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L_2 controller synthesis with L_∞-bounded closed-loop impulse response

Y. WILLIAM WANG† and DENNIS S. BERNSTEIN†

In this paper we consider an L_2 control problem with an L_∞ norm on the closed-loop impulse response. To do this we first construct an upper bound for the L_∞ norm of the impulse response of the closed-loop system. To perform controller synthesis, we then modify the standard LQG cost functional by including an additional penalty term that weights the L_∞ norm of the impulse response of the closed-loop system. A numerical example is given to illustrate the improved L_∞ response of the closed-loop system.

Notation

$\mathcal{R}$, $\mathcal{C}$	real numbers, complex numbers
$\mathrm{spec}(A)$	the set of eigenvalues of A
$\rho(A)$	spectral radius of A, $\max\{\lvert\lambda\rvert : \lambda \in \mathrm{spec}(A)\}$
$\sigma_{\max}(A)$, $\sigma_{\min}(A)$	maximum, minimum singular value of A
$\sigma_2(A)$	second largest singular value of A
$\alpha(A)$	spectral abscissa of A, $\max\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{spec}(A)\}$
$\lVert A\rVert_F$, $\lVert x(t)\rVert_2$	Frobenius norm of A, $(x^T(t)x(t))^{1/2}$

1. Introduction and problem formulation

Although standard LQG design optimizes the closed-loop system response from disturbance to performance by minimizing the L_2 norm of the impulse response, it does not necessarily minimize the L_∞ norm of the closed-loop impulse response. Rather, LQG design optimizes the time integral of the square of the Frobenius norm of the closed-loop impulse response. In many practical situations, however, it is desirable to enforce a pointwise-in-time constraint on the impulse response of the closed-loop system (Gilbert and Tan 1991). To address this problem, we consider the linear feedback system depicted in Fig. 1,

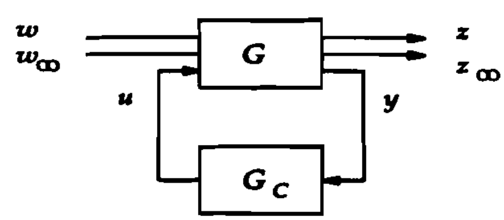

Figure 1. System schematic diagram.

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with plant G and controller G_c, where the signal $w(t)$ denotes white noise and $w_\infty(t)$ denotes an impulse signal. The performance variables are $z(t)$ and $z_\infty(t)$, respectively, for the disturbance inputs $w(t)$ and $w_\infty(t)$. With the closed-loop transfer functions $\tilde{G}$ and $\tilde{G}_\infty$ given by

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{D}w(t) + \tilde{D}_\infty w_\infty(t) \tag{1}$$

$$z(t) = \tilde{E}x(t) \tag{2}$$

$$z_\infty(t) = \tilde{E}_\infty x(t) \tag{3}$$

$\|\tilde{G}\|_2$ and $\|\tilde{G}_\infty\|_{L_\infty}$ are defined by

$$\|\tilde{G}\|_2 \triangleq \left[\int_0^\infty \|\tilde{E}\,e^{\tilde{A}t}\tilde{D}\|_F^2\,dt\right]^{1/2} \tag{4}$$

$$\|\tilde{G}_\infty\|_{L_\infty} \triangleq \sup_{t\geq 0} \sigma_{\max}(\tilde{E}_\infty\,e^{\tilde{A}t}\tilde{D}_\infty) \tag{5}$$

Mixed-norm optimization problem: Given G, obtain G_c such that $\|\tilde{G}\|_2$ is minimized with constrained $\|\tilde{G}_\infty\|_{L_\infty}$. □

A closely related problem involves L_2 disturbances $w_\infty(t)$ with an L_∞ norm on $z(t)$. In this case the induced norm is given by (Wilson 1989)

$$\|\tilde{G}\|_{\infty,2} = d_{\max}^{1/2}\left(\int_{-\infty}^{\infty} \tilde{G}(t)\tilde{G}^{\mathrm{T}}(t)\,dt\right) \tag{6}$$

where $d_{\max}$ denotes the largest diagonal entry. Controllers that minimize (6) are considered in (Rotea 1993). The norm (6), however, is more closely related to the L_2 norm (4) than the L_∞ norm (5). In fact, (6) and (4) are identical in the single-input and single-output cases, whereas (5) represents a pointwise-in-time constraint on the system response.

In this paper, we develop a method that constrains $\|\tilde{G}_\infty\|_{L_\infty}$ and thus the excursion of $z_\infty(t)$ when $w(t) = 0$ and $w_\infty(t)$ is an impulse. Since the size of $\tilde{E}_\infty e^{\tilde{A}t}\tilde{D}_\infty$ depends on the maximum singular value of $e^{\tilde{A}t}$, we first obtain upper bounds for the maximum singular value of $e^{\tilde{A}t}$. One such bound is based upon the maximum eigenvalue of $\tilde{A} + \tilde{A}^{\mathrm{T}}$ while the other involves the Frobenius norm of $\tilde{A}^{\mathrm{T}}\tilde{A} - \tilde{A}\tilde{A}^{\mathrm{T}}$. In general, there is no ordering between these bounds. Using synthesis techniques, we utilize the second bound to limit the L_∞ norm of $\tilde{E}_\infty e^{\tilde{A}t}\tilde{D}_\infty$. To do this, we modify the standard LQG performance measure by adding a penalty term to the standard LQG performance measure. We then perform controller design by optimizing this modified LQG performance measure.

