results of [10]. This note, therefore, unveils a link between polynomial hyperbolicity and stability.

Finally, as pointed out in [3, Sec. 18.9], the applications of frequency response convexity in robust control have only been minimally explored. The explicit LMI formulation described in this note may motivate further research along these lines.

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REFERENCES


I. INTRODUCTION

Internal model control is essential for both command following and disturbance rejection, where the exogenous command and disturbance signals are outputs of an unforced linear system. In short, the internal model principle states that asymptotic command following and disturbance rejection can be achieved by incorporating copies of the exogenous dynamics in the feedback loop. The most familiar case of internal model control is integral control for following step commands or rejecting constant disturbances.

Internal model control in continuous time was developed in [1]–[3]. Specifically, [1] treats the disturbance rejection problem for a class of matched systems in which the range of the disturbance input matrix is contained in the range of the control input matrix. Multiinput multioutput (MIMO) internal model control for both command following and disturbance rejection problems is given in [4]–[7], including the synthesis of a servocompensator. An alternative geometric approach is given in [8]. The converse problem is addressed in [9], and necessary conditions for asymptotic regulation are developed. Continuous-time internal model control is considered in [10] and [11], while [12] constructs internal model controllers for disturbances with known characteristics. Optimal H^2 control is combined in [13] and [14] with the internal model principle to develop controllers for both command following and disturbance rejection.

The results of [1]–[9] on internal model control are confined to continuous-time systems. Although the principle is used in [12] for single-input single-output (SISO) discrete-time systems, results for MIMO discrete-time systems are not available in the literature. Furthermore, the results of [4]–[7] use analytic tools specific to continuous-time models, and thus, do not extend to discrete-time systems. In the present note, we develop an alternative approach to internal model control that is applicable to continuous-time systems as well as discrete-time systems in both the shift and delta domains [15]. The MIMO problem that we consider includes both the command following and disturbance rejection problems as special cases, as well as mixed problems that include aspects of both. In contrast to the analytical approach of [4]–[7] and the geometric approach of [8] and [9], our approach is algebraic. An earlier version of this note appears as [16].

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where \( x \in \mathbb{R}^n \) is the state, \( y \in \mathbb{R}^l \) is the measured output available to the controller, \( z \in \mathbb{R}^l \) is the performance, \( u \in \mathbb{R}^r \) is the control, and \( w \in \mathbb{R}^r \) is the exogenous input. To simultaneously consider continuous-time and discrete-time systems, let \( D \) denote the differential operator \( d/dt \), the forward shift operator \( q \), or the delta operator \( \delta \), where

\[
D = \begin{cases} 
\frac{d}{dt} & \text{for continuous} \\
q & \text{for discrete} \\
\delta & \text{for delta}
\end{cases}
\]

The problem objective is to construct a feedback controller that stabilizes the open-loop system (1)–(3) and regulates the performance \( z \) to 0 when the exogenous input \( w \) is generated by an unforced linear system. This control problem includes both disturbance rejection and command following objectives, where the exogenous signal \( w \) may contain components to be rejected and components to be followed. The problem can be restricted to command following by letting \( D_1 = 0 \), or can be restricted to disturbance rejection by letting \( D_2 = 0 \) and \( E_0 = 0 \). Mixed problems can be considered as well. As in the general problem formulation (1)–(3), all signals in these specialized problems may be multivariable.

As a special case, the classical SISO command following problem in Fig. 1 can be written in the form (1)–(3), where the plant \( G \) has the realization

\[
G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix},
\]

where \( D_1 = 0, E_1 = C, E_2 = D, E_0 = D_2 = -1 \), and \( \hat{G} \) is the feedback controller. Then, \( z = y = Cx + Du + D_2w = y_{out} - w \) is the tracking error, where \( y_{out} = Cx + Du \). Similarly, the classical SISO disturbance rejection problem in Fig. 2 can be written in the form (1)–(3), where \( D_1 = B, E_1 = C, E_2 = D, E_0 = D_2 = 0 \), and

\[
G \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
\]
Next, consider the derogatory system

\[ w_r(k) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} q_r(k), \quad w_r(k) = w(k). \] (10)

