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Y. William Wang a; Dennis S. Bernstein a
a Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI, U.S.A.

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L₂ controller synthesis with $L_\infty$-bounded closed-loop impulse response

Y. WILLIAM WANG† and DENNIS S. BERNSTEIN†

In this paper we consider an $L_2$ control problem with an $L_\infty$ norm on the closed-loop impulse response. To do this we first construct an upper bound for the $L_\infty$ norm of the impulse response of the closed-loop system. To perform controller synthesis, we then modify the standard LQG cost functional by including an additional penalty term that weights the $L_\infty$ norm of the impulse response of the closed-loop system. A numerical example is given to illustrate the improved $L_\infty$ response of the closed-loop system.

Notation

- $\mathbb{R}$, $\mathbb{C}$: real numbers, complex numbers
- spec$(A)$: the set of eigenvalues of $A$
- $\rho(A)$: spectral radius of $A$, $\max \{|\lambda| : \lambda \in \text{spec}(A)\}$
- $\sigma_{\max}(A)$, $\sigma_{\min}(A)$: maximum, minimum singular value of $A$
- $\alpha(A)$: second largest singular value of $A$
- $\sigma(A)$: spectral abscissa of $A$, $\max \{|\Re \lambda| : \lambda \in \text{spec}(A)\}$
- $\|A\|_F$, $\|x(i)\|_2$: Frobenius norm of $A$, $(x^T(i)x(i))^{1/2}$

1. Introduction and problem formulation

Although standard LQG design optimizes the closed-loop system response from disturbance to performance by minimizing the $L_2$ norm of the impulse response, it does not necessarily minimize the $L_\infty$ norm of the closed-loop impulse response. Rather, LQG design optimizes the time integral of the square of the Frobenius norm of the closed-loop impulse response. In many practical situations, however, it is desirable to enforce a pointwise-in-time constraint on the impulse response of the closed-loop system (Gilbert and Tan 1991). To address this problem, we consider the linear feedback system depicted in Fig. 1.

![System schematic diagram](image-url)
with plant \( G \) and controller \( G_c \), where the signal \( w(t) \) denotes white noise and \( w_o(t) \) denotes an impulse signal. The performance variables are \( z(t) \) and \( z_o(t) \), respectively, for the disturbance inputs \( w(t) \) and \( w_o(t) \). With the closed-loop transfer functions \( \tilde{G} \) and \( \tilde{G}_o \) given by

\[
\begin{align*}
\dot{x}(t) &= \tilde{A}x(t) + \tilde{D}w(t) + \tilde{D}_o w_o(t) \\
z(t) &= \tilde{E}x(t) \\
z_o(t) &= \tilde{E}_o x(t)
\end{align*}
\]

(1) (2) (3)

\[ \|G\|_2 \text{ and } \|G_o\|_{L_\infty} \text{ are defined by} \]

\[ \|G\|_2 \triangleq \left( \int_0^\infty \|\tilde{E} e^{\tilde{A}t} \tilde{D}\|^2 dt \right)^{1/2} \]

(4)

\[ \|G_o\|_{L_\infty} \triangleq \sup_{t \geq 0} \sigma_{\text{max}}(\tilde{E}_o e^{\tilde{A}t} \tilde{D}_o) \]

(5)

**Mixed-norm optimization problem:** Given \( G \), obtain \( G_c \) such that \( \|\tilde{G}\|_2 \) is minimized with constrained \( \|\tilde{G}_o\|_{L_\infty} \).

A closely related problem involves \( L_2 \) disturbances \( w_o(t) \) with an \( L_{\infty} \) norm on \( z(t) \). In this case the induced norm is given by (Wilson 1989)

\[ \|G\|_{L_2} = d_{\max} \left( \int_{-\infty}^\infty \tilde{G}(t) \tilde{G}^T(t) dt \right)^{1/2} \]

(6)

where \( d_{\max} \) denotes the largest diagonal entry. Controllers that minimize (6) are considered in (Rotea 1993). The norm (6), however, is more closely related to the \( L_2 \) norm (4) than the \( L_{\infty} \) norm (5). In fact, (6) and (4) are identical in the single-input and single-output cases, whereas (5) represents a pointwise-in-time constraint on the system response.