We begin in § 2 by investigating properties of the Frobenius norm and maximum singular value of e^{At} for an arbitrary matrix A. The results given in this section illustrate the difficulty of characterizing the L_∞ norm (5). Then, in § 3, we construct two upper bounds for the L_∞ norm of the impulse response. In § 4, we consider controller synthesis where we address both static output feedback control and dynamic compensation. We then propose a numerical algorithm for obtaining controller gains in § 5. An example involving an F8 fighter is shown in § 5. Concluding remarks are given in § 6.

2. Properties of the Frobenius norm and maximum singular value of e^{At}

To bound the impulse response of a linear system, it is useful to investigate properties of the maximum singular value of e^{At}. To begin, consider the system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{7}$$

whose solution is given by

$$x(t) = e^{At}x_0 \tag{8}$$

If A is asymptotically stable then $e^{At} \to 0$ as $t \to \infty$ and thus $x(t) \to 0$ as $t \to \infty$. Furthermore, $\|x(t)\|_2$ is bounded by

$$\|x(t)\|_2 = \|e^{At}x_0\|_2 \leq \sigma_{\max}(e^{At})\|x_0\|_2 \tag{9}$$

Since $\sigma_{\max}(e^{At})$ is the norm induced by the euclidean norm $\|\cdot\|_2$, it follows that for every value of $t \geq 0$ there exists an initial condition x_0 such that equality holds in (9), that is

$$\|x(t)\|_2 = \|e^{At}x_0\|_2 = \sigma_{\max}(e^{At})\|x_0\|_2 \tag{10}$$

Thus, the maximum excursion of $x(t)$ from the origin depends upon the maximum size of $\sigma_{\max}(e^{At})$ for $t \geq 0$. Unfortunately, although $\sigma_{\max}(e^{At}) \to 0$ as $t \to \infty$, $\sigma_{\max}(e^{At})$ may not be decreasing for all t. To illustrate this, we consider the following example.

Example 1: Let

$$A = \begin{bmatrix} -0{\cdot}1 & -3 \\ 2 & -0{\cdot}2 \end{bmatrix}$$

As can be seen in Fig. 2, $\sigma_{\max}(e^{At})$ initially increases and is oscillatory. Therefore, there exists $x(0)$ such that $\|x(0)\|_2 = 1$ and $\|x(t)\|_2 \approx 1{\cdot}2$ for $t \approx 0{\cdot}8$. We also observe that due to the switching in magnitude between the eigenvalues of $e^{At}e^{A^{\mathrm{T}}t}$, $\sigma_{\max}(e^{At})$ is not smooth at its local minima. However, since $\sigma_{\max}(e^{At}) = \lambda_{\max}^{1/2}(e^{At}e^{A^{\mathrm{T}}t})$ involves switching between complex exponential functions, it can be seen that the one-sided derivative of $\sigma_{\max}(e^{At})$ always exists. □

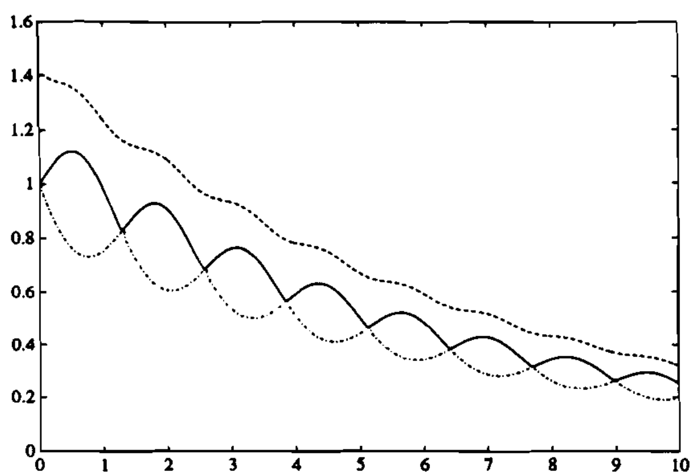

Figure 2. $\sigma_{\max}(e^{At})$ (solid curve), $\sigma_2(e^{At})$ (dash-dot curve) and $\|e^{At}\|_F$ (dashed curve).

Proposition 2.1: *Let $A \in \mathfrak{R}^{n\times n}$. Then the one-sided derivative of $\sigma_{\max}(e^{At})$ at $t = 0^+$ is given by*

$$\frac{d}{dt}\sigma_{\max}(e^{At})|_{t=0^+} = \lambda_{\max}(A + A^T)/2 \tag{11}$$

Proof: Note that

$$\begin{aligned}\frac{d}{dt}\sigma_{\max}(e^{At})|_{t=0^+} &= \lim_{t\to 0^+}\frac{\sigma_{\max}(e^{At}) - 1}{t}\\ &= \lim_{t\to 0^+}\frac{\lambda_{\max}^{1/2}(e^{A^Tt}e^{At}) - 1}{t}\\ &= \lim_{t\to 0^+}\frac{\{\lambda_{\max}[(I + A^Tt)(I + At)]\}^{1/2} - 1}{t}\\ &= \lim_{t\to 0^+}\frac{[1 + t\lambda_{\max}(A^T + A)]^{1/2} - 1}{t}\\ &= \lambda_{\max}(A^T + A)/2\end{aligned}$$