Next, suppose that \( q_r(0) \neq 0 \) and construct the system

\[ q_r(k + 1) = \lambda q_r(k), \quad w_r(k) = \begin{bmatrix} q_r(0) \\ q_r(0) \\ q_r(0) \end{bmatrix} \] (11)

Then, with \( q_r(0) = q_r(0) \), it follows that

\[ w_r(k) = \begin{bmatrix} \lambda^k q_r(0) \\ \lambda^k q_r(0) \\ \lambda^k q_r(0) \end{bmatrix} = w(k). \] (12)

Next, consider the derogatory system

\[ q(k + 1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} q(k), \quad w(k) = q(k) \] (13)

where \( q(k) \triangleq [q_1(k) \quad q_2(k) \quad q_3(k)]^T \). Hence

\[ q(k) = \begin{bmatrix} \lambda^k & 0 & 0 \\ 0 & \lambda^k & 0 \\ 0 & 0 & \lambda^k \end{bmatrix} q(0), \quad w(k) = q(k). \] (14)

If \( q_r(0) = 0 \), then \( q_r(k) \equiv 0 \), and an equivalent system can be constructed as in (8). Hence, assume \( q_r(0) \neq 0 \) and consider the cyclic system

\[ q_r(k + 1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} q_r(k), \quad w_r(k) = C_r q_r(k) \] (15)

where \( q_r(k) \triangleq [q_1(k) \quad q_2(k)]^T \) and

\[ C_r \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \] (16)

Then, with \( q_1(0) = q_2(0) = q_3(0) \), it follows that

\[ w_r(k) = \begin{bmatrix} \lambda^k q_1(0) + k \lambda^{k-1} q_2(0) \\ \lambda^k q_2(0) \\ \lambda^k q_3(0) \end{bmatrix} \]

\[ = \begin{bmatrix} \lambda^k q_1(0) + k \lambda^{k-1} q_2(0) \\ \lambda^k q_2(0) \\ \lambda^k q_3(0) \end{bmatrix} = w(k). \] (17)

Note that, in all of these cases, the cyclic system is observable. If, however, the cyclic system is not observable, then the unobservable dynamics can be truncated. This construction extends to derogatory systems of arbitrary order.

Furthermore, analogous results hold in continuous time as well as in discrete time with the delta operator. To see this, note that, in continuous time, the state-transition matrix has the same form as in discrete time with the shift operator, where \( \lambda^k \) is replaced by \( e^{\lambda t} \). For discrete time with the delta operator, note that, if \( A_r \) represents cyclic dynamics in discrete time with the shift operator, then \( A_r \triangleq (1/h) A_r - I \) represents cyclic dynamics in discrete time with the delta operator. Therefore, without the loss of generality, \( A_r \) can be assumed to be cyclic. This assumption is invoked in the proof of Theorem 4.1.

We now consider the feedback controller

\[ \hat{D} \hat{x} = \hat{A} \hat{x} + \hat{B} y \] (18)

\[ u = C \hat{x} \] (19)

where \( \hat{x} \in \mathbb{R}^n \). The closed-loop system (1)–(3) with the feedback controller (18), (19) is given by

\[ \hat{D} \hat{x} = \hat{A} \hat{x} + \hat{D} w \] (20)

\[ z = \hat{E} \hat{x} + e_0 w \] (21)

where

\[ \hat{A} \triangleq \begin{bmatrix} A & B C \\ \hat{B} C & \hat{A} + \hat{B} D C \end{bmatrix}, \quad \hat{D} \triangleq \begin{bmatrix} D_1 \\ \hat{B} D_2 \end{bmatrix}, \]

\[ \hat{E} \triangleq [E_1 \quad E_2 \hat{C}], \quad \hat{x} \triangleq \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \] (22)

The following result provides necessary and sufficient conditions for \( z \) to converge to 0 for arbitrary initial conditions. This result is presented in [9] for continuous-time systems, where a geometric proof is given. We provide an alternative proof that extends the result to discrete-time systems in the shift and delta domains.

