the inverse in (19) exists. For $K \in \mathbf{S}_s$, it now follows that $J(K,\alpha)$ is given by

$$J(K,\alpha) = \operatorname{tr}[(Q+mm^{\mathrm{T}})\tilde{R}] + \operatorname{tr}[K^{\mathrm{T}}R_2KV_2] + \delta^{\mathrm{T}}R_1\delta - 2m^{\mathrm{T}}L^{\mathrm{T}}R_1L\delta$$
$$+ 2m^{\mathrm{T}}L^{\mathrm{T}}R_{12}\alpha - 2\delta^{\mathrm{T}}R_{12}KCm - 2\delta^{\mathrm{T}}R_{12}\alpha + 2m^{\mathrm{T}}C^{\mathrm{T}}K^{\mathrm{T}}R_2\alpha + \alpha^{\mathrm{T}}R_2\alpha \quad (20)$$

Associated with Q is its dual $P \geqslant 0$ which is the unique solution of

$$P = \tilde{A}^{\mathrm{T}}P\tilde{A} + \sum_{i=1}^{p}\tilde{A}_i^{\mathrm{T}}P\tilde{A}_i + \tilde{R} \quad (21)$$

To obtain closed-form expressions for the feedback gain K, we further restrict consideration to the set

$$\mathbf{S}_s^+ \triangleq \{K \in \mathbf{S}_s : R_{2s} > 0,\ V_{2s} > 0 \text{ and } \Psi_s \text{ is invertible}\}$$

where

$$\Psi_s \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}R_1L\mathbf{A}^{-1}B + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}R_{12}(I_m + KC\mathbf{A}^{-1}B)$$
$$+ (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}R_{12}^{\mathrm{T}}L\mathbf{A}^{-1}B + (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}R_2(I_m + KC\mathbf{A}^{-1}B)$$
$$+ \sum_{i=1}^{p}[B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}\tilde{A}_i^{\mathrm{T}}P\tilde{A}_i\mathbf{A}^{-1}B + B_i^{\mathrm{T}}P\tilde{A}_i\mathbf{A}^{-1}B + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}A_i^{\mathrm{T}}PB_i$$
$$+ B_i^{\mathrm{T}}PB_i + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}C_i^{\mathrm{T}}K^{\mathrm{T}}R_2KC_i\mathbf{A}^{-1}B]$$

Furthermore, we assume that

$$[B_i \neq 0 \Rightarrow C_i = 0], \quad i = 1, \ldots, p \quad (22)$$

i.e. for each $i \in \{1, \ldots, p\}$, B_i and C_i are not both non-zero. Essentially, (22) expresses the condition that the control-dependent and measurement-dependent disturbances are independent. There are no constraints, however, on correlation with the state-dependent noise. For the statement of the main theorem define

$$\Lambda_s \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}(R_1L + R_{12}KC)\mathbf{A}^{-1} + (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}(R_{12}^{\mathrm{T}}L + R_2KC)\mathbf{A}^{-1}$$
$$+ \sum_{i=1}^{p}([A_i\mathbf{A}^{-1}B + B_i]^{\mathrm{T}}P\tilde{A}_i\mathbf{A}^{-1} + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}C_i^{\mathrm{T}}K^{\mathrm{T}}R_2KC_i\mathbf{A}^{-1})$$

$$\Omega \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}R_1L + R_{12}^{\mathrm{T}}L + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}C^{\mathrm{T}}K^{\mathrm{T}}R_{12}^{\mathrm{T}}L$$