□

Proposition 2.1 shows that if $A + A^T$ is indefinite then $\sigma_{\max}(e^{At})$ initially increases for $t \geq 0$. In this case, $\sigma_{\max}(e^{At})$ has a maximum value greater than unity. Since $\sigma_{\max}(e^{At})$ may not be differentiable at all $t > 0$, it is not easy to characterize the maximum value of $\sigma_{\max}(e^{At})$. Instead, we recall an upper bound for $\sigma_{\max}(e^{At})$.

Lemma 2.1: *Let $A \in \mathfrak{R}^{n\times n}$. Then, for all $t \in \mathfrak{R}$*

$$\sigma_{\max}(e^{At}) \leq e^{\lambda_{\max}(A+A^T)t/2} \tag{12}$$

Proof: The result is given by Lemma 1c of Strom (1975). □

In the more restrictive case in which A is dissipative, that is, $A + A^T < 0$, we have the following result.

Proposition 2.2: *Let $A \in \mathfrak{R}^{n\times n}$. Then A is dissipative if and only if $\sigma_{\max}(e^{At})$ is strictly decreasing for all $t \in \mathfrak{R}$.*

Proof: If A is dissipative, Lemma 2.1 implies that $\sigma_{\max}(e^{At}) \leq e^{\lambda_{\max}(A+A^T)t/2} < 1$, for $t > 0$. Thus, it follows that $e^{A^Tt}e^{At} < I$ for all $t > 0$. Letting $t = t_2 - t_1$, where $t_2 > t_1$, we obtain $e^{A^Tt_2}e^{At_2} < e^{A^Tt_1}e^{At_1}$. Hence, $\sigma_{\max}(e^{At_2}) < \sigma_{\max}(e^{At_1})$. Conversely, if $\sigma_{\max}(e^{At})$ is strictly decreasing for $t \geq 0$ then Proposition 2.1 implies that $\lambda_{\max}(A + A^T) < 0$. Thus, $A + A^T < 0$. □

Corollary 2.1: *Let $\dot{x}(t) = Ax(t)$ and $A \in \mathfrak{R}^{n\times n}$ be dissipative. Then $\|x(t)\|_2$ is strictly decreasing for all $t \in \mathfrak{R}$.*

Proof: Using Proposition 2.2, it can be seen that if $t_1 < t_2$ then

$$\|x(t_2)\|_2 = \|e^{A(t_2-t_1)}x(t_1)\|_2 \leq \sigma_{\max}(e^{A(t_2-t_1)})\|x(t_1)\|_2 < \|x(t_1)\|_2$$

□

We now consider the most restrictive case in which A is normal and asymptotically stable.

Proposition 2.3: *Let $A \in \mathfrak{R}^{n\times n}$ be normal and asymptotically stable. Then, for all $t > 0$,*

$$\sigma_{\max}(e^{At}) = e^{\frac{1}{2}\lambda_{\max}(A+A^{T})t} = e^{\alpha(A)t} < 1 \tag{13}$$

Proof: Since A is normal and asymptotically stable, there exist unitary $V \in \mathscr{C}^{n\times n}$ and diagonal $\Lambda \in \mathscr{C}^{n\times n}$ such that $A = V\Lambda V^*$ and $\Lambda + \Lambda^* < 0$. Thus, for $t > 0$

$$\sigma_{\max}(e^{At}) = [\lambda_{\max}(e^{A^{T}t}e^{At})]^{1/2} = e^{\frac{1}{2}\lambda_{\max}(A+A^{T})t} < 1 \qquad \square$$

We also consider $\|e^{At}\|_F$ which is a smooth function of t. As can be seen from Fig. 2, $\|e^{At}\|_F$ and $\sigma_{\max}(e^{At})$ may have very different characteristics. Let us consider the case in which A is dissipative.

Proposition 2.4: *Let $A \in \mathfrak{R}^{n\times n}$ be dissipative. Then $\|e^{At}\|_F$ is strictly decreasing for $t \geq 0$.*

Proof: Note that, for $t \geq 0$

$$\frac{d}{dt}\|e^{At}\|_F^2 = \operatorname{tr} e^{A^{T}t}(A + A^{T})e^{At} < 0 \qquad \square$$

Although Proposition 2.4 provides a sufficient condition for $\|e^{At}\|_F$ to be strictly decreasing, it can be seen from Fig. 2 that this condition is not necessary.

Finally, we consider the case in which A is normal.

Proposition 2.5: *Let $A \in \mathfrak{R}^{n\times n}$ be normal. Then*

$$\|e^{At}\|_F = \left[\sum_{i=1}^{n} e^{2t\,\mathrm{Re}\,\lambda_i}\right]^{1/2} \tag{14}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

3. An alternative upper bound for $\sigma_{\max}(e^{At})$

Lemma 2.1 provides an upper bound for $\sigma_{\max}(e^{At})$. Here, we state an alternative bound for $\sigma_{\max}(e^{At})$ based upon the Schur decomposition. Let the Schur decomposition of A be $A = Q(D + N)Q^*$, where D is diagonal, N is strictly upper triangular, and Q is unitary.

