Generalized Riccati equations for the full- and reduced-order mixed-norm $H_2/H_\infty$ standard problem *

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Abstract: This paper considers the mixed-norm $H_2/H_\infty$ standard problem. Specifically, an LQG control design problem involving a constraint on $H_\infty$ disturbance attenuation is addressed. It is shown that the $H_2/H_\infty$ dynamic compensator gains are completely characterized via coupled Riccati/Lyapunov equations. The principal result involves sufficient conditions for characterizing full- and reduced-order controllers that satisfy bounds on both $H_2$ and $H_\infty$ performance costs. As a special case of this unified result we obtain the full-order $H_\infty$ solution to the standard control problem and the pure reduced-order $H_\infty$ solution with no $H_2$ contribution. Further extensions include nonstrictly proper dynamics, a direct transmission term from disturbances to $H_\infty$ performance variables, cross-weighting and sensor noise/plant disturbance correlation, and a treatment of the pure reduced-order $H_\infty$ control problem.

Keywords: $H_2/H_\infty$ design; mixed norm; $H_\infty$ reduced-order controllers.

1. Introduction

In a recent paper [1] a unification of the $H_2$ (LQG) and $H_\infty$ control-design problems was obtained in terms of modified coupled algebraic Riccati equations. Specifically, the results of [1] address a unified solution of the $H_2/H_\infty$ standard problem for full- and reduced-order controllers. This mixed-norm problem thus permits design tradeoffs between $H_2$ performance and $H_\infty$ disturbance rejection.

The goal of the $H_2/H_\infty$ problem is to minimize an $H_2$ performance criterion subject to a prespecified $H_\infty$ constraint on the closed-loop transfer function. The $H_\infty$ constraint is embedded within the optimization process by replacing the closed-loop covariance Lyapunov equation by a Riccati equation whose solution leads to an upper bound on the $H_2$ performance. The key idea to this approach is to view this upper bound as an auxiliary cost and, for a fixed controller structure, seek compensator gains that minimize the $H_2$ bound and guarantee that the disturbance attenuation constraint is enforced. The principal result is a sufficient condition involving coupled modified Riccati equations whose solutions, when they exist, are used to explicitly construct feedback gains for characterizing full- and reduced-order controllers with bounded $H_2$ and $H_\infty$ costs. Note that, strictly speaking, the problem addressed is suboptimal in both the $H_2$ sense and the $H_\infty$ sense. However, solving the design equations for progressively smaller $H_\infty$ disturbance attenuation constraints should, in the limit, yield an $H_\infty$-optimal controller over the class of fixed-structure stabilizing controllers. Although our main result gives sufficient conditions, these conditions will also be necessary as long as the mixed-norm optimization problem possesses at least one extremal over the class of fixed-structure controllers (see Lemma 2.2 and [2]).

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The solution given in [1] however, was restricted to the case in which the plant was strictly proper and there was no direct transmission from disturbances to $H_\infty$ performance variables. The main contribution of the present paper is to extend the results of [1] to remove these restrictions and to allow further generalizations. First, a direct transmission term in the state space plant dynamics is included within the problem formulation along with a direct feedthrough term from exogenous disturbances to $H_2$ performance variables. Next, to allow for greater design flexibility we permit correlated plant and measurement noise. And, finally, we consider the dual design feature of cross weighting in both the $H_2$ and $H_\infty$ performance criteria. These generalizations have been studied in [14] for full-state feedback and in [4,5,11] for dynamic compensation. However, the results of [4,5,11] are limited to the ‘pure’ full-order $H_\infty$ standard problem without the $H_2/H_\infty$ unification. Furthermore, the results given in [4,5,11] are obtained by indirect transformation methods. In the present paper we derive the solution to mixed-norm $H_2/H_\infty$ fixed-order (i.e., full- and reduced-order) dynamic compensation problem without employing such transformations.

It should be noted that the approach developed in [4,5] is quite different from our fixed-structure optimization design approach. Specifically, the authors in [4,5] consider a general $H_\infty$ optimization problem of the form $\|T - UQV\|_\infty$, where $Q$ is a parameterization of all stabilizing controllers that give infinity norm better than $\gamma$. It is shown that the central member of this set minimizes an entropy functional at infinity and yields a set of decoupled Riccati equations that characterize full-order compensators satisfying an $H_\infty$ norm bound [5,8]. Furthermore, the results of [4,5,11] are necessary as well as sufficient. In contrast, the approach of [1] and the present paper is based upon Lagrange multiplier methods which permit the fixed-order-constraints as well as different $H_2$ and $H_\infty$ performance weights.

Finally, as a special case of the results given in the present paper we obtain the full-order $H_2$ solution (LQG), reduced-order $H_2$ solution [6], full-order $H_\infty$ solution [3,4,5,11], and the ‘pure’ reduced-order $H_\infty$ solution with no $H_2$ contribution. It is interesting to note that in the full-order $H_\infty$ controller case with no $H_2$ contribution our results specialize to [3,4,5,11]. Since the results of [3,4,5,11] are necessary as well as sufficient, these connections show that our sufficient conditions (at least in this special case) are also necessary.

