Adaptive stabilization of non-linear oscillators using direct adaptive control

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Direct adaptive controllers developed for linear systems are applied to non-linear oscillators. A wide range of non-linearities are considered, including stiffness non-linearities, input non-linearities, limit cycle oscillations and friction. Numerical results suggest that by increasing the speed of adaptation, these direct adaptive controllers are highly effective when applied to non-linear plants.

1. Introduction

The goal of both robust control and adaptive control is to achieve system performance without excessive reliance on plant models. While robust control seeks to desensitize a control system to plant uncertainty, the gains of a robust controller are fixed. On the other hand, an adaptive controller seeks to adjust controller gains during operation in order to permit greater uncertainty levels than can be tolerated by robust control and to improve system performance during operation, which is not possible with robust control.

This paper considers an output feedback adaptive stabilization problem with unknown constant disturbance rejection. Our results are closely related to those of Åström and Wittenmark (1995), Krstic et al. (1995), Ioannou and Sun (1996) and Kaufman et al. (1998) which focus on model reference adaptive control. The adaptive controller given by Theorem 1 requires that the disturbance satisfy a matching condition and that an output condition be satisfied. This condition is related to a positive real condition for the closed-loop system. Next we specialize this result in Corollary 1 and Corollary 2 to the case of full-state feedback, in which case the range condition is satisfied. By representing the system in controllable canonical form, we show that adaptive stabilization is possible without additional knowledge of the plant dynamics. However, this approach assumes that the sign of the input coefficient is known. If this assumption is violated then universal stabilization techniques are required (Ilchmann 1993).

The primary objective of the present paper is to apply the adaptive controller of Corollary 2 to non-linear systems. In particular, we consider non-linear oscillators possessing various non-linearities including stiffness non-linearities, input non-linearities, limit cycle oscillations and friction. As shown in the paper, the direct adaptive controller is remarkably effective in adaptively stabilizing these plants in spite of the broad range of non-linearities.

2. Adaptive stabilization with constant disturbance rejection

Consider the linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) + d \] (1)
\[ y(t) = Cx(t) \] (2)
\[ z(t) = Ex(t) \] (3)

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( d \in \mathbb{R}^p \), \( y(t) \in \mathbb{R}^q \) and \( z(t) \in \mathbb{R}^r \).

**Theorem 1:** Assume there exists \( K_s \in \mathbb{R}^{n_s \times n_s} \) such that \( A_s \equiv A + BK_sC \) is asymptotically stable and assume there exists \( \phi_s \in \mathbb{R}^{n_s} \) such that \( B \phi_s = d \). Let \( R \in \mathbb{R}^{n_s \times n_s} \) be positive semidefinite and assume \((A_s, R)\) is controllable. Let \( P \in \mathbb{R}^{n_s \times n_s} \) be the positive-definite solution to the Lyapunov equation \( 0 = A_s^T P + PA_s + R \), and assume there exists \( M \in \mathbb{R}^{n_s \times n_s} \) such that \( B^T P = M E \). Finally, let \( \Gamma \in \mathbb{R}^{n_s \times n_s} \) and \( A \in \mathbb{R}^{n_s \times n_s} \) be positive definite, and let \( \lambda > 0 \). Then (2.1)–(2.3) with the control law

\[ u(t) = K(t)y(t) + \phi(t) \] (4)

where

\[ \dot{K}(t) = -\Gamma M z(t) y^T(t) A \] (5)
\[ \dot{\phi}(t) = -\lambda M z(t) \] (6)

yields \( R x(t) \to 0 \) as \( t \to \infty \).