In this paper, we develop a method that constrains \( \|\tilde{G}_o\|_{L_\infty} \) and thus the excursion of \( z_o(t) \) when \( w(t) = 0 \) and \( w_o(t) \) is an impulse. Since the size of \( \tilde{E}_o e^{\tilde{A}t} \tilde{D}_o \) depends on the maximum singular value of \( e^{\tilde{A}t} \), we first obtain upper bounds for the maximum singular value of \( e^{\tilde{A}t} \). One such bound is based upon the maximum eigenvalue of \( \tilde{A} + \tilde{A}^T \) while the other involves the Frobenius norm of \( \tilde{A}^T \tilde{A} - \tilde{A} \tilde{A}^T \). In general, there is no ordering between these bounds.

Using synthesis techniques, we utilize the second bound to limit the \( L_{\infty} \) norm of \( \tilde{E}_o e^{\tilde{A}t} \tilde{D}_o \). To do this, we modify the standard LQG performance measure by adding a penalty term to the standard LQG performance measure. We then perform controller design by optimizing this modified LQG performance measure.

We begin in § 2 by investigating properties of the Frobenius norm and maximum singular value of \( e^{\tilde{A}t} \) for an arbitrary matrix \( \tilde{A} \). The results given in this section illustrate the difficulty of characterizing the \( L_{\infty} \) norm (5). Then, in § 3, we construct two upper bounds for the \( L_{\infty} \) norm of the impulse response. In § 4, we consider controller synthesis where we address both static output feedback control and dynamic compensation. We then propose a numerical algorithm for obtaining controller gains in § 5. An example involving an F8 fighter is shown in § 5. Concluding remarks are given in § 6.
L<sub>2</sub> controller synthesis

2. Properties of the Frobenius norm and maximum singular value of e<sup>At</sup>

To bound the impulse response of a linear system, it is useful to investigate properties of the maximum singular value of e<sup>At</sup>. To begin, consider the system

\[ \dot{x}(t) = Ax(t), \quad x(0) = x_0 \]  

whose solution is given by

\[ x(t) = e^{At}x_0 \]  

If A is asymptotically stable then e<sup>At</sup> → 0 as t → ∞ and thus x(t) → 0 as t → ∞. Furthermore, \( \|x(t)\|_2 \) is bounded by

\[ \|x(t)\|_2 = \|e^{At}x_0\|_2 \leq \sigma_{\max}(e^{At})\|x_0\|_2 \]  

Since \( \sigma_{\max}(e^{At}) \) is the norm induced by the euclidean norm \( \|\cdot\|_2 \), it follows that for every value of \( t \geq 0 \) there exists an initial condition \( x_0 \) such that equality holds in (9), that is

\[ \|x(t)\|_2 = \|e^{At}x_0\|_2 = \sigma_{\max}(e^{At})\|x_0\|_2 \]  

Thus, the maximum excursion of \( x(t) \) from the origin depends upon the maximum size of \( \sigma_{\max}(e^{At}) \) for \( t \geq 0 \). Unfortunately, although \( \sigma_{\max}(e^{At}) \) → 0 as \( t \to \infty \), \( \sigma_{\max}(e^{At}) \) may not be decreasing for all \( t \). To illustrate this, we consider the following example.

**Example 1:** Let

\[ A = \begin{bmatrix} -0.1 & -3 \\ 2 & -0.2 \end{bmatrix} \]

As can be seen in Fig. 2, \( \sigma_{\max}(e^{At}) \) initially increases and is oscillatory. Therefore, there exists \( x(0) \) such that \( \|x(0)\|_2 = 1 \) and \( \|x(t)\|_2 \approx 1.2 \) for \( t = 0.8 \). We also observe that due to the switching in magnitude between the eigenvalues of \( e^{At}e^{At^\top} \), \( \sigma_{\max}(e^{At}) \) is not smooth at its local minima. However, since \( \sigma_{\max}(e^{At}) = \lambda_{\max}^{1/2}(e^{At}e^{At^\top}) \) involves switching between complex exponential functions, it can be seen that the one-sided derivative of \( \sigma_{\max}(e^{At}) \) always exists.