Proposition 3.1: *Let $A \in \mathfrak{R}^{n\times n}$. Then, for all $t \in \mathfrak{R}$*

$$\sigma_{\max}(e^{At}) \leq e^{\alpha(A)t}\sum_{k=0}^{n-1}\frac{t^k}{k!}\sigma_{\max}^k(N) \tag{15}$$

Proof: The result appears as (2.11) in Van Loan (1977). $\square$

Although bounds (12) and (15) hold for all A, Fig. 3 shows with

$$A = \begin{bmatrix} -0{\cdot}5 & 2 \\ 0 & -0{\cdot}7 \end{bmatrix}$$

that there does not generally exist an ordering between these two bounds. Note that bound (12) approaches zero monotonically as $t \to \infty$ if and only if $A + A^{T} < 0$, while bound (15) approaches zero as $t \to \infty$ if and only if A is asymptotically stable. The following result shows that the exponential factor $e^{\alpha(A)t}$ in (15) is always better than the exponential factor $e^{\lambda_{\max}(A+A^{T})t/2}$ in (12).

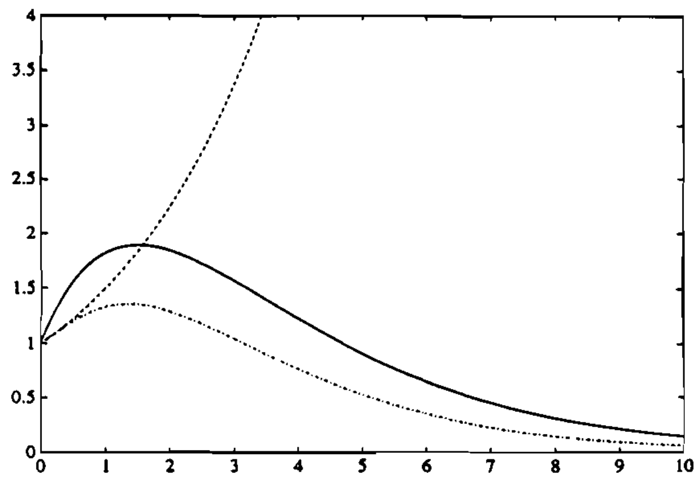

Figure 3. $\sigma_{\max}(e^{At})$ (dash–dot curve), Bound (12) (dashed curve) and Bound (15) (solid curve).

Proposition 3.2: *Let $A \in \mathfrak{R}^{n\times n}$. Then*

$$\alpha(A) \leqslant \lambda_{\max}(A + A^{T})/2 \tag{16}$$

Furthermore, equality holds if and only if A is normal.

Proof: See Theorem F.1 on p. 237 of Marshall and Olkin (1979). □

Now let us focus on the polynomial part of (15). Since N is upper triangular and nilpotent, we have the following bounds for $\|N\|_F$.

Lemma 3.1: *Let $A \in \mathfrak{R}^{n\times n}$. Then*

(i) $\|N\|_F = [\|A\|_F^2 - \sum_{i=1}^{n}|\lambda_i|^2]^{1/2} < \|A\|_F$

(ii) $\|N\|_F = [\frac{1}{2}\|A + A^T\|_F^2 - 2\sum_{i=1}^{n}(\mathrm{Re}\,\lambda_i)^2]^{1/2} \leqslant \dfrac{1}{\sqrt{2}}\|A + A^T\|_F$

(iii) $\|N\|_F = [\frac{1}{2}\|A - A^T\|_F^2 - 2\sum_{i=1}^{n}(\mathrm{Im}\,\lambda_i)^2]^{1/2} \leqslant \dfrac{1}{\sqrt{2}}\|A - A^T\|_F$

(iv) $\|N\|_F \leqslant \left(\dfrac{n^3 - n}{12}\right)^{1/4}\|A^TA - AA^T\|_F^{1/2}$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Proof: To prove (i), note that $\|A\|_F^2 = \|Q^*AQ\|_F^2 = \|D + N\|_F^2$. By direct expansion, we have

$$\|D + N\|_F^2 = \mathrm{tr}(D^*D + N^*N) = \|N\|_F^2 + \sum_{i=1}^{n}|\lambda_i|^2$$

Thus, $\|N\|_F = [\|A\|_F^2 - \sum_{i=1}^{n}|\lambda_i|^2]^{1/2} < \|A\|_F$. Statements (ii) and (iii) can be proved in a similar manner. The proof of (iv) is given in Henrici (1962). □

Using (iv) in Lemma 3.1, $\sigma_{\max}(e^{At})$ can be bounded as below.

Theorem 3.1: *Let $A \in \Re^{n\times n}$. Then, for all $t \in \Re$*

$$\sigma_{\max}(e^{At}) \leq e^{\alpha(A)t}\sum_{k=0}^{n-1}\frac{t^k}{k!}\left(\frac{n^3-n}{12}\right)^{k/4}\|A^TA - AA^T\|_F^{k/2} \tag{17}$$

Proof: Combining Lemma 3.1 and Proposition 3.1 with the fact that $\sigma_{\max}(N) \leq \|N\|_F$ yields (17). □

The following bound for the impulse response matrix will be useful for synthesis.