**Proof:** Define

\[ \tilde{K}(t) = K(t) - K_s \]
\[ \tilde{\phi}(t) = \phi(t) + \phi_s \]

so that (5) and (6) imply

\[ \dot{\tilde{K}}(t) = -\Gamma M z(t) y^T(t) A \] (7)
\[ \dot{\tilde{\phi}}(t) = -\lambda M z(t) \] (8)
Then the closed-loop system consists of (7) and (8) and
\[ x(t) = (A_s + B\dot{K}(t)C)x(t) + B\dot{\phi}(t) \quad (9) \]

Next, consider the positive-definite Lyapunov candidate
\[ V(x, \dot{K}, \dot{\phi}) = x^T P x + \tau \Gamma^{-1} \dot{K} A^{-1} \dot{K}^T + \tau \dot{\phi} \lambda^{-1} \dot{\phi}^T \]
The derivative of V along trajectories of the closed-loop system is given by
\[ \dot{V}(x, \dot{K}, \dot{\phi}) = x^T (A_s^T P + PA_s)x + 2x^T PB \dot{K} C x + 2x^T PB \dot{\phi} + 2 \tau \Gamma^{-1} \dot{K} A^{-1} \dot{K}^T + 2 \tau \dot{\phi} \lambda^{-1} \dot{\phi}^T \]
\[ = -x^T R x + 2 \tau \dot{K} (A_s^T \dot{K} T^{-1} + C x x^T P B) + 2 \tau \dot{\phi} (\lambda^{-1} \dot{\phi}^T + x^T P) \]
\[ = -x^T R x + 2 \tau \dot{K} (A_s^T \dot{K} T^{-1} + C x x^T E^T M^T) + 2 \tau \dot{\phi} (\lambda^{-1} \dot{\phi}^T + x^T E^T M^T) \]
\[ = -x^T R x + 2 \tau \dot{K} (A_s^T \dot{K} T^{-1} + \dot{x} x^T M^T) + 2 \tau \dot{\phi} (\lambda^{-1} \dot{\phi}^T + z^T M^T) \]
\[ = -x^T R x \]

It now follows from Theorem 4.4 in Khalil (1996) that, for every initial condition \( x(0), \dot{K}(0) \) and \( \dot{\phi}(0) \), the states of the closed-loop system are bounded, and \( x(t) R x(t) \to 0 \) as \( t \to \infty \). Since \( R \) is positive semidefinite, it follows that \( R x(t) \to 0 \) as \( t \to \infty \).

Theorem 1 requires that there exist \( K_s \) and \( \phi_s \) such that \( A_s = A + BK_s C \) is asymptotically stable and \( B\phi_s = d \). However, the control law (4)–(6) does not require explicit knowledge of \( K_s, \phi_s \) and \( d \). On the other hand, implementation of (5) and (6) requires that there exist a known matrix \( M \) such that \( B^T P = M E \). This condition and the Lyapunov equation \( 0 = A_s^T P + PA_s + R \) are KYP conditions that are equivalent to the assumption that \( (A_s, B, M, E) \) is the realization of a positive real transfer function.

Note that (6) is an integrator state which serves to reject the constant disturbance \( d \).

Next, we specialize Theorem 1 to the full-state feedback case. In this case \( C = E = I \) so that the assumptions of Theorem 1 are satisfied with \( M = B^T P \).

**Corollary 1:** Assume there exists \( K_s \in \mathbb{R}^{n_s \times n_s} \) such that \( A_s = A + BK_s C \) is asymptotically stable and assume there exists \( \phi_s \in \mathbb{R}^{n_s} \) such that \( B\phi_s = d \). Let \( R \in \mathbb{R}^{n_s \times n_s} \) be positive semidefinite and assume \( (A_s, R) \) is controllable. Let \( P \in \mathbb{R}^{n_s \times n_s} \) be the positive-definite solution to the Lyapunov equation \( 0 = A_s^T P + PA_s + R \). Finally, let
\[ \Gamma \in \mathbb{R}_{+}^{n_s \times n_s} \text{ and } A \in \mathbb{R}^{n_s \times n_s} \text{ be positive definite, and let } \lambda > 0. \text{ Then (1) with the control law } \]
\[ u(t) = K(t)x(t) + \phi(t) \quad (10) \]

where
\[ \dot{K}(t) = -\Gamma B^T P x(t)x^T(t) \lambda \]
\[ \dot{\phi}(t) = -B^T P x(t) \lambda \quad (11) \]
yields \( R x(t) \to 0 \) as \( t \to \infty \).