**Lemma 3.1:** Consider the closed-loop system (20)–(22) with the exogenous input (6), (7), and assume that \( \hat{A} \) is asymptotically stable. Then, for all initial conditions \( \hat{x}(0) \) and \( x_s(0) \), \( \lim_{t \to \infty} z(t) = 0 \) if and only if there exists \( S \in \mathbb{R}^{(n+h) \times n+w} \) such that

\[ \tilde{A} \hat{S} - \hat{A} A_r \hat{D} C_r = \hat{D} \hat{C}_w \] (23)

\[ \hat{E} S = E_0 C_w. \] (24)

**Proof:** The closed-loop system (20)–(22) with the exogenous input (6), (7) can be written as

\[ \hat{D} x_s = A_r x_s \] (25)

\[ z = E_s x_s \] (26)

where

\[ A_r \triangleq \begin{bmatrix} \hat{A} & \hat{D} C_r \\ 0 & A_r \end{bmatrix}, \quad E_s \triangleq [\hat{E} \ E_0 C_w], \quad x_s \triangleq \begin{bmatrix} x \\ x_w \end{bmatrix}. \] (27)

Since \( \hat{A} \) is asymptotically stable and \( A_r \) is antistable, the Sylvester equation (23) has a unique solution \( S \in \mathbb{R}^{(n+h) \times n+w} \). Now define

\[ Q \triangleq \begin{bmatrix} I & -S \\ 0 & I \end{bmatrix} \] (28)

and consider the change of basis

\[ \hat{A}_s \triangleq Q^{-1} A_r Q \] (29)

\[ E_s \triangleq E_s Q = [\hat{E} \ -\hat{E} S + E_0 C_w]. \] (30)

To prove necessity in the continuous-time case, suppose that \( \lim_{t \to \infty} z(t) = 0 \) for all initial conditions, so that \( \lim_{t \to \infty} [\hat{E} e^{\hat{A} t} + (-\hat{E} S + E_0 C_w) e^{A_r t}] = 0 \). Since \( \hat{A} \) is asymptotically stable, it follows that \( \lim_{t \to \infty} [\hat{E} e^{\hat{A} t}] = 0 \), and thus, \( \lim_{t \to \infty} (-\hat{E} S + E_0 C_w) e^{A_r t} = 0 \). Since \( A_r \) is antistable, every nonzero entry of \( (-\hat{E} S + E_0 C_w) e^{A_r t} \) is either a constant or involves exponentials of \( t \), where each coefficient of \( t \) has nonnegative real part. Therefore, \( \lim_{t \to \infty} (-\hat{E} S + E_0 C_w) e^{A_r t} = 0 \) implies that \( -\hat{E} S + E_0 C_w = 0 \).
To prove necessity in the discrete-time case with the shift operator, suppose that \( \lim_{t \to \infty} z(t) = 0 \) for all initial conditions, so that \( \lim_{t \to \infty} \{ E(A^t + (-E\varepsilon + E_0 C_\varepsilon) A_\varepsilon^t \} = 0 \). Since \( \dot{A} \) is asymptotically stable, it follows that \( \lim_{t \to \infty} E(A^t) = 0 \), and thus \( \lim_{t \to \infty} (-E\varepsilon + E_0 C_\varepsilon) A_\varepsilon^t = 0 \). As \( \dot{A} \) is antistable, \( \lim_{t \to \infty} A_\varepsilon^t \) does not exist, and, for all \( t \geq 0 \), \( A_\varepsilon^t \) is nonsingular. Assume that \( -E\varepsilon + E_0 C_\varepsilon \neq 0 \), let \( \sigma_{\min}() \) denote the minimum singular value, and let \( \| \cdot \|_F \) denote the Frobenius norm. Then, it follows that \( 0 = \lim_{t \to \infty} \| (-E\varepsilon + E_0 C_\varepsilon) A_\varepsilon^t \|_F \). But \( \lim_{t \to \infty} \sigma_{\min}(A_\varepsilon^t) = \infty \), which is a contradiction. Therefore, \( -E\varepsilon + E_0 C_\varepsilon = 0 \).