Theorem 2.1

Suppose K and α solve the non-zero set point problem with $K \in \mathbf{S}_s^+$. Then there exist $n \times n$ Q, $P \geqslant 0$ such that

$$K = -R_{2s}^{-1}(B^{\mathrm{T}}PAQC^{\mathrm{T}} + P_{s1}QC^{\mathrm{T}} + B^{\mathrm{T}}PQ_{s1})V_{2s}^{-1} \quad (23)$$

$$\alpha = -\Psi_s^{-1}[\Lambda_s\gamma + \Omega\delta] \quad (24)$$

and such that Q and P satisfy

$$Q = AQA^{\mathrm{T}} + V_1 + \sum_{i=1}^{p}[(A_i + B_iKC)Q(A_i + B_iKC)^{\mathrm{T}} + B_iKV_2K^{\mathrm{T}}B_i^{\mathrm{T}} + \tilde{A}_imm^{\mathrm{T}}\tilde{A}_i^{\mathrm{T}}$$
$$+ \tilde{B}m^{\mathrm{T}}\tilde{A}_i^{\mathrm{T}} + \tilde{A}_im\tilde{B}^{\mathrm{T}} + \tilde{B}_i\tilde{B}_i^{\mathrm{T}}]$$
$$+ (Q_s + BKV_{2s})V_{2s}^{-1}(Q_s + BKV_{2s})^{\mathrm{T}} - Q_sV_{2s}Q_s^{\mathrm{T}} \quad (25)$$

$$P = A^{\mathrm{T}}PA + R_1 + \sum_{i=1}^{p}[(A_i + BKC_i)^{\mathrm{T}}P(A_i + BKC_i) + C_i^{\mathrm{T}}K^{\mathrm{T}}R_2KC_i$$
$$+ (P_s + R_{2s}KC)^{\mathrm{T}}R_{2s}^{-1}(P_s + R_{2s}KC) - P_s^{\mathrm{T}}R_{2s}P_s] \quad (26)$$

Proof

The derivation of the necessary conditions is a straightforward application of the Lagrange multiplier technique. To optimize (20) over $\mathbf{S}_s^+$ subject to the constraints (18) and (19), form the lagrangian

$$\mathbf{L}(K, \alpha, Q, P, m) \triangleq \operatorname{tr}\left[\lambda_0 J(K, \alpha) + \left(\tilde{A}Q\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} [\tilde{A}_i Q \tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m m^{\mathrm{T}} \tilde{A}_i^{\mathrm{T}} + \tilde{B}_i m^{\mathrm{T}} \tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m \tilde{B}_i^{\mathrm{T}} + \tilde{B}_i \tilde{B}_i^{\mathrm{T}}] + \tilde{V} - Q\right)P + \lambda^{\mathrm{T}}(\tilde{A}m + \tilde{B} - m)\right]$$

where the Lagrange multipliers $\lambda_0 \geqslant 0$, $\lambda \in \mathbf{R}^n$ and $P \in \mathbf{R}^{n \times n}$ are not all zero. Setting $\partial\mathbf{L}/\partial Q = 0$ and using the second-moment stability assumption it follows that $\lambda_0 = 1$ without loss of generality. Thus the stationarity conditions are given by

$$\frac{\partial \mathbf{L}}{\partial Q} = \tilde{A}^{\mathrm{T}} P \tilde{A} + \sum_{i=1}^{p} \tilde{A}_i^{\mathrm{T}} P \tilde{A}_i + \tilde{R} - P = 0 \tag{27}$$

$$\frac{\partial \mathbf{L}}{\partial P} = \tilde{A}Q\tilde{A}^{\mathrm{T}} + \sum_{i=1}^{p} [\tilde{A}_i Q \tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m m^{\mathrm{T}} \tilde{A}_i^{\mathrm{T}} + \tilde{B}_i m^{\mathrm{T}} \tilde{A}_i^{\mathrm{T}} + \tilde{A}_i m \tilde{B}_i^{\mathrm{T}} + \tilde{B}_i \tilde{B}_i^{\mathrm{T}}] + \tilde{V} - Q = 0 \tag{28}$$

$$\frac{\partial \mathbf{L}}{\partial \lambda^{\mathrm{T}}} = \tilde{A}m + \tilde{B} - m = 0 \tag{29}$$