3. State feedback for uncertain systems

Consider the linear system (1) with
\[ A = \begin{bmatrix} A_0 & 0 \\ c & a \end{bmatrix} \quad B = \begin{bmatrix} 0 \end{bmatrix} \quad d = \begin{bmatrix} 0 \end{bmatrix} \quad (13) \]

where \( x(t) \in \mathbb{R}^{n_s}, u(t) \in \mathbb{R}, d \in \mathbb{R}^{n_s}, A_0 \in \mathbb{R}^{(n_s-1) \times n_s}, a \in \mathbb{R}^{1 \times n_s}, b, d_0 \in \mathbb{R} \) and \( b \neq 0 \). Define
\[ B_0 \triangleq \begin{bmatrix} 0^{(n_s-1) \times 1} \\ \text{sign } b \end{bmatrix} \]

**Corollary 2:** Assume there exists \( K_s \in \mathbb{R}^{1 \times n_s} \) such that \( A_s = A + BK_s \) is asymptotically stable. Let \( R \in \mathbb{R}^{n_s \times n_s} \) be positive semidefinite and assume \( (A_s, R) \) is controllable. Let \( P \in \mathbb{R}^{n_s \times n_s} \) be the positive-definite solution to the Lyapunov equation \( 0 = A_s^T P + PA_s + R \). Finally, let \( \Gamma > 0 \) and \( \lambda > 0 \) and let \( A \in \mathbb{R}^{n_s \times n_s} \) be positive definite. Then (1) with the control law
\[ u(t) = K(t)x(t) + \phi(t) \quad (14) \]

where
\[ \dot{K}(t) = -\Gamma B_0^T P x(t)x^T(t) \lambda \]
\[ \dot{\phi}(t) = -B_0^T P x(t) \lambda \quad (15) \]
yields \( R x(t) \to 0 \) as \( t \to \infty \).

**Proof:** First, note that because of the structure of \( B \) and \( d \), it follows that \( \phi_s = d_0 / b \) satisfies \( B \phi_s = d \). Second, since \( A \) and \( \lambda \) are arbitrary, \( A \) in (11) and \( \lambda \) in (12) can be replaced by \( |b|^{-1} A \) and \( |b|^{-1} \lambda \), respectively. Thus, (11) and (12) imply (15) and (16).

Note that (15) and (16) require the solution \( P \) of the Lyapunov equation \( 0 = A_s^T P + PA_s + R \). Since \( b \neq 0 \), let \( K_s = (1 / b)(a_s - a) \), where \( a_s \in \mathbb{R}^{1 \times n_s} \). It then follows that
\[ A_s = A + BK_s = \begin{bmatrix} A_0 & 0 \\ a_s & b \end{bmatrix} \]

Since \( a_s \) can be chosen to stabilize \( A_s \) without knowledge of either \( a \) or \( b \), it follows that \( P \) can be determined without knowledge of either \( a \) or \( b \). However, sign \( b \) must be known in order to implement (15) and (16).
To illustrate Corollary 2, consider the case $n_1 = 1$ and let $a_s < 0$ and $R = -2a_s$. Then $P = 1$, and (15) and (16) are given by

$$\dot{K}(t) = -(\text{sign } b)\lambda_1 x_1^2(t) \quad (17)$$

$$\dot{\phi}(t) = -(\text{sign } b)\lambda_2 x(t) \quad (18)$$

where $\lambda_1 \triangleq A/\Gamma$ and $\lambda_2 \triangleq \lambda$. Note that (17) and (18) yield $x(t) \to 0$ as $t \to \infty$ for all $\lambda_1, \lambda_2 > 0$.

Next, consider the case $n_1 = 2$, and let $A_0 = [0 \ 1]$, $p > 0$, $a_{s1} < 0$, $a_{s2} < -p$ and

$$R = \begin{bmatrix} -2pa_{s1} & 0 \\ 0 & -2p - 2a_{s2} \end{bmatrix}$$

Then

$$P = \begin{bmatrix} -pa_{s2} - a_{s1} & p \\ p & 1 \end{bmatrix}$$

satisfies $0 = A_0^T P + PA + R$ and (15) and (16) are given by

$$\dot{K}_1(t) = -(\text{sign } b)[\lambda_1 px_1^2(t) + (\lambda_1 + \lambda_2 p)x_1(t)x_2(t) + \lambda_2 x_2^2(t)] \quad (19)$$

$$\dot{K}_2(t) = -(\text{sign } b)[\lambda_2 px_2^2(t) + (\lambda_1 + \lambda_2 p)x_1(t)x_2(t) + \lambda_2 x_2^2(t)] \quad (20)$$

$$\dot{\phi}(t) = -(\text{sign } b)\lambda_3 [px_1(t) + x_2(t)] \quad (21)$$
Figure 2. Deadzone non-linearity.