To prove necessity in the discrete-time case with the shift operator, suppose that \( \lim_{t \to \infty} z(t) = 0 \) for all initial conditions, so that \( \lim_{t \to \infty} E(I + h\dot{A})^{y/h} = 0 \). Thus, \( \lim_{t \to \infty} E(I + h\dot{A})^{y/h} = 0 \). Since \( \dot{A} \) is asymptotically stable, it follows that \( \lim_{t \to \infty} E(I + h\dot{A})^{y/h} = 0 \) and thus \( \lim_{t \to \infty} (-E\varepsilon + E_0 C_\varepsilon) I + h\dot{A}^{y/h} = 0 \). Since \( \dot{A} \) is antistable, \( \lim_{t \to \infty} (-E\varepsilon + E_0 C_\varepsilon) I + h\dot{A}^{y/h} \) does not exist, and, for all \( t \geq 0 \), \( (I + h\dot{A})^{y/h} ) \) is nonsingular. Assuming that \( -E\varepsilon + E_0 C_\varepsilon \neq 0 \) it follows that \( 0 = \lim_{t \to \infty} \| E(I + h\dot{A}) I + h\dot{A}^{y/h} \|_F \geq \| -E\varepsilon + E_0 C_\varepsilon \|_F \lim_{t \to \infty} \sigma_{\min}(I + h\dot{A}^{y/h}) = \infty \), which is a contradiction. Therefore, \( -E\varepsilon + E_0 C_\varepsilon = 0 \).

Conversely, since \( E\varepsilon = E_0 C_\varepsilon \), we have \( z(t) = E\varepsilon \dot{A} \hat{x}(0) \) in continuous time, \( z(t) = E\varepsilon \dot{x}(0) \) in discrete time with the shift operator, and \( z(t) = E(I + h\dot{A})^{y/h} \hat{x}(0) \) in discrete time with the delta operator. Since \( \dot{A} \) is asymptotically stable, \( \lim_{t \to \infty} z(t) = 0 \).

\section{IV. Internal Model Control}

We now consider the MIMO command following and disturbance rejection problem for the linear system (1)–(3). We provide sufficient conditions for the existence of a feedback controller (18), (19) that stabilizes (20)–(22) and regulates the performance variable \( z \) to 0.

To describe the form of this controller, consider the open-loop system (1)–(3) and, as in [4]–[7], cascade its output with an internal model of the exogenous dynamics

\[
D \dot{x}_1 = A_W x_1 + B_W y \tag{31}
\]

where \( A_W \triangleq I_{t_y} \otimes A_e \in \mathbb{R}^{n_{xW} \times n_{yW}}, B_W \triangleq I_{t_y} \otimes B_e, \) and \( B_e \in \mathbb{R}^{n_{xW}} \) is chosen such that \((A_e, B_e)\) is controllable. The symbol \( \otimes \) represents the Kronecker product. There exists \( B_e \) such that \((A_e, B_e)\) is controllable because \( A_e \) is cyclic [17, Fact 5.12.6]. Note that the dynamics of (31) contain \( t_y \) copies of the exogenous dynamics \( A_W \). The cascade (1–3) and (31) is

\[
D \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_W C & A_W \end{bmatrix} \begin{bmatrix} x \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} B \\ B_W D \end{bmatrix} u + \begin{bmatrix} D_1 \\ B_W D_2 \end{bmatrix} w \tag{32}
\]

\[
\begin{bmatrix} y \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ \dot{x}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ D \end{bmatrix} u + \begin{bmatrix} D_2 \\ 0 \end{bmatrix} w \tag{33}
\]