**Notation.** Note: All matrices have real entries.

- $\mathbb{R}$, $\mathbb{R}^{r \times s}$, $\mathbb{R}^r$, $\mathbb{E}$: real numbers, $r \times s$ real matrices, $\mathbb{R}^{r \times 1}$, expected value.
- $I_r$, $I_r^T$, $(\cdot)^*$: $r \times r$ identity matrix, transpose, complex conjugate transpose.
- $\rho(\cdot)$: spectral radius.
- $\mathbb{S}^r$, $\mathbb{N}^r$, $\mathbb{P}^r$: $r \times r$ symmetric, nonnegative-definite, positive-definite matrices.
- $X, Y, X_i, \tilde{X}$: $n, m, l, n_c, \tilde{n}$-dimensional vectors.
- $A, B, C, D$: $n \times n$, $n \times m$, $l \times n$, $l \times m$ matrices.
- $A_c, B_c, C_c$: $n_c \times n_c$, $n_c \times l$, $m \times n_c$ matrices.
- $x, \tilde{x}$: $n \times 1$ vector.
- $\tilde{A}$: $n \times n$ matrix.
- $\gamma$: positive constant.
- $E_{\infty}$: $q_{\infty} \times d$ matrix.
- $M$: $I_{q_{\infty}} - \gamma^{-2} E_{\infty} E_{\infty}^T$, $M \in \mathbb{P}_{q_{\infty}}$.
- $N$: $I_{d} - \gamma^{-2} E_{\infty}^T E_{\infty}$, $N \in \mathbb{P}_{d}$.
- $w(\cdot)$: $d$-dimensional standard white noise or $L_2$ signal.
- $D_1, D_2$: $n \times d$, $l \times d$ matrices.
- $V_1, V_2, V_{12}$: $D_1 D_1^T, D_2 D_2^T, D_1 D_2^T, V_2 \in \mathbb{P}_l$.
- $V_{1\infty}, V_{2\infty}$: $D_1 N^{-1} D_1^T, D_2 N^{-1} D_2^T, D_1 N^{-1} D_2^T, V_{2\infty} \in \mathbb{P}_l$.
- $\tilde{D}, \tilde{V}$: $D_1 D_2^T, V_1 V_{12} B_c^T = D D^T$.
- $E_1, E_2$: $q \times n$, $q \times m$ matrices.
- $\tilde{E}$, $\tilde{R}_1, \tilde{R}_2$: $E_1 E_2 C_c, E_1^T E_1, E_2^T E_2, R_2 \in \mathbb{P}_m$.
- $R_{12}, \tilde{R}$: $E_1^T E_2, E_1^T \tilde{E}$.
2. Statement of the problem

In this section we introduce the LQG dynamic output-feedback control problem with constrained $H_\infty$ disturbance attenuation. Without the $H_2$ performance criterion the problem considered here is the standard $H_\infty$ control problem [3,4,5]. For simplicity, the first part of the paper addresses controllers of order $n_c = n$ only, i.e., controllers whose order is equal to the dimension of the plant. This constraint is removed in Section 6 where controllers of reduced order are considered. Hence, throughout Sections 2–5 the controller dimension $n_c$ and closed-loop plant dimension $\tilde{n} \triangleq n + n_c$ should be interpreted as $n$ and $2n$, respectively.

$H_\infty$-Constrained LQG Control Problem. Given the $n$-th-order stabilizable and detectable plant
\[
\dot{x}(t) = Ax(t) + Bu(t) + D_1w(t),
\]
\[
y(t) = Cx(t) + Du(t) + D_2w(t),
\]
determine an $n$-th-order dynamic compensator
\[
\dot{x}(t) = A_cx_c(t) + B_cy(t),
\]
\[
u(t) = C_cx_c(t),
\]
that satisfies the following design criteria:
(i) the closed-loop system (2.1)–(2.4) is asymptotically stable, i.e., $\tilde{A}$ is asymptotically stable;
(ii) the $q_\infty \times p$ nonstrictly proper transfer function
\[
H(s) \triangleq \tilde{E}_\infty(sI_\tilde{n} - \tilde{A})^{-1}\tilde{D} + E_\infty
\]
from $w(t)$ to $z_\infty(t) = E_1oxx(t) + Ez_\inftyU(t) + E_oo w(t)$ satisfies the constraint
\[
\|H(s)\|_\infty \leq \gamma,
\]
where $\gamma > 0$ is a given constant; and
(iii) the performance functional
\[
J(A_c, B_c, C_c) \triangleq \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left\{ \int_0^t \left[ x^T(t)R_1x(t) + 2x^T(t)R_12u(t) + u^T(t)R_2u(t) \right] dt \right\}
\]
is minimized.

Note that the closed-loop system (2.1)–(2.4) can be written as
\[
\dot{x}(t) = \tilde{A}x(t) + \tilde{D}w(t)
\]
and that (2.7) becomes
\[
J(A_c, B_c, C_c) = \lim_{t \to \infty} \mathbb{E} \left\{ [\tilde{E}\tilde{x}(t)]^T [\tilde{E}\tilde{x}(t)] \right\} = \lim_{t \to \infty} \mathbb{E} \left[ \tilde{x}^T(t)\tilde{R}\tilde{x}(t) \right].
\]
Furthermore, by defining the transfer function
\[
\hat{H}(s) \triangleq \tilde{E}(sI_\tilde{n} - \tilde{A})^{-1}\tilde{D},
\]
it can be shown that when $\tilde{A}$ is asymptotically stable, (2.8) is given by

$$J(A_c, B_c, C_c) = \| \tilde{H}(s) \|^2_2.$$  

(2.10)

Note that the problem statement involves both $H_2$ and $H_\infty$ performance weights. In particular, the matrices $R_1$ and $R_2$ are the $H_2$ weights for the state and control variables. By introducing the variables

$$z(t) = E_1x(t), \quad v(t) = E_2u(t),$$

(2.11)

the $H_2$ cost (2.7) can be written as

$$J(A_c, B_c, C_c) = \lim_{t \to \infty} \mathbb{E}[z^T(t)z(t) + 2z^T(t)v(t) + v^T(t)v(t)].$$

(2.12)

For convenience we thus define $R_1 = E_1 E_1^T$ and $R_2 = E_2 E_2^T$ which appear in subsequent expressions. Note that $R_{12} = E_1 E_2^T$ is an $H_2$ cross-weighting term which is included for greater design flexibility.

For the $H_\infty$ performance constraint, the transfer function (2.5) involves weighting matrices $E_{1\infty}$, $E_{2\infty}$, and $E_{\infty}$ for the state, control, and disturbance variables. The matrices $R_{1\infty} = E_{1\infty} E_{1\infty}^T$ and $R_{2\infty} = E_{2\infty} E_{2\infty}^T$ are thus the $H_\infty$ counterparts of the $H_2$ weights $R_1$ and $R_2$. Here $M = I_q - \gamma^{-2} E_{\infty} E_{\infty}^T$ arises due to the feedthrough term to the $H_\infty$ performance variables. Although we do not require that $R_{1\infty}$ and $R_{2\infty}$ be equal to $R_1$ and $R_2$, we shall assume for simplicity that $R_2 = \alpha^2 \tilde{R}_2$ and $R_{2\infty} = \beta^2 \tilde{R}_2$, where the nonnegative scalars $\alpha$, $\beta$ are design variables such that $\alpha^2 + \beta^2 \neq 0$. As in the $H_2$ case we allow an $H_\infty$ cross-weighting term $R_{12\infty} = E_{1\infty} M^{-1} E_{2\infty}$. Finally, the dual design feature of plant disturbance and sensor noise correlation is also permitted. As in [1], $w(t)$ is interpreted as white noise for the $H_2$ design aspect and as an $L_2$ signal for the $H_\infty$ design aspect. Note that without the $H_2$ performance criterion, i.e., $R_1 = 0$ and $\alpha = 0$, the problem considered here reduces to the 'pure' $H_\infty$ standard problem (see Figure 1).