Thus the controller (14)–(16) can be used for this problem.

4. Non-linear stiffness

Consider an oscillator with non-linear stiffness modelled by

$$\ddot{r}(t) + c\dot{r}(t) + f(r(t)) = bu(t)$$

where $f: \mathbb{R} \to \mathbb{R}$. The control objective is to require $r(t)$ to approach $r_{\text{des}}$ without knowledge of $c, f(\cdot)$, and $b$, except the sign of $b$ which is taken to be positive. If $c > 0$ and $rf(r) > 0$ for all $r \in \mathbb{R}$, then (27) is a stable oscillator. However, we do not invoke these assumptions. Defining the error signal $x_1(t) \triangleq r(t) - r_{\text{des}}$ and the state $x \triangleq [x_1 \ x_1^T]$, equation (27) becomes

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -f(x_1(t)) & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ d_0 \end{bmatrix}$$

where $f(x_1)x_1 = f(x_1 + r_{\text{des}}) - f(r_{\text{des}})$ and $d_0 = -f(r_{\text{des}})$. Equation (28) is of the form (26) with $a_1$ replaced by the unknown state-dependent coefficient $-f(x_1)$ and $a_2 = -c$. The controller (14) with (22)–(24) is applied to this problem.

First we let $f(\cdot)$ be a hardening spring modelled by

$$f(r) = k_1r + k_3r^3$$

where $k_1 > 0$ and $k_3 > 0$. In this case $\hat{f}(\cdot)$ and $d_0$ are given by

$$\hat{f}(x_1) = k_1 + k_3(x_1^2 + 3r_{\text{des}}x_1 + 3r_{\text{des}}^2)$$

$$d_0 = -k_1r_{\text{des}} - k_3r_{\text{des}}^3$$

We apply controller (14) with (22)–(24) and $k_1 = 1, c = 0.2, k_3 = 1, b = 3,$ and $r_{\text{des}} = 0.5$. Let $r(0) = -1, \dot{r}(0) = 0, K_1(0) = 0, K_2(0) = 0, \phi(0) = 0$ and choose adaptation weights $p = 1, \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 50$. Furthermore, at $t = 20, k_3$ is changed from 1 to 4, and $r_{\text{des}}$ is changed from 0.5 to 3.

Next, we let $f(\cdot)$ be the deadzone function shown in figure 2. We apply controller (14) with (22)–(24) and with $e_1 = -0.5$, $e_2 = 0.5$, $c = 1$, $b = 5$, and $r_{\text{des}} = 0$. Let $r(0) = 0.1, \dot{r}(0) = 0.2, K_1(0) = 0, K_2(0) = 0, \phi(0) = 0$ and choose adaptation weights $p = 1, \lambda_1 = 10, \lambda_2 = 10, \lambda_3 = 10$. Initially, the adaptation is stopped. As can be seen from figure 3, $r(t)$ approaches 0.3 due to the deadzone. At $t = 10$, the adaptation is started, and, as can be seen from figure 3, $r(t)$ approaches 0.

Next, we let $f(\cdot)$ be the relay function shown in figure 4. We apply controller (14) with (22)–(24) and with $c = 1$, $b = 5$, and $r_{\text{des}} = 0$. Let $r(0) = 0.6, \dot{r}(0) = 0.9, K_1(0) = 0, K_2(0) = 0, \phi(0) = 0$ and choose adaptation weights $p = 1, \lambda_1 = 100, \lambda_2 = 100, \lambda_3 = 100$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 5, the phase plot
shows a limit cycle. At $t = 30$, the adaptation is started, and, as can be seen from the solid line in figure 5, $r(t)$ and $\dot{r}(t)$ approach 0. Finally, we let $f(\cdot)$ be the backlash/hysteresis function shown in figure 6. We apply controller (14) with (22)–(24) and with $h = 1$, $c = 1$, $b = 5$ and $r_{\text{des}} = 0$. Let $r(0) = 0.6$, $\dot{r}(0) = 0.9$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 100$, $\lambda_2 = 100$, $\lambda_3 = 100$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 7, the phase plot shows a limit cycle. At $t = 30$, the adaptation is started and, as can be seen from the solid line in figure 7, $r(t)$ and $\dot{r}(t)$ approach 0.