Next, for the augmented system (32) and (33), we consider a feedback controller of the form

\[
D \dot{x}_2 = A_c x_2 + [B_{c_1} \ B_{c_2}] \begin{bmatrix} y \\ \dot{x}_1 \end{bmatrix} \tag{34}
\]

\[
u = C_c \hat{x}_2 \tag{35}
\]

where \( A_c \in \mathbb{R}^{(n + n_u t_y) \times (n + n_u t_y)}, B_{c_1} \in \mathbb{R}^{(n + n_u t_y) \times t_y}, B_{c_2} \in \mathbb{R}^{(n + n_u t_y) \times n_{xW}}, \) and \( C_c \in \mathbb{R}^{t_y \times (n + n_u t_y)} \). Then, the closed-loop system consisting of (32)–(35), which is shown in Fig. 4, is given by

\[
D \begin{bmatrix} x \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & BC_c \\ B_W C & A_W & B_W DC_c \\ B_{c_1} & B_{c_2} & A_c + B_{c_1} DC_c \end{bmatrix} \begin{bmatrix} x \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} D_1 \\ B_W D_2 \\ B_{c_1} D_2 \end{bmatrix} w \tag{36}
\]

\[
z = \begin{bmatrix} E_1 & 0 & E_2 C_c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + E_0 w. \tag{37}
\]

We now present the main result of this section. Note that stabilizability and detectability for delta-domain systems is defined in [15].

\textit{Theorem 4.1:} Assume that the following conditions hold.

1) \((A, B, C)\) is stabilizable and detectable.

2) \(l_y \geq l_y \).

3) For all \( \lambda \in \text{spec}(A_e) \), rank \( \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + l_y \).

4) There exists \( R \in \mathbb{R}^{l_y \times l_y} \) such that \( z = R y \).

Then, there exists a linear time-invariant controller of the form (18) and (19) such that the closed-loop system (20)–(22) is asymptotically stable, and, for all initial conditions \( x(0) \) and \( x(w) \), \( \lim_{t \to \infty} z(t) = 0 \). Furthermore, one such linear time-invariant controller is given by (18) and (19) with

\[
\hat{A} \triangleq \begin{bmatrix} A_W & 0 \\ B_{c_2} & A_c \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} B_W \\ B_{c_1} \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} 0 & C_c \end{bmatrix} \tag{38}
\]

where \( A_c, B_{c_1}, B_{c_2}, \) and \( C_c \) are chosen such that (34) and (35) stabilize (32) and (33).

\textit{Proof:} First, we show that the augmented system (32)–(33) with \( w = 0 \) is stabilizable and detectable. Let \( s \in \mathcal{U} \) and \( \lambda \in \text{spec}(A_e) \subset \mathcal{U} \).

Since \((A, B)\) is stabilizable, it follows that

\[
\begin{bmatrix} \text{rank} \begin{bmatrix} A - s I & B \\ B_W C & B_W D \end{bmatrix} \\ A_W - s I \end{bmatrix} \geq \begin{bmatrix} \text{rank} \begin{bmatrix} A - \lambda I & B \\ B_W C & B_W D \end{bmatrix} \\ A_W - \lambda I \end{bmatrix} \geq \begin{bmatrix} \text{rank} \begin{bmatrix} I_n & 0 \\ 0 & B_W \end{bmatrix} \\ A_W - \lambda I \end{bmatrix} \end{bmatrix} \tag{39}
\]
Conditions (2) and (3) imply that
\[
\text{rank } \begin{bmatrix} A - \lambda I & B \\ C & D \\ 0 & 0 \end{bmatrix} = n + l_y + l_y n_w
\]
which is full row rank. Therefore,
\[
n + l_y n_w \geq \text{rank } \begin{bmatrix} A - s I & B \\ B W C & B W D & A W - s I \end{bmatrix} \geq \text{rank } \begin{bmatrix} I_n & 0 & 0 \\ 0 & B W & A W - \lambda I \end{bmatrix}.
\]
(40)

Since \((A_W, B_W)\) is controllable, \(n + l_y n_w\) and thus
\[
\text{rank } \begin{bmatrix} A - s I & B \\ B W C & B W D & A W - s I \end{bmatrix} = n + l_y n_w.
\]
(41)

Hence,
\[
\begin{bmatrix} A & B \\ B W C & B W D \end{bmatrix}
\]
is stabilizable. Furthermore, since \((A, C)\) is detectable, it follows that
\[
\begin{bmatrix} A & 0 \\ B W C & B W D \end{bmatrix}
\]
is detectable.