Corollary 3.1: *Let $E \in \Re^{p\times n}$, $A \in \Re^{n\times n}$ and $D \in \Re^{n\times q}$. Then, for all $t \in \Re$*

$$\sigma_{\max}(E\,e^{At}D) \leq e^{\alpha(A)t}\|E\|_F\left[\sum_{k=0}^{n-1}\frac{t^k}{k!}\left(\frac{n^3-n}{12}\right)^{k/4}\|A^TA - AA^T\|_F^{k/2}\right]\|D\|_F \tag{18}$$

4. Controller synthesis for bounded closed-loop impulse response

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t) + D_{1\infty}w_\infty(t) \tag{19}$$

$$y(t) = Cx(t) + D_2w(t) + D_{2\infty}w_\infty(t) \tag{20}$$

$$z(t) = E_1x(t) + E_2u(t) \tag{21}$$

$$z_\infty(t) = E_{1\infty}x(t) + E_{2\infty}u(t) \tag{22}$$

where A, B, D_1, $D_{1\infty}$, C, D_2, $D_{2\infty}$, E_1, E_2, $E_{1\infty}$ and $E_{2\infty}$ are $n\times n$, $n\times m$, $n\times l_1$, $n\times l_2$, $p\times n$, $p\times l_1$, $p\times l_2$, $q\times n$, $q\times m$, $r\times n$ and $r\times m$ real matrices. We seek a static output feedback controller $u(t) = Ky(t)$ such that

(i) the closed-loop system

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{D}w(t) \tag{23}$$

$$z(t) = \tilde{E}x(t) \tag{24}$$

where $\tilde{A} \triangleq A + BKC$ is asymptotically stable, $\tilde{D} \triangleq D_1 + BKD_2$ and $\tilde{E} \triangleq E_1 + E_2KC$;

(ii) the H_2 performance

$$J(K) \triangleq \lim_{t\to\infty}\mathscr{E}[z^T(t)z(t)] \tag{25}$$

is minimized; and

(iii) the closed-loop impulse response defined by (5) is bounded, where $\tilde{D}_\infty \triangleq D_{1\infty} + BKD_{2\infty}$ and $\tilde{E}_\infty \triangleq E_{1\infty} + E_{2\infty}KC$.

Thus, an auxiliary minimization problem can be formulated as follows. Given $\alpha > 0$, minimize

$$\mathscr{J}(K) \triangleq J(K) + \alpha f(K) \tag{26}$$

In (26), $f(K)$ is a non-negative function of the closed-loop gain K and is included to penalize the maximum value of $\|z_\infty(t)\|_2$. Using Corollary 3.1 and setting

$$f(K) = \|\tilde{E}_\infty\|_F\left[\sum_{k=0}^{n-1}\beta_k\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^k\right]\|\tilde{D}_\infty\|_F \tag{27}$$

where β_k, $k = 0, \ldots, n-1$, are positive numbers, we consider the following modified linear quadratic optimization problem: minimize

$$\mathcal{J}(K) = \operatorname{tr} Q\tilde{R} + \alpha\|\tilde{E}_\infty\|_F\left[\sum_{k=0}^{n-1}\beta_k\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^k\right]\|\tilde{D}_\infty\|_F \tag{28}$$

subject to the closed-loop Lyapunov equation

$$0 = \tilde{A}Q + Q\tilde{A}^T + \tilde{V} \tag{29}$$

where $\tilde{V} \triangleq \tilde{D}\tilde{D}^T$ and $\tilde{R} \triangleq \tilde{E}^T\tilde{E}$. The number β_k will be chosen within the optimization problem to represent the maximum value of $t^k/k!(n^3 - n/12)^{k/4}$ for $t \geq 0$. Also note that for convenience $k/2$ is replaced by k in the exponent of $\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F$.

Theorem 4.1: *Let $K \in \mathcal{R}^{m\times p}$ be such that $\tilde{A}$ is asymptotically stable and $\mathcal{J}(K)$ is minimized. Then there exist P, $Q \geq 0$ satisfying*

$$0 = \tilde{A}Q + Q\tilde{A}^T + \tilde{V}$$
$$0 = \tilde{A}^TP + P\tilde{A} + \tilde{R}$$

such that K satisfies

$$\begin{aligned}0 = E_2^T\tilde{E}QC^T &+ \alpha\|\tilde{E}_\infty\|_F\left[\sum_{k=0}^{n-2}(k+1)\beta_{k+1}B^T(2\tilde{A}\tilde{A}^T\tilde{A} - \tilde{A}^T\tilde{A}^2 - \tilde{A}^2\tilde{A}^T)C^T\right.\\ &\left.\times \|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^{k-1}\right]\|\tilde{D}_\infty\|_F + \alpha E_{2\infty}^T\tilde{E}_\infty C^T\|\tilde{E}_\infty\|_F^{-1}\\ &\times\left[\sum_{k=0}^{n-1}\beta_k\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^k\right]\|\tilde{D}_\infty\|_F\\ &+ \alpha\|\tilde{E}_\infty\|_F\left[\sum_{k=0}^{n-1}\beta_k\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^k\right]B^T\tilde{D}_\infty D_{2\infty}^T\|\tilde{D}_\infty\|_F^{-1} + B^TPQC^T\\ &+ B^TP\tilde{D}D_2^T\end{aligned} \tag{30}$$