$$\frac{\partial \mathbf{L}}{\partial K} = R_{2s} K V_{2s} + R_{12}^{\mathrm{T}} Q C^{\mathrm{T}} + B^{\mathrm{T}} P A Q C^{\mathrm{T}} + \sum_{i=1}^{p} [B_i P A_i Q C + B^{\mathrm{T}} P A_i (Q + m m^{\mathrm{T}}) C_i^{\mathrm{T}}] + B^{\mathrm{T}} P V_{12} = 0 \tag{30}$$

$$\frac{\partial \mathbf{L}}{\partial \alpha} = \sum_{i=1}^{p} [B_i^{\mathrm{T}} P A_i m + B_i^{\mathrm{T}} P B_i K C + B_i^{\mathrm{T}} P B_i \alpha] + \frac{1}{2} B^{\mathrm{T}} \lambda + R_{12}^{\mathrm{T}} L m - R_{12}^{\mathrm{T}} \delta + R_2 K C m + R_2 \alpha = 0 \tag{31}$$

$$\frac{\partial \mathbf{L}}{\partial m} = \tilde{R}m + \sum_{i=1}^{p} [\tilde{A}_i^{\mathrm{T}} P B_i + \tilde{A}_i^{\mathrm{T}} P \tilde{A}_i m] - \frac{1}{2}\mathbf{A}^{\mathrm{T}} \lambda - L^{\mathrm{T}} R_1 \delta + L^{\mathrm{T}} R_{12} \alpha - C^{\mathrm{T}} K^{\mathrm{T}} R_{12}^{\mathrm{T}} \delta + C^{\mathrm{T}} K^{\mathrm{T}} R_2 \alpha = 0 \tag{32}$$

$$\lambda = 2\mathbf{A}^{-\mathrm{T}}\left(\tilde{R}m + \sum_{i=1}^{p} [\tilde{A}_i^{\mathrm{T}} P \tilde{B}_i + \tilde{A}_i P \tilde{A}_i^{\mathrm{T}} m] - L^{\mathrm{T}} R_1 \delta + L^{\mathrm{T}} R_{12} \alpha - C^{\mathrm{T}} K^{\mathrm{T}} R_{12}^{\mathrm{T}} \delta + C^{\mathrm{T}} K^{\mathrm{T}} R_2 \alpha\right) \tag{33}$$

Using the definitions for Q_{s1} and P_{s1} along with (33), we obtain (23) and (24). Substituting the expressions for the optimal gains into (27) and (28) yields (25) and (26). □

Remark 1

Because of the presence of δ in (25) via m in both Q_{s1} and V_{2s} and in (25) via $\tilde{B}$ (in m) and $\tilde{B}_i$, the closed-loop component of the control law (23) cannot be determined independently of the open-loop component. As shown in the following section, independence is recovered when the multiplicative noise terms are absent.

Remark 2

To specialize Theorem 2.1 to the standard regulation problem, set $\delta = 0$ and $\gamma = 0$ yielding Theorem 2.1 of Bernstein and Haddad (1987 a).

3. Specializations of Theorem 2.1

A series of specializations of Theorem 2.1 is now given. We begin by deleting all multiplicative white noise terms, i.e.

$$A_i, B_i, C_i = 0, \quad i = 1, \ldots, p \tag{34}$$

In this case the stabilizing set $\mathbf{S}_s$ can be characterized by

$$\mathbf{S} = \{K : \tilde{A} \text{ is asymptotically stable}\}$$

and, furthermore, $\mathbf{S}_s^+$ becomes

$$\mathbf{S}^+ \triangleq \{K \in \mathbf{S} : R_{2a} > 0, V_{2a} > 0 \text{ and } \Psi_a \text{ is invertible}\}$$

where

$$\Psi_a \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}R_1 L\mathbf{A}^{-1}B + B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}R_{12}(I_m + KC\mathbf{A}^{-1}B)$$
$$+ (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}R_{12}^{\mathrm{T}}L\mathbf{A}^{-1}B + (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}R_2(I_m + KC\mathbf{A}^{-1}B)$$