Before continuing, it is useful to note that if $\tilde{A}$ is asymptotically stable for a given compensator $(A_c, B_c, C_c)$ then the $H_2$ performance (2.8) is given by

$$J(A_c, B_c, C_c) = \text{tr} \tilde{Q} \tilde{R},$$

(2.13)

where the steady-state closed-loop state covariance defined by

$$\tilde{Q} = \lim_{t \to \infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^T(t)]$$

(2.14)

satisfies the $\tilde{n} \times \tilde{n}$ algebraic Lyapunov equation

$$0 = A \tilde{Q} + \tilde{Q} A^T + \tilde{V}.$$  

(2.15)

The key step in enforcing the disturbance attenuation constraint (2.6) is to replace the algebraic Lyapunov equation (2.15) by an algebraic Riccati equation that overbounds the closed-loop steady-state covariance. Justification for this technique is provided by the following result.
Lemma 2.1. Let \((A, B, C, \mathcal{Q})\) be given and assume there exists \(\mathcal{Q} \in \mathbb{R}^{n \times n}\) satisfying
\[\mathcal{Q} \in \mathbb{C}^n\]
and
\[0 = \tilde{A} \mathcal{Q} + \mathcal{Q} \tilde{A}^\top + \gamma^{-2}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top) \mathcal{M}^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top)^\top + \tilde{V}.\]  
(2.17)

Then
\[(\tilde{A}, \tilde{D})\text{ is stabilizable}\]
(2.18)
if and only if
\[\tilde{A} \text{ is asymptotically stable.}\]
(2.19)
In this case,
\[\|H(s)\|_\infty \leq \gamma\]
(2.20)
and
\[\bar{Q} \leq \mathcal{Q}.\]
(2.21)
Consequently,
\[J(A, B, C) \leq J(A, B, C, \mathcal{Q}),\]
(2.22)
where
\[J(A, B, C, \mathcal{Q}) \triangleq \text{tr } \mathcal{Q} \tilde{R}.\]
(2.23)

Proof. If follows from [13, Theorem 3.6] that (2.18) implies that
\[\left(\tilde{A}, \left[\gamma^{-2}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top) \mathcal{M}^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top)^\top + \tilde{V}\right]^{1/2}\right)\]
is also stabilizable. Using the assumed existence of a nonnegative-definite solution to (2.17) and [13, Lemma 12.2] it now follows that \(\tilde{A}\) is asymptotically stable. The converse is immediate. To prove (2.20), replace \(\tilde{V}\) by \(\tilde{D} \tilde{D}^\top\) and add and subtract \(\text{tr } \mathcal{Q} \mathcal{I}\) to (2.17) so that (2.17) becomes
\[0 = (-j\omega I + \tilde{A}) \mathcal{Q} + \mathcal{Q} (-j\omega I + \tilde{A})^\top + \gamma^{-2}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top) \mathcal{M}^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top)^\top + \tilde{D} \tilde{D}^\top\]
or, equivalently,
\[\tilde{D} \tilde{D}^\top = (j\omega I - \tilde{A}) \mathcal{Q} + \mathcal{Q} (-j\omega I - \tilde{A})^\top - \gamma^{-2}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top) \mathcal{M}^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top)^\top.\]
(2.25)
Next, forming
\[\tilde{E}(j\omega I - \tilde{A})^{-1}(2.25)(-j\omega I - \tilde{A})^{-\top} \tilde{E}^\top\]
yields
\[\tilde{E}(j\omega I - \tilde{A})^{-1} \tilde{D} \tilde{D}^\top (j\omega I - \tilde{A})^{-\top} \tilde{E}^\top\]
\[= \tilde{E}(j\omega I - \tilde{A})^{-1} \tilde{E}^\top + \tilde{E}(j\omega I - \tilde{A})^{-\top} \tilde{E}^\top\]
\[= \gamma^{-2} \tilde{E}(j\omega I - \tilde{A})^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top) \mathcal{M}^{-1}(\tilde{D} \mathcal{E}^\top + 2 \mathcal{E} \mathcal{E}^\top)^\top (j\omega I - \tilde{A})^{-\top} \tilde{E}^\top.\]  
(2.26)
Now adding $\tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \tilde{D} \tilde{E}_\infty^T + E_\infty \tilde{D}^T (-j\omega I_n - \tilde{A})^{-T} + E_\infty E_\infty^T$ to both sides of (2.26) yields

$$
\tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \tilde{D} \tilde{D}^T (-j\omega I_n - \tilde{A})^{-T} \tilde{E}_\infty^T + \tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \tilde{D} \tilde{E}_\infty^T + E_\infty \tilde{D}^T (-j\omega I_n - \tilde{A})^{-T} + E_\infty E_\infty^T
$$
$$
= \tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \left[ \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right] + \left[ \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right] (-j\omega I_n - \tilde{A})^{-T} + E_\infty E_\infty^T
$$
$$
+ \gamma^{-2} \tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \left[ (\tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T) M^{-1} (\tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T) \right] (-j\omega I_n - \tilde{A})^{-T} \tilde{E}_\infty^T.
$$

(2.27)