As can be seen from figures 1, 3, 5 and 7, $r(t)$ tracks $r_{\text{des}}$, that is, the controller (14) with (22)–(24) is able to compensate for the non-linear stiffness in (27).

5. Non-linear damping

Consider an oscillator with position-dependent damping modelled by

$$\ddot{r}(t) + g(r(t)) \dot{r}(t) + k r(t) = bu(t)$$

(30)

where $g: \mathbb{R} \to \mathbb{R}$. The control objective is to require $r(t)$ to approach $r_{\text{des}}$ without knowledge of $g(\cdot), k$ and $b$, except the sign of $b$, which is taken to be positive. Defining the error signal $x_1(t) = r(t) - r_{\text{des}}$ and the state $x = [x_1 \ x_1^T]$, equation (30) becomes
Equation (31) has the form (26) with $a_1 = -k$, $a_2$ replaced by the unknown state-dependent coefficient $-g(x_1 + r_{\text{des}})$, and $d_0$ replaced by the unknown constant $-kr_{\text{des}}$. The controller (14) with (22)–(24) is applied to this problem.

As a special case, consider the Van der Pol equation modelled by

$$g(r) = \varepsilon (1 - r^2)$$

with $k = 1$ and $b = 1$. We apply controller (14) with (22)–(24) and with $\varepsilon = 0.4$ and $r_{\text{des}} = 0$. Let $r(0) = 1$, $r'(0) = 0$, $K_1(0) = 0$, $K_2(0) = 0$, $\phi(0) = 0$ and choose adaptation weights $p = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 1$. Initially, the adaptation is stopped. As can be seen from the dashed line in figure 8, the phase plot shows a limit cycle. At $t = 20$, the adaptation is started and, as can be seen from the solid line in figure 8, $r(t)$ and $\dot{r}(t)$ approach 0.

Next, we consider a time-varying command $r_{\text{des}}(t)$ and define the error signal $x_1(t) = r(t) - r_{\text{des}}(t)$ and the state $x = [x_1 \ x_1']^T$. Then equation (30) becomes

\[
\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -g(x_1 + r_{\text{des}}) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -kr_{\text{des}} \end{bmatrix}
\]
where\(x\) is the state of the system.

The equations of motion for the mass-spring system shown in Figure 10 are given by

\[
\dot{x}_1(t) = -\varepsilon x_1(t) + 2 \sin \omega t,
\]

with adaptation weights \(a_1(t) = 1\) and \(a_2(t) = 0\). Equation (33) has the form (26) with \(a_1\) and \(a_2\) replaced by unknown state-dependent, time-varying coefficients and with time-varying \(d_0(t)\). The controller (14) with (22)–(24) is applied to this problem.

As a special case, consider the Van der Pol equation (32) with the time-varying command \(\sin \omega t\) and \(k = 1\) and \(b = 1\). Then \(\hat{g}(\cdot)\) and \(d_0\) are given by

\[
\hat{g}(x_1(t)) = -\varepsilon x_1(t) + 2 \sin \omega t,
\]

where \(\hat{g}(x_1) = g(x_1 + r_{des}) - g(r_{des})\) with \(d_0(t) = -\dot{r}_{des}(t) - g(r_{des}(t))\dot{r}_{des}(t) - k r_{des}(t)\).