![Figure 2. \( \sigma_{\max}(e^{At}) \) (solid curve), \( \sigma_{2}(e^{At}) \) (dash-dot curve) and \( \|e^{At}\|_F \) (dashed curve).](image-url)
Proposition 2.1: Let $A \in \mathbb{R}^{n \times n}$. Then the one-sided derivative of $\sigma_{\max}(e^{At})$ at $t = 0^+$ is given by

$$\frac{d}{dt}\sigma_{\max}(e^{At})|_{t=0^+} = \lambda_{\max}(A + A^T)/2$$  \hspace{1cm} (11)

Proof: Note that

$$\frac{d}{dt}\sigma_{\max}(e^{At})|_{t=0^+} = \lim_{t \to 0^+} \frac{\sigma_{\max}(e^{At}) - 1}{t}$$

$$= \lim_{t \to 0^+} \frac{\lambda_{\max}^{1/2}(e^{AT} e^{At}) - 1}{t}$$

$$= \lim_{t \to 0^+} \frac{\{\lambda_{\max}[(I + AT)(I + At)]\}^{1/2} - 1}{t}$$

$$= \lim_{t \to 0^+} \frac{[1 + t\lambda_{\max}(A^T + A)]^{1/2} - 1}{t}$$

$$= \lambda_{\max}(A^T + A)/2$$

Proposition 2.1 shows that if $A + A^T$ is indefinite then $\sigma_{\max}(e^{At})$ initially increases for $t \geq 0$. In this case, $\sigma_{\max}(e^{At})$ has a maximum value greater than unity. Since $\sigma_{\max}(e^{At})$ may not be differentiable at all $t > 0$, it is not easy to characterize the maximum value of $\sigma_{\max}(e^{At})$. Instead, we recall an upper bound for $\sigma_{\max}(e^{At})$.

Lemma 2.1: Let $A \in \mathbb{R}^{n \times n}$. Then, for all $t \in \mathbb{R}$

$$\sigma_{\max}(e^{At}) \leq e^{\sigma_{\max}(A + A^T)/2}$$  \hspace{1cm} (12)

Proof: The result is given by Lemma 1c of Strom (1975).

In the more restrictive case in which $A$ is dissipative, that is, $A + A^T < 0$, we have the following result.

Proposition 2.2: Let $A \in \mathbb{R}^{n \times n}$. Then $A$ is dissipative if and only if $\sigma_{\max}(e^{At})$ is strictly decreasing for all $t \in \mathbb{R}$.

Proof: If $A$ is dissipative, Lemma 2.1 implies that $\sigma_{\max}(e^{At}) \leq e^{\sigma_{\max}(A + A^T)/2} < 1$, for $t > 0$. Thus, it follows that $e^{AT} e^{At} < I$ for all $t > 0$. Letting $t = t_2 - t_1$, where $t_2 > t_1$, we obtain $e^{AT_2} e^{At_2} < e^{AT_1} e^{At_1}$. Hence, $\sigma_{\max}(e^{At_2}) < \sigma_{\max}(e^{At_1})$. Conversely, if $\sigma_{\max}(e^{At})$ is strictly decreasing for $t \geq 0$ then Proposition 2.1 implies that $\lambda_{\max}(A + A^T) < 0$. Thus, $A + A^T < 0$.

Corollary 2.1: Let $\dot{x}(t) = Ax(t)$ and $A \in \mathbb{R}^{n \times n}$ be dissipative. Then $\|x(t)\|_2$ is strictly decreasing for all $t \in \mathbb{R}$.

Proof: Using Proposition 2.2, it can be seen that if $t_1 < t_2$ then

$$\|x(t_2)\|_2 = \|e^{A(t_2-t_1)}x(t_1)\|_2 \leq \sigma_{\max}(e^{A(t_2-t_1)})\|x(t_1)\|_2 < \|x(t_1)\|_2$$

We now consider the most restrictive case in which $A$ is normal and asymptotically stable.
Proposition 2.3: Let $A \in \mathbb{R}^{n \times n}$ be normal and asymptotically stable. Then, for all $t > 0$,

$$\sigma_{\text{max}}(e^{At}) = e^{\frac{1}{2} \lambda_{\text{max}}(A+A^T)t} = e^{\alpha(A)t} < 1$$ (13)

Proof: Since $A$ is normal and asymptotically stable, there exist unitary $V \in \mathbb{C}^{n \times n}$ and diagonal $\Lambda \in \mathbb{C}^{n \times n}$ such that $A = V\Lambda V^*$ and $A + A^* < 0$. Thus, for $t > 0$

$$\sigma_{\text{max}}(e^{At}) = [\lambda_{\text{max}}(e^{A^T}e^{At})]^{1/2} = e^{\frac{1}{2} \lambda_{\text{max}}(A+A^T)t} < 1$$

We also consider $\|e^{At}\|_F$ which is a smooth function of $t$. As can be seen from Fig. 2, $\|e^{At}\|_F$ and $\sigma_{\text{max}}(e^{At})$ may have very different characteristics. Let us consider the case in which $A$ is dissipative.