Note that the left hand side of (2.27) is equal $H(j\omega)H^*(j\omega)$ and the right hand side of (2.27) can be written as

$$
S + S^* - \gamma^{-2} SM^{-1} S^* + \gamma^2 (I_{q_s} - M)
$$

(2.28)

where

$$
S \triangleq \tilde{E}_\infty (j\omega I_n - \tilde{A})^{-1} \left[ \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right]
$$

and $E_\infty E_\infty^T$ is replaced by $\gamma^2 (I_{q_s} - M)$. Hence, it follows from (2.27) and (2.28) that

$$
H(j\omega)H^*(j\omega) = -\left[ \left( \gamma M^{1/2} - \gamma^{-1} SM^{1/2} \right) \left( \gamma M^{1/2} - \gamma^{-1} SM^{1/2} \right)^* \right] + \gamma^2 I_{q_s} \geq 0,
$$

(2.29)

which implies $H(j\omega)H^*(j\omega) \leq \gamma^2 I_{q_s}$. This proves (2.20). To prove (2.21), subtract (2.15) from (2.17) to obtain

$$
0 = \vec{A} (2 - \vec{Q}) + (2 - \vec{Q}) \vec{A}^T + \gamma^{-2} \left( \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right) M^{-1} \left( \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right)^T
$$

(2.30)

which, since $\vec{A}$ is asymptotically stable, is equivalent to

$$
2 - \vec{Q} = \int_0^\infty e^{\vec{A} t} \left[ \gamma^{-2} \left( \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right) M^{-1} \left( \tilde{D} \tilde{E}_\infty^T + 2 \tilde{E}_\infty^T \right)^T \right] e^{\vec{A}^T t} dt \geq 0.
$$

Finally, (2.22) follows immediately from (2.21). □

**Remark 2.1.** An equivalent form of (2.17) is given by

$$
0 = \left( \vec{A} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \right) \vec{Q} + \gamma^{-2} \left( \vec{A} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \right)^T \vec{Q} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \vec{Q} + \tilde{D} N^{-1} \tilde{D}^T.
$$

(2.31)

The equivalence of (2.17) and (2.31) is easily shown by noting that (2.17) can be rewritten as

$$
0 = \left( \vec{A} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \right) \vec{Q} + \gamma^{-2} \left( \vec{A} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \right)^T \vec{Q} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty \vec{Q} + \tilde{D} N^{-1} \tilde{D}^T
$$

(2.32)

and noting that $\tilde{D} \left[ \gamma^{-2} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty + I_d \right] \tilde{D}^T$ is equal to $\tilde{D} N^{-1} \tilde{D}^T$ since $E_\infty^T M^{-1} = N^{-1} E_\infty^T$ and $N^{-1} (\gamma^{-2} E_\infty^T E_\infty + N) = N^{-1}$.

Lemma 2.1 shows that $H_\infty$ disturbance attenuation is automatically enforced when a nonnegative-definite solution to (2.17) is known to exist and $\vec{A}$ is asymptotically stable. Furthermore, all such solutions provide upper bounds for the actual closed-loop state covariance $\vec{Q}$ along with a bound on the $H_2$ performance criterion. Next, we present a partial converse of Lemma 2.1 that guarantees the existence of a unique minimal nonnegative-definite solution to (2.17) when (2.20) is satisfied. The minimal solution is desirable since it yields the tightest performance bound in (2.22). This was first pointed out in [7].

**Lemma 2.2.** Let $(A_c, B_c, C_c)$ be given, suppose $\vec{A}$ is asymptotically stable, and assume the disturbance attenuation constraint (2.20) is satisfied. Then there exists a unique nonnegative-definite solution $\vec{Q}$ satisfying (2.17) and such that the eigenvalues of $\vec{A} + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty + \gamma^{-2} \tilde{D} \tilde{E}_\infty^T M^{-1} \tilde{E}_\infty$ lie in the closed left half plane. Furthermore, this solution is also minimal.
Proof. The result is an immediate extension of [2, pp. 150 and 167], using Theorems 3 and 2. The proof of minimality of given in [12]. □

3. The Auxiliary Minimization Problem

As shown in the previous section, replacing (2.15) by (2.17) enforces the $H_\infty$ disturbance attenuation constraint and yields an upper bound for the $H_2$ performance criterion. That is, given a compensator $(A_c, B_c, C_c)$ for which there exists a nonnegative-definite solution to (2.17), the actual $H_2$ performance $J(A_c, B_c, C_c)$ of the compensator is guaranteed to be no worse than the bound given by $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$. Hence, $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ can be interpreted as an auxiliary cost which leads to the following optimization problem.

Auxiliary Minimization Problem. Determine $(A_c, B_c, C_c, \mathcal{Q})$ which minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$ subject to (2.17) with $\mathcal{Q} \in \mathbb{N}^n$.

It follows from Lemma 2.1 that the satisfaction of (2.16) and (2.17) along with the generic condition (2.18) lead to: (1) closed-loop stability; (2) prespecified $H_\infty$ performance attenuation; (3) an upper bound for the $H_2$ performance criterion. Hence, it remains to determine $(A_c, B_c, C_c, \mathcal{Q})$ that minimizes $\mathcal{J}(A_c, B_c, C_c, \mathcal{Q})$, and thus provides an optimized bound for the actual $H_2$ performance $J(A_c, B_c, C_c)$.

4. Sufficient conditions for $H_\infty$ disturbance attenuation

In this section we state sufficient conditions for characterizing full-order controllers guaranteeing closed-loop stability, constrained $H_\infty$ disturbance attenuation, and an optimized $H_2$ performance bound. For arbitrary $Q$, $P$, $\hat{Q} \in \mathbb{R}^{n \times n}$ and $\alpha, \beta \geq 0$ define the notation

$$Q_a \triangleq QC^T + V_{120}, \quad P_a \triangleq \begin{bmatrix} B^T + \gamma^{-2} R_{020}^T D_1^T + \gamma^{-2} R_{120}^T (Q + \hat{Q}) \end{bmatrix} P + R_{12}^T,$$

$$S \triangleq (\alpha^2 I_n + \beta^2 \gamma^{-2} \hat{Q} P)^{-1}$$

when the indicated inverse exists.