6. Stick-slip friction

The equations of motion for the mass-spring system shown in Figure 10 are given by

\[
m \ddot{r}(t) + k r(t) = k \zeta(t) + k L - F_f(t)
\]

\[
k \zeta(t) = k r(t) - k L + u(t)
\]

where \(m, k > 0, L\) is the distance between the mass and the massless bar when the spring is relaxed and the stick
slip frictional force \( F_s(t) \) is given by (Canudas de Wit et al. 1993)

\[
F_s(t) = (1 - \kappa(t))F_s(t) + \kappa(t)F_d(t)
\]  

(36)

The stick friction \( F_s(t) \) is given by

\[
F_s(t) = \text{sat}_{(\omega_1 + \omega_1)}(k_s\dot{r}(t) + d_s\ddot{r}(t))
\]

(37)

and the dynamic friction \( F_d(t) \) is given by

\[
F_d(t) = \alpha_0 \text{sgn}(\dot{r}(t)) + \alpha_2\ddot{r}(t)
\]

(38)

where \( \alpha_0, \alpha_1, \alpha_2, k_s, d_s > 0 \)

\[
\tau_s\ddot{r}(t) = -\kappa(t) + 1 - e^{-(\dot{r}(t)/\dot{r}(0))^2}
\]

(39)

and

\[
i(t) = (1 - \kappa(t))\dot{x}(t) - \kappa(t)\frac{1}{\tau_r} \eta(t)
\]

(40)

where \( \tau_s, \tau_r > 0 \). The sat function is defined by

\[
\text{sat}_\alpha(\eta) = \begin{cases} 
\alpha & \text{if } \eta \geq \alpha \\
\eta & \text{if } |\eta| < \alpha \\
-\alpha & \text{if } \eta \leq \alpha 
\end{cases}
\]

(41)

As can be seen in figure 11, the magnitude of stick friction, which affects the initial movement of the mass, is greater than the magnitude of the slip friction, which is the frictional force when the mass is moving. By
defining the error signal \( x_1(t) = \dot{r}(t) - r_{\text{des}} \) and eliminating the internal physical variable \( \zeta(t) \), (34) becomes

\[
\dot{x}_2(t) = \dot{x}_1(t) = \frac{1}{m} u(t) - \frac{1}{m} F_i(t)
\]

The controller (14) with (22)–(24) is applied to this problem.

Figure 12 shows the response of the mass-spring system with stick-slip friction with \( m = 1, k = 100, L = 10, r_{\text{des}} = 1, \alpha_0 = 1, \alpha_1 = 1.5, \alpha_2 = 0.6, \tau_s = 0.01, \tau_r = 0.001, k_s = 10000 \) and \( d_s = 1100 \). Let \( r(t) = 0, \dot{r}(0) = 0.04, K_1(0) = 0, K_2(0) = 0, \phi(0) = 0 \) and choose adaptation weights \( p = 40, \lambda_1 = 500, \lambda_2 = 1, \lambda_3 = 100 \). As can be seen in figure 12(d), \( r(t) \) approaches the commanded position. However, due to stick friction, figure 12(d) shows overshoot at the beginning of control. A critical aspect is the distance between the mass and the massless bar, which has increased due to the control action.

7. Input non-linearity

Consider an oscillator with input non-linearity modelled by the Hammerstein system

\[
\ddot{r}(t) + c\dot{r}(t) + kg(t) = bf(u(t))
\]

The control objective is to require \( \dot{r}(t) \) to approach \( r_{\text{des}} \) without knowledge of \( c, k, f(\cdot) \) and \( b \), except the sign of
which is taken to be positive. Defining the error signal
\[ x_1(t) = r(t) - r_{\text{des}} , \]
equation (7.1) becomes
\[ \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b f(u(t)) \end{bmatrix} \]  
which has the form (26) with \( a_1 = -k \) and \( a_2 = -c \) and with \( bu(t) \) replaced by \( bf(u(t)) \). The controller (14) with (22)–(24) is applied to this problem.

First, we let \( f(\cdot) \) be the deadzone non-linearity shown in figure 2. In this case, \( uf(u) > 0 \), and thus \( u \) and \( f(u) \) have the same sign. We apply controller (14) with (22)–(24) and with \( e_1 = -0.5, e_2 = 0.5, c = -2, k = -1, b = 1, r_{\text{des}} = 0, r(0) = -0.3 \), and \( \hat{r}(0) = 0.5 \). For comparison, a stabilizing linear controller is designed for the system (44) with \( f(u) = u \), which is \( u(t) = -2x_1(t) - 4x_2(t) \). This controller is applied to the system (44) with the deadzone non-linearity \( f(\cdot) \). It can be seen from the solid line in figure 13(b) that \( r(t) \) does not approach \( r_{\text{des}} \) when a linear controller is used. However, by choosing adaptation weights \( p = 1, \lambda_1 = 10^3, \lambda_2 = 10^3, \lambda_3 = 10^3 \) and letting \( K_1(0) = 0, K_2(0) = 0, \) and \( \phi(0) = 0 \), figure 13(d) shows that \( r(t) \) approaches \( r_{\text{des}} \) when the adaptive controller is used.