Since (32) and (33) are stabilizable and detectable, there exist observer-based controllers that stabilize the augmented systems (32) and (33). Hence, consider the controllers (34) and (35) with the parameters \(A_1, B_{c1}, B_{c2}, C_c\) chosen to stabilize (32) and (33). The closed-loop system consisting of (32) and (33) and the feedback controller (34) and (35) is thus asymptotically stable and is given by (36) and (37). Furthermore, the closed-loop system (36) and (37) is in the form of the closed-loop system (20)–(22) with the controller (18) and (19), where
\[
\dot{x} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}
\]
and \((\hat{A}, \hat{B}, \hat{C})\) are given by (38). Therefore, the closed-loop system (20)–(22) with (38) is asymptotically stable.

Next, we show that \(\lim_{t \to \infty} z(t) = 0\). Define
\[
T \triangleq \begin{bmatrix} I_n & 0 \\ 0 & I_{l_y n_w} \end{bmatrix}
\]
and consider the change of basis \(\bar{x} = T \hat{x}\). In the new basis, the closed-loop system (20)–(22) with (38) has the form
\[
\begin{aligned}
D \bar{x} & = \bar{A} \bar{x} + \bar{D} w \\
z & = \bar{E} \bar{x} + E_0 w
\end{aligned}
\]
(43)
(44)
where
\[
\begin{bmatrix} \bar{A} & BC_c \\ B_{c1} C & A_c + B_{c1} D_c c \\ B W C & B W D \end{bmatrix} = \begin{bmatrix} A & B \\ B W C & B W D \\ 0 & 0 \end{bmatrix} T A^{-1}
\]
(45)
\[
\begin{bmatrix} D_1 & D_1 D_2 \\ B_{c1} D_2 \\ B W D_2 \end{bmatrix} = \begin{bmatrix} D_1 & D_2 \\ B W D_2 \end{bmatrix} T D
\]
(46)
\[
\begin{bmatrix} E & E_2 C_c \end{bmatrix} \triangleq ET^{-1} = \begin{bmatrix} E_1 \\ E_2 C_c \end{bmatrix}
\]
(47)

Next, let
\[
S \triangleq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}
\]
be the unique solution to the Sylvester equation
\[
AS - SA_w = DC_w.
\]
(48)

Now, it follows from Lemma A1 with
\[
F_1 = \begin{bmatrix} A & BC_c \\ B_{c1} C & A_c + B_{c1} D_c c \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ B_{c2} \end{bmatrix}, \quad F_3 = A_w,
\]
\[
G = B_w, \quad H = [C \quad DC_c], \quad J = [D_c C_w B_{c1} D_2 C_w],
\]
and \(K = D_2 C_w\) that \([C \quad DC_c] S_1 = D_2 C_w\). Condition (4) implies that there exists \(R \in \mathbb{R}^{m \times \alpha}\) such that \(E_1 = RC, E_2 = RD,\) and \(E_0 = RD_2\). Therefore, \([E_1 \quad E_2 C_c], S_1 = E_0 C_w,\) or equivalently
\[
E S = E_0 C_w.
\]
(49)

Thus, there exists \(S\) satisfying (48) and (49), and Lemma 3.1 implies that \(\lim_{t \to \infty} z(t) = 0\).