Theorem 4.1. Suppose there exist $Q$, $P$, $\hat{Q} \in \mathbb{N}^n$ satisfying

$$0 = (A + \gamma^{-2} D_1 R_{010}^T) Q + Q (A + \gamma^{-2} D_1 R_{010}^T)^T + \gamma^{-2} QR_{120}^T Q + V_{120} - Q_a V_{202}^T Q_a^T,$$  \hspace{1cm} (4.1)

$$0 = (A + \gamma^{-2} \begin{bmatrix} Q + \hat{Q} \end{bmatrix} R_{120} + \gamma^{-2} D_1 R_{010}^T - \gamma^{-2} \hat{Q} S \hat{Q}^T P_a \hat{R}_{2}^{-1} R_{120}^T) P$$

$$+ P (A + \gamma^{-2} \begin{bmatrix} Q + \hat{Q} \end{bmatrix} R_{120} + \gamma^{-2} D_1 R_{010}^T - \gamma^{-2} \hat{Q} S \hat{Q}^T P_a \hat{R}_{2}^{-1} R_{120}^T) + R_{12} - S \hat{Q}^T \hat{R}_{2}^{-1} P_a S,$$  \hspace{1cm} (4.2)

$$0 = (A - B \hat{R}_{2}^{-1} P_a S + \gamma^{-2} Q \begin{bmatrix} R_{100} - R_{120} \hat{R}_{2}^{-1} P_a S \end{bmatrix} + \gamma^{-2} \begin{bmatrix} D_1 R_{010} - D_1 R_{020} \hat{R}_{2}^{-1} P_a S \end{bmatrix} \hat{Q}$$

$$+ \hat{Q} \begin{bmatrix} A - B \hat{R}_{2}^{-1} P_a S + \gamma^{-2} Q \begin{bmatrix} R_{100} - R_{120} \hat{R}_{2}^{-1} P_a S \end{bmatrix} + \gamma^{-2} \begin{bmatrix} D_1 R_{010} - D_1 R_{020} \hat{R}_{2}^{-1} P_a S \end{bmatrix} \hat{Q} \end{bmatrix} \hat{Q} + Q_a V_{202}^T Q_a,$$  \hspace{1cm} (4.3)

and let $(A_c, B_c, C_c, \mathcal{Q})$ be given by

$$A_c = A - B \hat{R}_{2}^{-1} P_a S - Q_a V_{202}^T C - Q_a V_{202}^T D \hat{R}_{2}^{-1} P_a S$$

$$+ \gamma^{-2} (Q R_{100} + D_1 R_{010} - D_1 R_{020} \hat{R}_{2}^{-1} P_a S - Q R_{120} \hat{R}_{2}^{-1} P_a S$$

$$- Q_a V_{202}^T D_2 R_{010} + Q_a V_{202}^T D_2 R_{020} \hat{R}_{2}^{-1} P_a S),$$  \hspace{1cm} (4.4)
Then \((\tilde{A}, \tilde{D})\) is stabilizable if and only if \(\tilde{A}\) is asymptotically stable. In this case, the closed-loop transfer function \(H(s)\) satisfies the \(H_\infty\) disturbance attenuation constraint (2.20) and the \(H_2\) performance criterion (2.7) satisfies the bound

\[
J(A_c, B_c, C_c) \leq \text{tr}\left[ (Q + \hat{Q}) R_1 - 2R_{12} \hat{R}_2^{-1} P_a S \hat{Q} + S^T P_a \hat{R}_2^{-1} R_2 \hat{R}_2^{-1} P_a S \hat{Q} \right].
\]  

Proof. The proof follows as in the proof given in [1]. 

Remark 4.1. Theorem 4.1 presents sufficient conditions for designing controllers with a prespecified \(H_\infty\) constraint on the closed-loop transfer function. These sufficient conditions comprise a system of three modified algebraic Riccati equations in variables \(Q\), \(P\), and \(\hat{Q}\). The \(Q\) and \(P\) equations are similar to the estimator and regulator Riccati equations of LQG theory, while the \(\hat{Q}\) equation has no counterpart in the standard theory. Note that the \(Q\) equation is decoupled from the \(P\) and \(\hat{Q}\) equations and thus can be solved independently. The \(P\) equation, however depends on \(Q\). Thus, regulator/estimator separation holds in only one direction which clearly shows that the certainty equivalence principle is no longer valid for the mixed \(H_2/H_\infty\) design problem. Finally, note that if the \(H_\infty\) disturbance attenuation constraint is sufficiently relaxed, i.e., \(\gamma \to \infty\), then the \(P\) equation becomes decoupled from the \(\hat{Q}\) equation and thus the \(\hat{Q}\) equation becomes superfluous. Furthermore, the remaining \(Q\) and \(P\) equations separate and coincide with the standard LQG result. Alternatively, note that if both \(\beta = 0\) and \(R_{1\infty} = 0\), then Theorem 4.1 also specializes to the standard LQG result.

Remark 4.2. The results of [1] are a special case of Theorem 4.1. To see this set the plant/measurement noise correlation to zero \((V_{12} = 0)\), set both the \(H_2\) and \(H_\infty\) cross weighting terms to zero \((R_{12}, R_{12\infty} = 0)\), set the direct transmission term in the plant dynamics to zero \((D = 0)\) and set the feedthrough term from disturbances to \(H_\infty\) performance variables to zero \((E_\infty = 0)\). This yields Theorem 3.1 of [1].

Remark 4.3. When solving (4.1)–(4.3) numerically, the \(H_\infty\) constraint can be adjusted to examine tradeoffs between \(H_2\) performance and disturbance rejection. Specifically, \(\gamma\) can be varied systematically to determine the region of solvability of the design equations (4.1)–(4.3) and to study tradeoffs between the \(H_2/H_\infty\) performance criteria (see [1]).