Proposition 2.4: Let $A \in \mathbb{R}^{n \times n}$ be dissipative. Then $\|e^{At}\|_F$ is strictly decreasing for $t \geq 0$.

Proof: Note that, for $t \geq 0$

$$\frac{d}{dt}\|e^{At}\|_F = \text{tr} e^{A^T}(A + A^T)e^{At} < 0$$

Although Proposition 2.4 provides a sufficient condition for $\|e^{At}\|_F$ to be strictly decreasing, it can be seen from Fig. 2 that this condition is not necessary.

Finally, we consider the case in which $A$ is normal.

Proposition 2.5: Let $A \in \mathbb{R}^{n \times n}$ be normal. Then

$$\|e^{At}\|_F = \left[ \sum_{k=1}^{n} e^{2t \text{Re} \lambda_k} \right]^{1/2}$$ (14)

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$.

3. An alternative upper bound for $\sigma_{\text{max}}(e^{At})$

Lemma 2.1 provides an upper bound for $\sigma_{\text{max}}(e^{At})$. Here, we state an alternative bound for $\sigma_{\text{max}}(e^{At})$ based upon the Schur decomposition. Let the Schur decomposition of $A$ be $A = Q(D + N)Q^*$, where $D$ is diagonal, $N$ is strictly upper triangular, and $Q$ is unitary.

Proposition 3.1: Let $A \in \mathbb{R}^{n \times n}$. Then, for all $t \in \mathbb{R}$

$$\sigma_{\text{max}}(e^{At}) \leq e^{\sigma_{\text{max}}(D)t} \sum_{k=0}^{n-1} \frac{t^k}{k!} \sigma_{\text{max}}^k(N)$$ (15)

Proof: The result appears as (2.11) in Van Loan (1977).

Although bounds (12) and (15) hold for all $A$, Fig. 3 shows with

$$A = \begin{bmatrix} -0.5 & 2 \\ 0 & -0.7 \end{bmatrix}$$

that there does not generally exist an ordering between these two bounds. Note that bound (12) approaches zero monotonically as $t \to \infty$ if and only if $A + A^T < 0$, while bound (15) approaches zero as $t \to \infty$ if and only if $A$ is asymptotically stable. The following result shows that the exponential factor $e^{\sigma_{\text{max}}(D)t}$ in (15) is always better than the exponential factor $e^{\sigma_{\text{max}}(A+A^T)t/2}$ in (12).
Proposition 3.2: Let $A \in \mathbb{R}^{n \times n}$. Then
\[ \alpha(A) \leq \lambda_{\text{max}}(A + A^T)/2 \]  
Furthermore, equality holds if and only if $A$ is normal.


Now let us focus on the polynomial part of (15). Since $N$ is upper triangular and nilpotent, we have the following bounds for $\|N\|_F$.

Lemma 3.1: Let $A \in \mathbb{R}^{n \times n}$. Then
\[
\text{(i) } \|N\|_F = \left\|A\right\|_F^2 - \sum_{i=1}^n |\lambda_i|^2)^{1/2} < \|A\|_F \\
\text{(ii) } \|N\|_F = \left\|A + A^T\right\|_F^2 - 2\sum_{i=1}^n (\text{Re} \lambda_i)^2)^{1/2} \leq \frac{1}{\sqrt{2}} \|A + A^T\|_F \\
\text{(iii) } \|N\|_F = \left\|A - A^T\right\|_F^2 - 2\sum_{i=1}^n (\text{Im} \lambda_i)^2)^{1/2} \leq \frac{1}{\sqrt{2}} \|A - A^T\|_F \\
\text{(iv) } \|N\|_F \leq \left(\frac{n^3 - n}{12}\right)^{1/4} \|A^T A - A A^T\|_F^{1/2}
\]
where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$.