Proof: The results follow from straightforward algebraic manipulation. □

Now we consider dynamic compensation. With the controller

$$\dot{x}_c(t) = A_cx_c(t) + B_cy(t) \tag{31}$$

$$u(t) = C_cx_c(t) \tag{32}$$

the closed-loop system (19)–(22), (31), (32) can be written as

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_c(t)\end{bmatrix} = \begin{bmatrix}A & BC_c\\ B_cC & A_c\end{bmatrix}\begin{bmatrix}x(t)\\ x_c(t)\end{bmatrix} + \begin{bmatrix}D_1\\ B_cD_2\end{bmatrix}w(t) + \begin{bmatrix}D_{1\infty}\\ B_cD_{2\infty}\end{bmatrix}w_\infty(t) \tag{33}$$

Thus, the auxiliary minimization problem for dynamic compensation can be stated as follows: minimize

$$\mathcal{J}(A_c, B_c, C_c) = \operatorname{tr}\tilde{Q}\tilde{R} + \alpha\|\tilde{E}_\infty\|_F\left[\sum_{k=0}^{n-1}\beta_k\|\tilde{A}\tilde{A}^T - \tilde{A}^T\tilde{A}\|_F^k\right]\|\tilde{D}_\infty\|_F \tag{34}$$

subject to

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} \tag{35}$$

where

$$\widetilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}, \quad \widetilde{D} \triangleq \begin{bmatrix} D_1 \\ B_cD_2 \end{bmatrix}, \quad \widetilde{E} \triangleq [E_1 \;\; E_2C_2]$$

$$R_1 \triangleq E_1^TE_1, \quad R_2 \triangleq E_2^TE_2$$

$$\widetilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^TR_2C_c \end{bmatrix}, \quad V_2 \triangleq D_2D_2^T, \quad \widetilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_cV_2B_c^T \end{bmatrix}$$

$$\widetilde{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}$$

$$\widetilde{D}_\infty \triangleq \begin{bmatrix} D_{1\infty} \\ B_cD_{2\infty} \end{bmatrix}, \quad \widetilde{E}_\infty \triangleq [E_{1\infty} \;\; E_{2\infty}C_c], \quad V_{2\infty} \triangleq D_{2\infty}D_{2\infty}^T, \quad R_{2\infty} \triangleq E_{2\infty}^TE_{2\infty}$$

Theorem 4.2: *Let $A_c \in \Re^{n\times n}$, $B_c \in \Re^{m\times n}$ and $C_c \in \Re^{n\times p}$ be such that $\widetilde{A}$ is asymptotically stable and $\mathcal{J}(A_c, B_c, C_c)$ is minimized. Then there exist*

$$\widetilde{P} \triangleq \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \geq 0 \quad \textit{and} \quad \widetilde{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \geq 0$$

such that A_c, B_c, C_c, $\widetilde{P}$, $\widetilde{Q}$ satisfy

$$0 = \widetilde{A}\widetilde{Q} + \widetilde{Q}\widetilde{A}^T + \widetilde{V}$$

$$0 = \widetilde{A}^T\widetilde{P} + \widetilde{P}\widetilde{A} + \widetilde{R}$$

$$0 = P_{12}^TQ_{12} + P_2Q_2 + \alpha\|\widetilde{E}_\infty\|_F\left[\sum_{k=0}^{n-2}(k+1)\beta_{k+1}(\Phi A_c - A_c\Phi + \Lambda BC_c - B_cC\Lambda^T) \times\|\widetilde{A}\widetilde{A}^T - \widetilde{A}^T\widetilde{A}\|_F^{k-1}\right]\|\widetilde{D}_\infty\|_F$$

$$0 = P_2B_cV_2 + P_{12}^TQ_1C^T + P_2Q_{12}^TC^T + \alpha\|\widetilde{E}_\infty\|_F\left[\sum_{k=0}^{n-2}(k+1)\beta_{k+1} \times(\Phi B_cCC^T + \Lambda AC^T - A_c\Lambda C^T - B_cC\Pi C^T)\|\widetilde{A}\widetilde{A}^T - \widetilde{A}^T\widetilde{A}\|_F^{k-1}\right]\|\widetilde{D}_\infty\|_F + \alpha\|\widetilde{E}_\infty\|_F\left[\sum_{k=0}^{n-1}\beta_k\|\widetilde{A}\widetilde{A}^T - \widetilde{A}^T\widetilde{A}\|_F^k\right]B_cV_{2\infty}\|\widetilde{D}_\infty\|_F^{-1}$$