5. The pure \(H_\infty\) standard problem

As shown in Theorem 4.1, the Riccati equations (4.1)–(4.3) provide sufficient conditions for explicitly synthesizing controllers \((A_c, B_c, C_c)\) satisfying both \(H_2\) and \(H_\infty\) performance bounds. The main purpose of this section is to completely eliminate the \(H_2\) aspect in the design problem. This section also provides connections between our approach and the recent results obtained in [3,4,6]. In [1] it was shown that by equalizing the \(H_2/H_\infty\) weights the three coupled Riccati equation form could be transformed into two decoupled Riccati equations as in [3,7]. Furthermore, it was shown in [7] that the auxiliary cost (2.23) is equivalent to an entropy integral. However, it is important to note that, as noticed in Remark 2.1, the results of [7] cannot consider a general direct transmission term from disturbances to \(H_\infty\) performance variables in order to guarantee that the minimum value of the entropy evaluated at infinity is finite. In the present paper we utilize a simpler approach wherein we eliminate the \(H_2\) contribution by letting \(R_1, R_{12}, \alpha\) (and thus \(R_2\)) approach zero. By eliminating the \(H_2\) contribution to the problem, the resulting setting
corresponds to the $H_\infty$ standard problem. In order to state the main result we require some additional notation. For arbitrary $Y_\infty \in \mathbb{R}^{n \times n}$ define the notation

$$ Y_\infty \triangleq B^T Y_\infty + \gamma^{-2} R_{02\infty}^T D_1^T Y_\infty + R_{12\infty}^T. $$

**Theorem 5.1.** Suppose there exist $Q \in \mathbb{R}^n$ and $Y_\infty \in \mathbb{R}^n$ satisfying

$$ 0 = (A + \gamma^{-2} D_1 R_{01\infty})^T + V_{1\infty} + \gamma^{-2} Q R_{1\infty} Q - Q_a V_{2\infty}^{-1} Q_a^T, $$

$$ 0 = (A + \gamma^{-2} D_1 R_{01\infty})^T Y_\infty + Y_\infty (A + \gamma^{-2} D_1 R_{01\infty}) + R_{1\infty} + \gamma^{-2} Y_\infty Y_{1\infty} - Y_{ao} R_{2\infty} Y_{ao}, $$

$$ \rho(Q Y_\infty) < \gamma^2, $$

and let $(A_0, B_0, C_0)$ be given by

$$ A_0 = A - B R_{2\infty}^{-1} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1} - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D R_{2\infty} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1} $$

$$ + \gamma^{-2} \left( Q R_{1\infty} + D_1 R_{01\infty} - D_1 R_{02\infty} R_{2\infty}^{-1} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1} \right) $$

$$ - Q R_{12\infty} R_{2\infty}^{-1} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1} - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} $$

$$ + Q_a V_{2\infty}^{-1} D_2 R_{02\infty} R_{2\infty}^{-1} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1}; $$

$$ B_0 = Q_a V_{2\infty}^{-1}, \quad C_0 = -R_{2\infty} Y_{ao} \left( I_n - \gamma^{-2} Q Y_\infty \right)^{-1}. $$

Then $(\hat{A}, \hat{D})$ is stabilizable if and only if $\hat{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the $H_\infty$ disturbance attenuation constraint (2.20).

**Proof.** First let $R_1, R_{12}, \alpha \to 0$ in equations (4.1)-(4.3) so that $S \to \beta^{-2} \gamma^2 P^{-1} \hat{Q}^{-1}$. Next, note that $P_a S = \beta^{-2} \gamma^2 \Sigma \hat{Q}^{-1},$ where

$$ \Sigma \triangleq B^T + \gamma^{-2} R_{02\infty}^T D_1^T + \gamma^{-2} R_{12\infty}^T (Q + \hat{Q}). $$

Now define $Z_\infty \triangleq \gamma^2 \hat{Q}^{-1}$ and substitute into (4.3) to obtain

$$ 0 = (A + \gamma^{-2} Q R_{1\infty} + \gamma^{-2} D_1 R_{01\infty})^T Z_\infty + Z_\infty (A + \gamma^{-2} Q R_{1\infty} + \gamma^{-2} D_1 R_{01\infty}) + R_{1\infty} $$

$$ + \gamma^2 Z_{ao} Q_a V_{2\infty}^{-1} Q_a Z_{ao} - Z_{ao} R_{2\infty}^{-1} Z_{ao}, $$

where $Z_{ao} \triangleq \Sigma Z_{ao}.$ Now note that (5.2) follows by forming $Y_\infty \triangleq (Z_{ao}^{-1} + \gamma^2 Q)^{-1}.$ The gain expressions (5.4)-(5.6) follow as a direct consequence. $\square$

**Remark 5.1.** The solutions $Q$ and $Y_\infty$ of (5.1) and (5.2) are analogous to the matrices $S$ and $P$ of [5] and $Y_\infty$ and $X_\infty$ of [4], while (5.3) corresponds to condition 5.2 (iii) of [4].

**Remark 5.2.** By setting $R_{12\infty}, E_{ao},$ and $D$ to zero, the results of Theorem 5.1 specialize to Theorem 6 of [3] and Proposition 5.7 of [1] without the $H_2$ performance bound.

### 6. Mixed-norm reduced-order dynamic compensation

In this section we extend Theorem 4.1 by expanding the formulation of Sections 2 and 3 to allow the compensator to be of fixed dimension $n_c$, which may be less than the plant order $n$. Hence, in this section define $\tilde{n} = n + n_c$, where $n_c \leq n$. As in [1,6] this additional constraint leads to an oblique projection that introduces additional coupling in the design equations along with an additional equation. The following lemma is required for the statement of the main theorem (see [1].)
Lemma 6.1. Let $\hat{Q}, \hat{P} \in \mathbb{N}^n$ and suppose rank $\hat{Q} \hat{P} = n_c$. Then there exist $n_c \times n_c$ $G, \Gamma$, and $n_c \times n_c$ invertible $M$, unique except for a change of basis in $\mathbb{R}^{n_c}$, such that
\[
\hat{Q} \hat{P} = G^T \Gamma M G, \quad \Gamma G^T = I_{n_c}. \tag{6.1}, (6.2)
\]
Furthermore, the $n \times n$ matrices
\[
\tau = G^T \Gamma, \quad \tau_\perp = I_n - \tau \tag{6.3}, (6.4)
\]
are idempotent and have rank $n_c$ and $n - n_c$.