Proof: To prove (i), note that $\|A\|_F^2 = \|Q^* A Q\|_F^2 = \|D + N\|_F^2$. By direct expansion, we have
\[
\|D + N\|_F^2 = \text{tr} (D^* D + N^* N) = \|N\|_F^2 + \sum_{i=1}^n |\lambda_i|^2
\]
Thus, $\|N\|_F = \left\|A\right\|_F^2 - \sum_{i=1}^n |\lambda_i|^2)^{1/2} < \|A\|_F$. Statements (ii) and (iii) can be proved in a similar manner. The proof of (iv) is given in Henrici (1962).

Using (iv) in Lemma 3.1, $\alpha_{\text{max}}(e^{At})$ can be bounded as below.
Theorem 3.1: Let \( A \in \mathbb{R}^{n \times n} \). Then, for all \( t \in \mathbb{R} \)

\[
\sigma_{\text{max}}(e^{A t}) \leq e^{\alpha(A) t} \sum_{k=0}^{n-1} \frac{t^k}{k!} \left( \frac{n^3 - n}{12} \right)^{k/4} \|A^T A - AA^T\|_F^{k/2}
\]

(17)

Proof: Combining Lemma 3.1 and Proposition 3.1 with the fact that \( \sigma_{\text{max}}(N) \leq \|N\|_F \) yields (17).

The following bound for the impulse response matrix will be useful for synthesis.

Corollary 3.1: Let \( E \in \mathbb{R}^{n \times n} \), \( A \in \mathbb{R}^{n \times n} \) and \( D \in \mathbb{R}^{n \times q} \). Then, for all \( t \in \mathbb{R} \)

\[
\sigma_{\text{max}}(E e^{A t} D) \leq e^{\alpha(A) t} \|E\|_F \left[ \sum_{k=0}^{n-1} \frac{t^k}{k!} \left( \frac{n^3 - n}{12} \right)^{k/4} \|A^T A - AA^T\|_F^{k/2} \right] \|D\|_F
\]

(18)

4. Controller synthesis for bounded closed-loop impulse response

Consider the system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) + D_1w(t) + D_{1\infty}w_\infty(t) \\
y(t) &= Cx(t) + D_2w(t) + D_{2\infty}w_\infty(t) \\
z(t) &= E_1x(t) + E_2u(t) \\
z_\infty(t) &= E_{1\infty}x(t) + E_{2\infty}u(t)
\end{align*}
\]

where \( A, B, D_1, D_{1\infty}, C, D_2, D_{2\infty}, E_1, E_2, E_{1\infty} \) and \( E_{2\infty} \) are \( n \times n, n \times m, n \times l_1, n \times l_2, p \times n, p \times l_1, p \times l_2, q \times n, q \times m, r \times n \) and \( r \times m \) real matrices. We seek a static output feedback controller \( u(t) = Ky(t) \) such that

(i) the closed-loop system

\[
\begin{align*}
\dot{x}(t) &= \bar{A}x(t) + \bar{D}w(t) \\
z(t) &= \bar{E}x(t)
\end{align*}
\]

where \( \bar{A} \triangleq A + BKC \) is asymptotically stable, \( \bar{D} \triangleq D_1 + BKD_2 \) and \( \bar{E} \triangleq E_1 + E_2KC \);

(ii) the \( H_2 \) performance

\[
J(K) \triangleq \lim_{t \to \infty} \mathbb{E}[z^T(t)z(t)]
\]

is minimized; and

(iii) the closed-loop impulse response defined by (5) is bounded, where \( \bar{D}_\infty \triangleq D_{1\infty} + BKD_{2\infty} \) and \( \bar{E}_\infty \triangleq E_{1\infty} + E_{2\infty}KC \).