$$0 = R_2C_cQ_2 + B^TP_1Q_{12} + B^TP_{12}Q_2 + \alpha\|\widetilde{E}_\infty\|_F\left[\sum_{k=0}^{n-2}(k+1)\beta_{k+1} \times(B^T\Pi BC_c + B^T\Lambda^TA_c - B^TA\Lambda^T - B^TBC_c\Phi)\|\widetilde{A}\widetilde{A}^T - \widetilde{A}^T\widetilde{A}\|_F^{k-1}\right]\|\widetilde{D}_\infty\|_F + \alpha R_{2\infty}C_c\|\widetilde{E}_\infty\|_F^{-1}\left[\sum_{k=0}^{n-1}\beta_k\|\widetilde{A}\widetilde{A}^T - \widetilde{A}^T\widetilde{A}\|_F^k\right]\|\widetilde{D}_\infty\|_F^2$$

where

$$\Phi \triangleq B_cCC^TB_c^T + A_cA_c^T - C_c^TB^TBC_c - A_c^TA_c$$

$$\Lambda \triangleq B_cCA^T + A_cC_c^TB^T - C_c^TB^TA - A_c^TB_cC$$

$$\Pi \triangleq AA^T - A^TA - C^TB_c^TB_cC + BC_cC_c^TB^T$$

5. Numerical algorithm and illustrative examples

To compute the controller gains (A_c, B_c, C_c), we first rewrite the dynamic compensation gains (A_c, B_c, C_c) as decentralized gains so that each gain in the decentralized format is regarded as a static output feedback gain for an equivalent problem (Wang and Bernstein 1993, Bernstein *et al.* 1989, Seinfeld *et al.* 1991). We then use a standard quasi-Newton algorithm to perform the optimization and thus obtain controller gains (A_c, B_c, C_c). In the numerical calculation, the LQG controller gains are used as the initial condition for the numerical algorithm.

In the following we consider an example as a numerical illustration of Theorem 4.2. Consider the F8 aircraft plant model in Gilbert and Tan (1991) with disturbance and measurement noise

$$A = \begin{bmatrix} -0{\cdot}8 & -0{\cdot}0006 & -12 & 0 \\ 0 & -0{\cdot}014 & -16{\cdot}64 & -32{\cdot}2 \\ 1 & -0{\cdot}0001 & -1{\cdot}5 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -19 & -3 \\ -0{\cdot}66 & -0{\cdot}5 \\ -0{\cdot}16 & -0{\cdot}5 \\ 0 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} -2{\cdot}1 & -0{\cdot}6 & 0 & 0 \\ -0{\cdot}3 & -0{\cdot}25 & 0 & 0 \\ -0{\cdot}1 & -0{\cdot}2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 & 0 & 0{\cdot}1 & 0 \\ 0 & 0 & 0 & 0{\cdot}2 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 10 & 0 \\ 0 & 10 \end{bmatrix}$$

The state vector is $x = [x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}$, where $x_1(t) =$ pitch rate in rad s^{-1}, $x_2(t) =$ forward velocity in ft s^{-1}, $x_3(t) =$ angle of attack in radians, and $x_4(t) =$ pitch angle in radians. The input vector is such that $u = [\theta_e\ \theta_f]^{\mathrm{T}}$, where θ_e is the elevator angle in radians and θ_f is the flaperon angle in radians. The measurements are $y_1(t) = z_1(t) = x_4(t)$, pitch angle in radians, and $y_2(t) = z_2(t)$, flight path angle in radians. The performance variables $z_3(t)$ and $z_4(t)$ are the weighted control signals.

In this model, our goal is to constrain the maximum response of the pitch angle due to the worst initial condition $x(0)$, that is, to constrain $\|z_{1\infty}(t)\|_2 = \sigma_{\max}(\tilde{E}_\infty e^{\tilde{A}t})\|x(0)\|_2$. Thus, we set

$$E_{1\infty} = [0\ 0\ 0\ 1], \quad E_{2\infty} = 0_{1\times 2}, \quad D_{1\infty} = I_4, \quad D_{2\infty} = 0_{2\times 4}$$

Using the resulting LQG gains A_c, B_c, C_c as the initializing gains, we applied the BFGS quasi-Newton method to compute controller gains with $\alpha = 1$ and $\beta_k = (t^k_{\max}/k!)(n^3 - n/12)^{k/4}$, where $n = 4$ is the plant dimension, $k = 0, \ldots, 3$, and $t = t_{\max} = 0{\cdot}98$ s is the time at which $\sigma_{\max}(\tilde{E}\,e^{\tilde{A}t})$ achieves its maximum value over $[0, \infty)$ using the LQG gains. Figure 4 shows the maximum singular value of $\tilde{E}_\infty e^{\tilde{A}t}$ which corresponds to the worst case pitch angle $\|z_{1\infty}(t)\|_2 = \|x_4(t)\|_2$ for all $\|x(0)\|_2$ satisfying $\|x(0)\|_2 = 1$. Note that the maximum excursion of $\|x_4(t)\|_2$ has been reduced by 42·55%. The closed-loop poles of the LQG design are at $\{-2{\cdot}772 \pm j4{\cdot}29,\ -1{\cdot}1868 \pm j3{\cdot}464,\ -1{\cdot}26,\ -0{\cdot}2784,\ -0{\cdot}0122,\ -0{\cdot}013\}$, whereas the modified design places the closed-loop poles at $\{-2{\cdot}851 \pm j3{\cdot}538,\ -1{\cdot}1092 \pm j4{\cdot}2757,\ -1{\cdot}3954,\ -0{\cdot}0799 \pm j0{\cdot}1496,\ -0{\cdot}012\}$. Finally, in Fig.