For the statement of Corollary 3.1 define

$$\Lambda_a \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}L^{\mathrm{T}}(R_1 L + R_{12}KC)\mathbf{A}^{-1} + (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}(R_{12}^{\mathrm{T}}L + R_2 KC)\mathbf{A}^{-1}$$

Corollary 3.1

Assume (34) is satisfied and suppose K and α solve the non-zero set point problem with $K \in \mathbf{S}^+$. Then there exist $n \times n$ $Q, P \geqslant 0$ such that

$$K = -R_{2a}^{-1}(B^{\mathrm{T}}PAQC^{\mathrm{T}} + R_{12}^{\mathrm{T}}QC^{\mathrm{T}} + B^{\mathrm{T}}PV_{12})V_{2a}^{-1} \tag{35}$$

$$\alpha = -\Psi_a^{-1}[\Lambda_a\gamma + \Omega\delta] \tag{36}$$

and such that Q and P satisfy

$$Q = AQA^{\mathrm{T}} + V_1 + (Q_a + BKV_{2a})V_{2a}^{-1}(Q_a + BKV_{2a})^{\mathrm{T}} - Q_a V_{2a} Q_a^{\mathrm{T}} \tag{37}$$

$$P = A^{\mathrm{T}}PA + R_1 + (P_a + R_{2a}KC)^{\mathrm{T}}R_{2a}^{-1}(P_a + R_{2a}KC) - P_a^{\mathrm{T}}R_{2a}P_a \tag{38}$$

Finally, setting

$$\gamma = 0, \quad R_{12} = 0, \quad V_{12} = 0, \quad r = n, \quad L = I_n \tag{39}$$

we obtain the discrete-time version of Artstein and Leizarowitz (1985) for the case of output feedback. Define

$$\mathbf{S}_1^+ \triangleq \{K \in \mathbf{S} : R_{2a} > 0, V_{2a} > 0 \text{ and } \Psi_1 > 0\}$$

where

$$\Psi_1 \triangleq B^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}R_1\mathbf{A}^{-1}B + (I_m + KC\mathbf{A}^{-1}B)^{\mathrm{T}}R_2(I_m + KC\mathbf{A}^{-1}B)$$

Corollary 3.2

Assume (34) and (39) are satisfied and suppose K and α solve the non-zero set point problem with $K \in \mathbf{S}_1^+$. Then there exist $n \times n$ $Q, P \geqslant 0$ such that

$$K = -R_{2a}^{-1} B^{\mathrm{T}} PAQC^{\mathrm{T}} V_{2a}^{-1} \tag{40}$$

$$\alpha = -\Psi_1^{-1} B^{\mathrm{T}} \mathbf{A}^{-\mathrm{T}} R_1 \delta \tag{41}$$

and such that Q and P satisfy

$$\begin{aligned} Q = {} & AQA^{\mathrm{T}} + V_1 + (AQC^{\mathrm{T}} + BKV_{2a}) V_{2a}^{-1} (AQC^{\mathrm{T}} + BKV_{2a})^{\mathrm{T}} \\ & - AQC^{\mathrm{T}} V_{2a} CQA^{\mathrm{T}} \end{aligned} \tag{42}$$

$$\begin{aligned} P = {} & A^{\mathrm{T}} PA + R_1 + (B^{\mathrm{T}} PA + R_{2a} KC)^{\mathrm{T}} R_{2a}^{-1} (B^{\mathrm{T}} PA + R_{2a} KC) \\ & - A^{\mathrm{T}} PBR_{2a} B^{\mathrm{T}} PA \end{aligned} \tag{43}$$

4. Directions for further research

The extension to fixed-order dynamic compensation for non-zero set point regulation appears possible using the approach of Hyland and Bernstein (1984) and Haddad (1987). A generalization of Theorem 2.1 to design periodic tracking controllers (either static or dynamic) via the parameter optimization approach is being developed.

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