Thus, an auxiliary minimization problem can be formulated as follows. Given \( \alpha > 0 \), minimize

\[
\mathcal{J}(K) \triangleq J(K) + \alpha f(K)
\]

(26)

In (26), \( f(K) \) is a non-negative function of the closed-loop gain \( K \) and is included to penalize the maximum value of \( \|z_\infty(t)\|_2 \). Using Corollary 3.1 and setting

\[
f(K) = \|\bar{E}_\infty\|_F \left[ \sum_{k=0}^{n-1} \beta_k \|\bar{A}\bar{A}^T - \bar{A}^T\bar{A}\|_F^k \right] \|\bar{D}_\infty\|_F
\]

(27)
where $\beta_k$, $k = 0, \ldots, n - 1$, are positive numbers, we consider the following modified linear quadratic optimization problem: minimize

$$J(K) = \text{tr} \, Q \, \tilde{R} + \alpha \| \bar{E}_\omega \|_F \left[ \sum_{k=0}^{n-1} \beta_k \| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F^2 \right] \| \bar{D}_\omega \|_F$$

subject to the closed-loop Lyapunov equation

$$0 = \bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \tilde{V} \tag{29}$$

where $\tilde{V} \triangleq \bar{D} \bar{D}^T$ and $\bar{R} \triangleq \bar{E}^T \bar{E}$. The number $\beta_k$ will be chosen within the optimization problem to represent the maximum value of $r^k / k! (n^3 - n/12)^k/4$ for $t \geq 0$. Also note that for convenience $k/2$ is replaced by $k$ in the exponent of $\| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F$.

**Theorem 4.1:** Let $K \in \mathbb{R}^{m \times p}$ be such that $\bar{A}$ is asymptotically stable and $J(K)$ is minimized. Then there exist $P, Q \neq 0$ satisfying

$$0 = \bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \tilde{V}$$

$$0 = \bar{A}^T P + P \bar{A} + \bar{R}$$

such that $K$ satisfies

$$0 = E_2^T \bar{E} Q C^T + \alpha \| \bar{E}_\omega \|_F \left[ \sum_{k=0}^{n-2} (k + 1) \beta_{k+1} B^T (2 \bar{A} \bar{A}^T \bar{A} - \bar{A}^T \bar{A}^2 - \bar{A}^2 \bar{A}^T) C^T \right.$$

$$\times \left[ \| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F^k \right] \| \bar{D}_\omega \|_F + \alpha E_2^T \bar{E} Q C^T \| \bar{E}_\omega \|_F^k$$

$$\times \left[ \sum_{k=0}^{n-1} \beta_k \| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F^k \right] \| \bar{D}_\omega \|_F$$

$$+ \alpha \| \bar{E}_\omega \|_F \left[ \sum_{k=0}^{n-1} \beta_k \| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F^k \right] B^T \bar{D}_\omega D_2^T \| \bar{D}_\omega \|_F^k + B^T P Q C^T$$

$$+ B^T P \bar{D} \bar{D}^T$$

**Proof:** The results follow from straightforward algebraic manipulation. \qed

Now we consider dynamic compensation. With the controller

$$\dot{x}_c(t) = A_c x_c(t) + B_c y(t) \tag{31}$$

$$u(t) = C_c x_c(t) \tag{32}$$

the closed-loop system (19)-(22), (31), (32) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} A & B C_c \\ B_c C & A_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} + \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix} w(t) + \begin{bmatrix} D_{1w} \\ B_c D_{2w} \end{bmatrix} w_\omega(t) \tag{33}$$

Thus, the auxiliary minimization problem for dynamic compensation can be stated as follows: minimize

$$J(A_c, B_c, C_c) = \text{tr} \, \bar{Q} \, \tilde{R} + \alpha \| \bar{E}_\omega \|_F \left[ \sum_{k=0}^{n-1} \beta_k \| \bar{A} \bar{A}^T - \bar{A}^T \bar{A} \|_F^k \right] \| \bar{D}_\omega \|_F$$

subject to

$$0 = \bar{A} \bar{Q} + \bar{Q} \bar{A}^T + \tilde{V} \tag{35}$$
where

\[ \tilde{A} \triangleq \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}, \quad \tilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \quad \tilde{E} \triangleq \begin{bmatrix} E_1 & E_2 C_2 \\ 0 & E_2 \end{bmatrix} \]

\[ R_1 \triangleq E_1^T E_1, \quad R_2 \triangleq E_2^T E_2 \]

\[ \tilde{R} \triangleq \begin{bmatrix} R_1 & 0 \\ 0 & C_c^T R_2 C_c \end{bmatrix}, \quad V_2 \triangleq D_2 D_2^T, \quad \tilde{V} \triangleq \begin{bmatrix} V_1 & 0 \\ 0 & B_c V_2 B_c^T \end{bmatrix} \]

\[ \tilde{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \]

\[ \tilde{D}_\infty \triangleq \begin{bmatrix} D_1 \infty \\ B_c D_2 \infty \end{bmatrix}, \quad \tilde{E}_\infty \triangleq \begin{bmatrix} E_1 \infty & E_2 \infty C_c \end{bmatrix}, \quad V_{2 \infty} \triangleq D_2 \infty D_2 \infty^T, \quad R_{2 \infty} \triangleq E_2 \infty^T E_2 \]