Next, we let \( f(\cdot) \) be the relay non-linearity shown in figure 4. Note that in this case \( u \) and \( f(u) \) do not always have the same sign. We apply controller (14) with (22)–(24) and with \( c = -2, k = -1, b = 1, r_{\text{des}} = 0, r(0) = -0.4 \), and \( \hat{r}(0) = 0.5 \). For comparison, a
stabilizing linear controller is designed for the system (44) with \( f(u) = u \), which is \( u(t) = -2x_1(t) - 4x_2(t) \).

This linear controller is applied to the system (44) with relay \( f(\cdot) \). Choose adaptation weights \( p = 1, \lambda_1 = 10^3, \lambda_2 = 10^3, \lambda_3 = 10^3 \) and let \( K_1(0) = 0, K_2(0) = 0, \) and \( \phi(0) = 0 \). As can be seen from the solid line in figure 14(b), \( r(t) \) does not approach \( r_{\text{des}} \) when the linear controller is used. However, figure 14(d) shows that \( r(t) \) approaches \( r_{\text{des}} \) when the adaptive controller is used.

Next, we let \( f(\cdot) \) be the backlash/hysteresis non-linearity shown in figure 6. Note that in this case \( u \) and \( f(u) \) do not always have the same sign. We apply controller (14) with (22)–(24) and with backlash/hysteresis with \( h = 1, \ c = -2, \ k = -1, \ b = 1, \ r_{\text{des}} = 0, \ r(0) = -0.4, \) and \( \dot{r}(0) = 0.5 \). For comparison, the stabilizing linear controller \( u(t) = -2x_1(t) - 4x_2(t) \) is designed for the system (44) with \( f(u) = u \). This linear controller is applied to the system (44) with the backlash/hysteresis non-linearity \( f(\cdot) \). Choose adaptation weights \( p = 1, \lambda_1 = 10^4, \lambda_2 = 10^4, \lambda_3 = 10^4 \) and let \( K_1(0) = 0, K_2(0) = 0, \) and \( \phi(0) = 0 \). As can be seen from the solid line in figure 15(b), \( r(t) \) does not approach \( r_{\text{des}} \) when a linear controller is used. However, figure 15(d) shows that \( r(t) \) approaches \( r_{\text{des}} \) when the adaptive controller is used.
Finally, we let \( f(x) = \text{sign}(x)x^2 \). In this case, \( uf(u) > 0 \), and thus \( u \) and \( f(u) \) have the same sign. We apply controller (14) with (22)–(24) and with \( c = 0.1 \), \( k = 5 \), \( b = 1 \), \( r_{\text{des}} = 1 \), \( \dot{r}(0) = 0 \) and \( \ddot{r}(0) = 0 \). Choosing adaptation weights \( p = 1 \), \( \lambda_1 = 1 \), \( \lambda_2 = 1 \), \( \lambda_3 = 5 \) and let \( K_1(0) = 0 \), \( K_2(0) = 0 \) and \( \phi(0) = 0 \), figure 16 shows the response of the adaptive controller. For comparison, figure 16 shows also the response of the same system with \( f(u) = u \). As can be seen from figure 16(a), \( u(t) \) has negative values from time to time when a linear input is used. However, figure 16(c) shows that \( u(t) \) remains positive when the odd quadratic non-linearity is present.

8. Conclusion

In this paper we applied a direct adaptive control law derived for linear systems to non-linear oscillators possessing dynamic and input non-linearities. The adaptive controller was shown to be effective in all cases considered for the problems of adaptive stabilization and command following. Finally, it was shown by Roup and Bernstein (2000) that the controller given by Theorem 1 is guaranteed to stabilize a class of non-linear systems.

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References