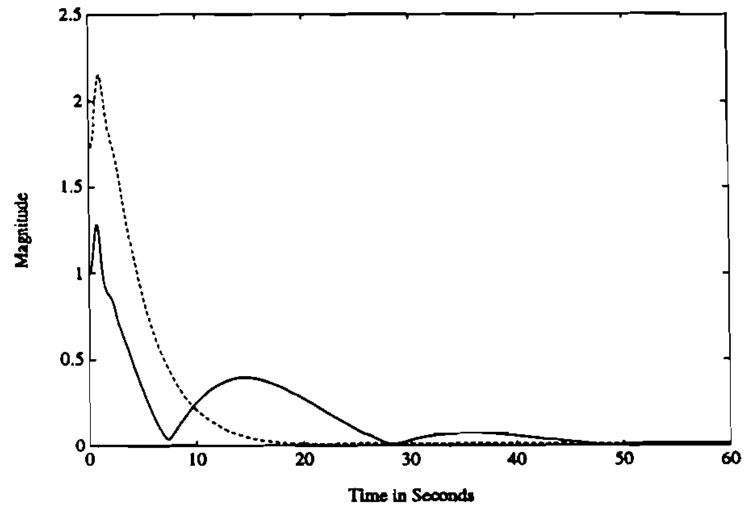

Figure 4. Excursion of pitch angle $\|x_4(t)\|_2$ due to the worst case initial condition $x(0)$ satisfying $\|x(0)\|_2 = 1$; modified design (solid curve) versus LQG design (dashed curve).

5 we compare the normalized H_2 cost with the normalized L_∞ cost of three modified controllers.

6. Conclusions

We present a synthesis method to constrain the closed-loop impulse response. A penalty cost is added to the standard LQG cost to obtain an auxiliary optimization problem that constrains the closed-loop impulse response. This auxiliary cost is an upper bound for the maximum singular value of the closed-loop impulse response. We then applied the method to improve the

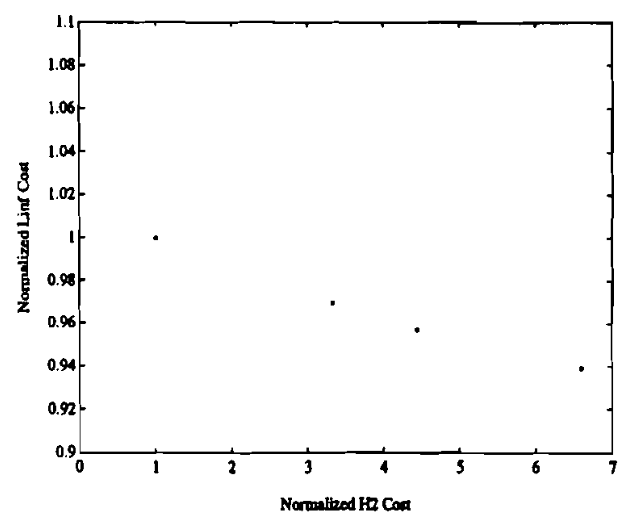

Figure 5. Cost comparison between H_2 cost and L_∞ cost for the LQG controller and three modified controllers.

closed-loop impulse response of the F8 fighter plane as measured by the L_∞ norm (5).

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References

Bernstein, D. S., Haddad, W. M., and Nett, C. N., 1989, Minimal complexity control law synthesis—Part 2: Problem solution via H_2/H_∞ optimal static output feedback. *Proceedings of the American Control Conference*, Pittsburgh, Pennsylvania, pp. 2506–2512.

Gilbert, E. G., and Tan, K. T., 1991, Linear systems with state and control constraints: the theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, **36**, 1008–1020.

Henrici, P., 1962, Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices. *Numerische Mathematik*, **8**, 24–40.

Horn, R. A., and Johnson, C. R., 1985, *Matrix Analysis* (Cambridge, U.K.: Cambridge University Press).

Marshall, A. W., and Olkin, I., 1979, *Inequalities: Theory of Majorization and Its Applications* (New York: Academic Press).

Rotea, M. A., 1993, The generalized H_2 control problem. *Automatica*, **29**, 373–385.

Seinfeld, D. R., Haddad, W. M., Bernstein, D. S., and Nett, C. N., 1991, H_2/H_∞ controller synthesis: illustrative numerical results via quasi-Newton methods. *Proceedings of American Control Conference*, Boston, Massachusetts, pp. 1155–1156.

Strom, T., 1975, On logarithmic norms. *SIAM Journal on Numerical Analysis*, **12**, 741–753.

Van Loan, C., 1977, The sensitivity of the matrix exponential. *SIAM Journal on Numerical Analysis*, **14**, 971–981.

Wang, Y. W., and Bernstein, D. S., 1993, H_2 Optimal control with an α-shifted pole constraint. *International Journal of Control*, **58**, 1201–1214.

Wilson, D. A., 1989, Convolution and Hankel operator norms for linear systems. *IEEE Transactions on Automatic Control*, **34**, 94–97; 1990, Extended optimality properties of the linear quadratic regulator and stationary Kalman filter. *IEEE Transactions on Automatic Control*, **35**, 583–585.