**Theorem 4.2:** Let \( A_c \in \mathbb{R}^{n \times n} \), \( B_c \in \mathbb{R}^{m \times n} \) and \( C_c \in \mathbb{R}^{n \times p} \) be such that \( \tilde{A} \) is asymptotically stable and \( \mathcal{J}(A_c, B_c, C_c) \) is minimized. Then there exist

\[ \tilde{P} \triangleq \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix} \geq 0 \quad \text{and} \quad \tilde{Q} \triangleq \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \geq 0 \]

such that \( A_c, B_c, C_c, \tilde{P}, \tilde{Q} \) satisfy

\[ 0 = \tilde{A} \tilde{Q} + \tilde{Q} \tilde{A}^T + \tilde{V} \]

\[ 0 = \tilde{A}^T \tilde{P} + \tilde{P} \tilde{A} + \tilde{R} \]

\[ 0 = P_{12} Q_{12} + P_2 Q_2 + \alpha \| \tilde{E}_\infty \|_F \left[ \sum_{k=0}^{\infty} (k + 1) \beta_{k+1} \right] \| \tilde{D}_\infty \|_F \]

\[ \times \| \tilde{A} - \tilde{A}^T \tilde{A} \|_F^{-1} \| \tilde{D}_\infty \|_F \]

\[ 0 = P_2 B_c V_2 + P_{12} Q_1 C_c^T + P_2 Q_2 C_c^T + \alpha \| \tilde{E}_\infty \|_F \left[ \sum_{k=0}^{\infty} (k + 1) \beta_{k+1} \right] \]

\[ \times (\Phi B_c C C^T \Lambda A_c^T + A_c \Lambda C^T - B_c C P C^T) \| \tilde{A} - \tilde{A}^T \tilde{A} \|_F^{-1} \] \( \| \tilde{D}_\infty \|_F^{-1} \)

\[ 0 = R_2 C_c Q_2 + B^T P_1 Q_{12} + B^T P_{12} Q_2 + \alpha \| \tilde{E}_\infty \|_F \left[ \sum_{k=0}^{\infty} (k + 1) \beta_{k+1} \right] \]

\[ \times (B^T \Pi B C_c + B^T \Lambda A_c - B^T A \Lambda - B^T B C_c \Phi) \| \tilde{A} - \tilde{A}^T \tilde{A} \|_F^{-1} \] \( \| \tilde{D}_\infty \|_F \)

\[ + \alpha R_2 C_c \| \tilde{E}_\infty \|_F \left[ \sum_{k=0}^{\infty} \beta_k \| \tilde{A} - \tilde{A}^T \tilde{A} \|_F \right] \| \tilde{D}_\infty \|_F^{-1} \]

where

\[ \Phi \triangleq B_c C C^T B_c^T + A_c A_c^T - C_c B^T B C_c - A_c^T A_c \]

\[ \Lambda \triangleq B_c C A^T + A_c C_c^T B - C_c B^T A - A_c^T B_c C \]

\[ \Pi \triangleq A A^T - A^T A - C^T B_c^T B_c C + B C_c^T B^T \]
5. Numerical algorithm and illustrative examples

To compute the controller gains \((A_e, B_e, C_e)\), we first rewrite the dynamic compensation gains \((A_e, B_e, C_e)\) as decentralized gains so that each gain in the decentralized format is regarded as a static output feedback gain for an equivalent problem (Wang and Bernstein 1993, Bernstein et al. 1989, Seinfeld et al. 1991). We then use a standard quasi-Newton algorithm to perform the optimization and thus obtain controller gains \((A_e, B_e, C_e)\). In the numerical calculation, the LQG controller gains are used as the initial condition for the numerical algorithm.