Theorem 6.1. Let $n_c \leq n$, suppose there exist $Q, P, \hat{Q}, \hat{P} \in \mathbb{N}^n$ satisfying
\[
0 = (A + \gamma^{-2}D_1 R_{01\infty})Q + Q(A + \gamma^{-2}D_1 \hat{R}_{01\infty})^T + \gamma^{-2}Q \hat{R}_{1\infty} \hat{Q}
+ V_{1\infty} - Q_a V_{2\infty}^{-1} \hat{Q}_a^T \tau_\perp Q_a V_{2\infty}^{-1} Q_a^T \tau_\perp, \tag{6.5}
\]
\[
0 = (A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2}D_1 R_{01\infty} - \gamma^{-2}\hat{Q} \hat{S}^T P_a \hat{R}_{1\infty}^{-1} R_{1\infty}^{-1})^T P
+ P(A + \gamma^{-2}[Q + \hat{Q}] R_{1\infty} + \gamma^{-2}D_1 R_{01\infty} - \gamma^{-2}\hat{Q} \hat{S}^T P_a \hat{R}_{1\infty}^{-1} R_{1\infty}^{-1})
+ R_1 - S^T P_a \hat{R}_{1\infty}^{-1} P_a S + \tau_\perp S^T P_a \hat{R}_{1\infty}^{-1} P_a S \tau_\perp, \tag{6.6}
\]
\[
0 = (A - B \hat{R}_{1\infty}^{-1} P_a S + \gamma^{-2}Q[R_{1\infty} - R_{12\infty} \hat{R}_{1\infty}^{-1} P_a S] + \gamma^{-2}D_1 [R_{01\infty} - R_{02\infty} \hat{R}_{1\infty}^{-1} P_a S]) \hat{Q}
+ \hat{Q} \hat{R}_1 - B \hat{R}_{1\infty}^{-1} P_a S + \gamma^{-2}Q[R_{1\infty} - R_{12\infty} \hat{R}_{1\infty}^{-1} P_a S]
+ \gamma^{-2}\hat{Q} \hat{R}_1 - B \hat{R}_{1\infty}^{-1} P_a S + \gamma^{-2}Q \hat{R}_{1\infty}^{-1} P_a S
+ Q_a V_{2\infty}^{-1} Q_a^T \tau_\perp Q_a V_{2\infty}^{-1} Q_a^T \tau_\perp, \tag{6.7}
\]
\[
0 = (A - Q_a V_{2\infty}^{-1} C + \gamma^{-2}D_1 R_{01\infty} + \gamma^{-2}Q \hat{R}_{1\infty} - \gamma^{-2}Q_a V_{2\infty}^{-1} D_2 R_{01\infty}) \hat{P}
+ \hat{P} \hat{R}_1 - B \hat{R}_{1\infty}^{-1} P_a S + \gamma^{-2}Q[R_{1\infty} - R_{12\infty} \hat{R}_{1\infty}^{-1} P_a S]
+ S^T P_a \hat{R}_{1\infty}^{-1} P_a S - \tau_\perp S^T P_a \hat{R}_{1\infty}^{-1} P_a S \tau_\perp, \tag{6.8}
\]
\[
\text{rank } \hat{Q} = \text{rank } \hat{P} = \text{rank } \hat{Q} \hat{P} = n_c, \tag{6.9}
\]
and let $(A_c, B_c, C_c, \mathcal{Z})$ be given by
\[
A_c = \Gamma \left[ A - B \hat{R}_{1\infty}^{-1} P_a S - Q_a V_{2\infty}^{-1} C + Q_a V_{2\infty}^{-1} D \hat{R}_{1\infty}^{-1} P_a S + \gamma^{-2}(Q R_{1\infty} + D_1 R_{01\infty})
- D_1 R_{02\infty} \hat{R}_{1\infty}^{-1} P_a S - Q R_{1\infty} \hat{R}_{1\infty}^{-1} P_a S - Q_a V_{2\infty}^{-1} D_2 R_{01\infty} + Q_a V_{2\infty}^{-1} D_2 R_{02\infty} \hat{R}_{1\infty}^{-1} P_a S \right] G^T, \tag{6.10}
\]
\[
B_c = \Gamma Q_a V_{2\infty}^{-1}, \quad C_c = - \hat{R}_{1\infty}^{-1} P_a S G^T, \tag{6.11}, (6.12)
\]
\[
\mathcal{Z} = \begin{bmatrix} Q + \hat{Q} & \hat{Q} \hat{I}^T \\ \hat{Q} \hat{I} & \hat{Q} \hat{I} \hat{Q} \hat{I}^T \end{bmatrix}. \tag{6.13}
\]
Then $(\tilde{A}, \tilde{D})$ is stabilizable if and only if $\tilde{A}$ is asymptotically stable. In this case, the closed-loop transfer function $H(s)$ satisfies the $H_{\infty}$ disturbance attenuation constraint (2.20) and the $H_2$ performance criterion (2.7) satisfies the bound
\[
J(A_c, B_c, C_c) \leq \text{tr} \left[ (Q + \hat{Q}) R_1 - 2 R_{12} \hat{R}_{1\infty}^{-1} P_a S \hat{Q} + S^T P_a \hat{R}_{1\infty}^{-1} P_a S \hat{Q} \right]. \tag{6.14}
\]

Proof. The proof follows as in [1] with the additional terms arising due to cross weighting, disturbance/measurement noise correlation, and direct feedthrough terms. □
Remark 6.1. It is easy to see that Theorem 6.1 is a direct generalization of Theorem 4.1. To recover Theorem 4.1, set $n_e = n$ so that $\tau = G = \Gamma = I_n$ and $\tau_\perp = 0$. In this case the last term in each of (6.5)–(6.8) can be deleted and (6.8) becomes superfluous. Furthermore, (6.5)–(6.7) now reduce to (4.1)–(4.3), as expected. Alternatively, setting $\gamma = \infty$ and retaining the reduced-order constraint $n_e < n$ yields the result of [6]. Finally, to recover Theorem 6.1 of [1] set $V_{12} = 0$, $R_{12} = 0$, $R_{12\infty} = 0$, $D = 0$, and $E_{\infty} = 0$.