V. DISCUSSION OF NECESSITY

Theorem 4.1 provides sufficient conditions for the existence of a linear time-invariant controller that stabilizes (1)–(3) in the differential, shift, and delta domains and regulates the performance to zero. The case \(z = y\) is considered in [5]–[7], where it is claimed that conditions (1)–(3) are necessary and sufficient for continuous-time systems. However, it is possible to construct examples for which (2) and (3) are not necessary. For example, consider the SISO disturbance rejection problem
\[
y = G(s)(u + w), \quad G(s) \triangleq \frac{s^2 + \alpha^2}{p(s)}
\]
where \(z = y, \alpha \in \mathbb{R}, \deg p(s) \geq 2,\) and \(p(s)\) does not have roots at \(\pm \alpha j\). Furthermore, assume that \(w\) is the output of the linear system (6) and (7), where \(A_w\) has the characteristic polynomial \(p_w(s) = s^2 + \alpha^2\). Therefore, for every minimal realization of \(G(s)\), condition (3) does not hold since \(\pm \alpha j\) are eigenvalues of \(A_w\) and zeros of \(G(s)\). However, consider the feedback controller \(u = -G(s)y = -(\hat{q}(s)/\hat{p}(s))y\), where \(\hat{q}(s)\) and \(\hat{p}(s)\) are selected so that \(\hat{p}(s) = \frac{p_w(s)}{\hat{p}(s)}\). Then, the final value theorem implies that
\[
\lim_{t \to \infty} z(t) = \lim_{s \to 0} \frac{G(s)}{1 + G(s)\mathcal{L}(w(t))} = \lim_{s \to 0} \frac{\hat{p}(s)\hat{q}(s)}{\hat{p}(s)} = 0
\]
(51)
where \(\mathcal{L}(\cdot)\) is the Laplace transform and \(\mathcal{L}(w(t)) = q_w(s)/\hat{p}(s)\). In this case, every stabilizing controller drives the performance to zero because the disturbance frequency corresponds to the zeros of the open-loop system.

APPENDIX A

Lemma 5.1: Let \(F_1 \in \mathbb{R}^{r \times q}, F_2 \in \mathbb{R}^{r \times m_p}, F_3 \in \mathbb{R}^{m \times m}, G \in \mathbb{R}^{m \times q}, H \in \mathbb{R}^{r \times q}, J \in \mathbb{R}^{r \times m},\) and \(K \in \mathbb{R}^{r \times m}\). Assume that
\[
\text{spec} \left[ \begin{bmatrix} F_1 \\ (I_p \otimes \mathcal{G}) H \quad I_p \otimes F_3 \end{bmatrix} \right] \cap \text{spec}(F_3) = \emptyset
\]
(46)
and the pair \((F_3, G)\) is controllable. Let
\[
S \triangleq \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}
\]
be the unique solution to the Sylvester equation
\[
\begin{bmatrix} F_1 \\ (I_p \otimes G)H \\ I_p \otimes F_3 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} - \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} F_3 = \begin{bmatrix} J \\ (I_p \otimes G)K \end{bmatrix}.
\tag{A1}
\]
Then,
\[HS_1 = K.\tag{A2}\]

**Proof:** The Sylvester equation (A1) is equivalent to
\[F_1 S_1 + F_2 S_2 - S_1 F_3 = J \tag{A3}\]
\[(I_p \otimes G)HS_1 + (I_p \otimes F_3)S_2 - S_2 F_3 = (I_p \otimes G)K.\tag{A4}\]

Next let \[S_2 = \begin{bmatrix} S_{2,1} \\ \vdots \\ S_{2,p} \end{bmatrix},\]
where, for all \[i = 1, \ldots, p, S_{2,i} \in \mathbb{R}^{m \times m}.\]
It follows from (A4) that, for all \[i = 1, \ldots, p, F_3 S_{2,i} - S_{2,i} F_3 = GA_i,\]
where \[\Lambda_i = e_i(K - HS_1)\] and \[e_i = [0_1, \ldots, 0_i, 1, 0_1, \ldots, 0_{(p-i)}].\]

Let \[M \in \mathbb{R}^{m \times m}\] be such that \[\tilde{F} \triangleq M^{-1} F_3 M\] is in Jordan canonical form, that is, for some \[\mu \leq m, \tilde{F} = \text{diag}(F_1, \ldots, F_p),\] where, for \[j = 1, \ldots, \mu,\]
\[
\begin{bmatrix} \lambda_j & 1 \\ \vdots & \ddots \\ & & 1 \\ & & & \lambda_j \end{bmatrix} \in \mathbb{R}^{(j-1)\times(j-1)}
\]
and \[\lambda_j \in \text{spec}(F_j).\] Furthermore, define
\[
\tilde{G} \triangleq M^{-1}G = \begin{bmatrix} \tilde{G}_1 \\ \vdots \\ \tilde{G}_p \end{bmatrix},
\tag{A5}\]
and, for \[i = 1, \ldots, p,\]
\[
S_i \triangleq M^{-1} S_{2,i} M = \begin{bmatrix} \tilde{S}_{i,1} & \cdots & \tilde{S}_{i,n} \\ \vdots & \ddots & \vdots \\ \tilde{S}_{i,n,1} & \cdots & \tilde{S}_{i,n,n} \end{bmatrix},
\tag{A6}\]
\[
\tilde{\Lambda}_i \triangleq \Lambda_i M = [\phi_{i1}, \ldots, \phi_{i,m}].
\tag{A7}\]