In the following we consider an example as a numerical illustration of Theorem 4.2. Consider the F8 aircraft plant model in Gilbert and Tan (1991) with disturbance and measurement noise

\[
A = \begin{bmatrix}
-0.8 & -0.0006 & -12 & 0 \\
0 & -0.014 & -16.64 & -32.2 \\
1 & -0.0001 & -1.5 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-19 & -3 \\
-0.66 & -0.5 \\
-0.16 & -0.5 \\
0 & 0
\end{bmatrix}
\]

\[
D_1 = \begin{bmatrix}
-2.1 & -0.6 & 0 & 0 \\
-0.3 & -0.25 & 0 & 0 \\
-0.1 & -0.2 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad D_2 = \begin{bmatrix}
0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0.2
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
E_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad E_2 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
10 & 0 \\
0 & 10
\end{bmatrix}
\]

The state vector is \(x = [x_1 \ x_2 \ x_3 \ x_4]^T\), where \(x_1(t)\) = pitch rate in \(\text{rad s}^{-1}\), \(x_2(t)\) = forward velocity in \(\text{ft s}^{-1}\), \(x_3(t)\) = angle of attack in radians, and \(x_4(t)\) = pitch angle in radians. The input vector is such that \(u = [\theta_e \ \theta_f]^T\), where \(\theta_e\) is the elevator angle in radians and \(\theta_f\) is the flap angle in radians. The measurements are \(y_1(t) = z_1(t) = x_4(t)\), pitch angle in radians, and \(y_2(t) = z_2(t)\), flight path angle in radians. The performance variables \(z_3(t)\) and \(z_4(t)\) are the weighted control signals.

In this model, our goal is to constrain the maximum response of the pitch angle due to the worst initial condition \(x(0)\), that is, to constrain \(\|z_{1\omega}(t)\|_{2} = \sigma_{\max}(\tilde{E}_\omega e^{\tilde{A}_\omega t})\|x(0)\|_{2}\). Thus, we set

\[
E_{1\omega} = [0 \ 0 \ 0 \ 1], \quad E_{2\omega} = 0_{1x2}, \quad D_{1\omega} = I_4, \quad D_{2\omega} = 0_{2x4}
\]

Using the resulting LQG gains \(A_e, B_e, C_e\) as the initializing gains, we applied the BFGS quasi-Newton method to compute controller gains with \(\alpha = 1\) and \(\beta_k = (t_{\text{max}}/k!)(n^3 - n/12)^{k/4}\), where \(n = 4\) is the plant dimension, \(k = 0, \ldots, 3\), and \(t = t_{\text{max}} = 0.98\ \text{s}\) is the time at which \(\sigma_{\max}(\tilde{E} e^{\tilde{A} t})\) achieves its maximum value over \([0, \infty)\) using the LQG gains. Figure 4 shows the maximum singular value of \(\tilde{E}_\omega e^{\tilde{A}_\omega t}\) which corresponds to the worst case pitch angle \(\|z_{1\omega}(t)\|_{2} = \|x_4(t)\|_{2}\) for all \(\|x(0)\|_{2}\) satisfying \(\|x(0)\|_{2} = 1\). Note that the maximum excursion of \(\|x_4(t)\|_{2}\) has been reduced by 42-55%. The closed-loop poles of the LQG design are at \((-2.772 \pm j4.29, -1\cdot1868 \pm j3.464, -1\cdot26, -0\cdot2784, -0\cdot0122, -0\cdot013)\), whereas the modified design places the closed-loop poles at \((-2\cdot851 \pm j3\cdot538, -1\cdot1092 \pm j4\cdot2757, -1\cdot3954, -0\cdot0799 \pm j0\cdot1496, -0\cdot012)\). Finally, in Fig.
5 we compare the normalized $H_2$ cost with the normalized $L_\infty$ cost of three modified controllers.

6. Conclusions

We present a synthesis method to constrain the closed-loop impulse response. A penalty cost is added to the standard LQG cost to obtain an auxiliary optimization problem that constrains the closed-loop impulse response. This auxiliary cost is an upper bound for the maximum singular value of the closed-loop impulse response. We then applied the method to improve the
closed-loop impulse response of the F8 fighter plane as measured by the $L_{\infty}$ norm (5).

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**References**