where, for \(j = 1, \ldots, \mu, \tilde{G}_j \in \mathbb{R}^{f_j \times 1}\), and, for \(i = 1, \ldots, p, S_{i,j,k} \in \mathbb{R}^{f_j \times f_i}\). Therefore, for \(i = 1, \ldots, p, \) premultiplying \[F_3 S_{2,i} - S_{2,i} F_3 = GA_i\] by \(M^{-1}\) and postmultiplying by \(M\) yields
\[F \tilde{S}_i - \tilde{S}_i F = \tilde{G}_i \tilde{\Lambda}_i E_j. \tag{A8}\]

Substituting (A5)–(A6) into (A8) and considering only the block-diagonal terms imply that, for all \(i = 1, \ldots, p,\) and for all \(j = 1, \ldots, \mu,\)
\[F_j \tilde{S}_{i,j,j} - \tilde{S}_{i,j,j} F_j = \tilde{G}_j \tilde{\Lambda}_i E_j \tag{A9}\]
Next, for all \(i = 1, \ldots, p,\) and for all \(j = 1, \ldots, \mu,\)
\[\begin{bmatrix} s_{ij,1,1} & \cdots & s_{ij,1,f_j} \\ \vdots & \ddots & \vdots \\ s_{ij,j,f_j} & \cdots & s_{ij,j,j} \end{bmatrix},\]
so that (A10) holds, as shown at the bottom of the page.

For all \(j = 1, \ldots, \mu,\) let \(g_j \in \mathbb{R}\) denote the last entry of \(\tilde{G}_j.\) For all \(i = 1, \ldots, p,\) and all \(j = 1, \ldots, \mu,\) combining (A7), (A9), and (A10) yields (A11), as shown at the bottom of the page, where * denotes an inconsequential entry.

Now, since \((F_1, G)\) is controllable, it follows that \((\tilde{F}, \tilde{G})\) is controllable, and thus, for all \(j = 1, \ldots, \mu, (\tilde{F}_j, \tilde{G}_j)\) is controllable. Therefore, for all \(j = 1, \ldots, \mu, g_j \neq 0.\)

First, consider \(j = 1.\) Since \(g_j \neq 0,\) inspecting the \((f_j, 1)\) entry of (A11) yields that, for all \(i = 1, \ldots, p, \phi_{i1+f_j+\cdots+f_{j-1}} = 0,\) and thus, for all \(k = 2, \ldots, f_j, s_{ij,k-1} = 0.\) Now, since \(g_j \neq 0\) and \(s_{ij,j-1} = 0,\) inspecting the \((f_j, 2)\) entry of (A11) yields that, for all \(i = 1, \ldots, p, \phi_{i1+f_j+\cdots+f_{j-2}} = 0,\) and thus, for all \(k = 3, \ldots, f_j, s_{ij,k-1} = 0.\) Now, since \(g_j \neq 0\) and \(s_{ij,j-2} = 0,\) inspecting the \((f_j, 3)\) entry of (A11) yields that, for all \(i = 1, \ldots, p, \phi_{i1+f_j+\cdots+f_{j-1}} = 0,\) and thus, for all \(k = 4, \ldots, f_j, s_{ij,k-1} = 0.\) Continuing in this manner yields, for all \(i = 1, \ldots, p, [\phi_{i1+f_j+\cdots+f_{j-2}} \cdots \phi_{i1+f_j+\cdots+f_{j-1}}] = 0.\) Repeating this for all \(j = 2, \ldots, \mu,\) yields for all \(i = 1, \ldots, p, \tilde{\Lambda}_i = 0,\) which implies that \(HS_1 = K = 0,\) thus proving (A2).
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REFERENCES