Remark 6.2. As was noted earlier, the assumption that $R_2 = \alpha^2 \tilde{R}_2$ and $R_{2\infty} = \beta^2 \tilde{R}_2$ was made for simplicity. If it is desired that $R_2$ and $R_{2\infty}$ be independent then (6.12) is given by

$$C_e = -\text{vec}^{-1}\left[\Omega\text{vec}(P_2G^T)\right],$$

where

$$\Omega \triangleq R_2 \otimes I_{n_e} + \gamma^{-2}R_{2\infty} \otimes \Gamma \tilde{Q}PG^T,$$

'vec' is the column stacking operation, and $\otimes$ denotes Kronecker product. In this case, the compensator dynamics (6.10) along with the design equations (6.5)–(6.8) have to be changed accordingly. However, due to lack of space this result is not given. Similar remarks apply to the full-order mixed-norm problem given by Theorem 4.1.

7. The pure $H_\infty$ reduced-order dynamic compensation problem

In this section we eliminate the $H_2$ aspect of the reduced-order design problem to obtain reduced-order controllers for the pure $H_\infty$ standard problem. As in the full-order controller case (Section 5) we eliminate the $H_2$ contribution by letting $R_1$, $R_{12}$, $\alpha$ (and thus $R_2$) approach zero. In order to state the main result we require some additional notation. For arbitrary $\tilde{Q}$, $\tilde{P}, \tilde{Q}, P \in \mathbb{N}^n$, and $G, \Gamma \in \mathbb{R}^{n \times n}$ define

$$P_{\infty} \triangleq B^TP + \gamma^{-2}R_{02\infty}D_{1\infty}^TP + \gamma^{-2}R_{12\infty}^T(Q + \tilde{Q})P,$$

$$M_\infty \triangleq (\Gamma \tilde{Q}G^T)^{-1}, \quad N_\infty \triangleq (GPG^T)^{-1},$$

$$S_\infty \triangleq \gamma^2N_\infty M_\infty, \quad W_\infty \triangleq \gamma^4\Gamma^T S_\infty \gamma \Gamma^T S_\infty \gamma R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma.$$

Theorem 7.1. Suppose there exist $Q, P, \tilde{Q}, \tilde{P} \in \mathbb{N}^n$ satisfying (6.9), $GPG^T > 0$, and

$$0 = (A + \gamma^{-2}D_1 R_{01\infty})Q + Q(A + \gamma^{-2}D_1 R_{01\infty})^T + V_{1\infty} + \gamma^{-2}QR_{1\infty}Q - QaV_{2\infty}^{-1} Qa^T + \tau_1 QaV_{2\infty}^{-1} Qa^-1 T_a,$$

$$0 = (A + \gamma^{-2}[Q + \tilde{Q}] R_{1\infty} + \gamma^{-2}D_1 R_{01\infty} - \gamma^{-2}\tilde{Q} \Gamma^T S_\infty \gamma GPG^T P_{\infty} R_{2\infty}^{-1} R_{12\infty} + \gamma^{-2}\tilde{Q} W_\infty)^TP + P(A + \gamma^{-2}[Q + \tilde{Q}] R_{1\infty} + \gamma^{-2}D_1 R_{01\infty} - \gamma^{-2}\tilde{Q} \Gamma^T S_\infty \gamma GPG^T P_{\infty} R_{2\infty}^{-1} R_{12\infty} + \gamma^{-2}\tilde{Q} W_\infty) + R_{1\infty} - P_{\infty} R_{2\infty}^{-1} P_{\infty} + (I_n - G^T S_\infty \gamma)^TP_{\infty} R_{2\infty}^{-1} P_{\infty}(I_n - G^T S_\infty \gamma),$$

$$0 = (A - BR_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma + \gamma^{-2}Q[R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma]\gamma^{-2}[D_1 R_{01\infty} - R_{02\infty} R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma])\tilde{Q} + \tilde{Q}(A - BR_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma + \gamma^{-2}Q[R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma])\gamma^{-2}[D_1 R_{01\infty} - R_{02\infty} R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma] + \gamma^{-2}\tilde{Q}(R_{1\infty} - R_{12\infty} R_{2\infty}^{-1} P_{\infty} G^T S_\infty \gamma - \Gamma^T S_\infty \gamma P_{\infty} R_{2\infty}^{-1} R_{12\infty} + W_\infty)\tilde{Q} + QaV_{2\infty}^{-1} Qa^T - \tau_\perp Q_{a}V_{2\infty}^{-1} Q_{a}^{-1} T_a,$$

(7.8)
\[ 0 = \left( A - Q_2 V^{-1}_2 C + \gamma^{-2} D_1 R_{01} - \gamma^{-2} Q R_{10} - \gamma^{-2} Q_2 V^{-1}_2 D_2 R_{01} \right) \hat{\mathbf{p}} + \hat{\mathbf{p}} \left( A - Q_2 V^{-1}_2 C + \gamma^{-2} D_1 R_{01} - \gamma^{-2} Q R_{10} - \gamma^{-2} Q_2 V^{-1}_2 D_2 R_{01} \right) + P^T_{\infty} R^{-1}_{\infty} P_{\infty} (I_n - G^T S_\infty \Gamma) P^T_{\infty} R^{-1}_{\infty} P_{\infty} (I_n - G^T S_\infty \Gamma) - \gamma^{-2} \left( W_\infty \hat{\mathbf{Q}} + P \hat{\mathbf{Q}} W_\infty \right), \] (7.9)

and let \((A_c, B_c, C_c)\) be given by

\[ A_c = \Gamma \left[ A - R^{-1}_{2 \infty} P_{\infty} G^T S_\infty \Gamma - Q_2 V^{-1}_2 C + Q_2 V^{-1}_2 D R^{-1}_{2 \infty} P_{\infty} G^T S_\infty \Gamma \right] + \gamma^{-2} \left( Q R_{10} + D_1 R_{01} - D_1 R_{02} R^{-1}_{2 \infty} P_{\infty} G^T S_\infty \Gamma - Q R_{12} R^{-1}_{2 \infty} P_{\infty} G^T S_\infty \Gamma \right) - Q_2 V^{-1}_2 D_2 R_{01} + Q_2 V^{-1}_2 D_2 R_{02} R^{-1}_{2 \infty} P_{\infty} G^T S_\infty \Gamma \right] \Gamma^T, \] (7.10)

\[ B_c = \Gamma Q_2 V^{-1}_2, \quad C_c = -R^{-1}_{2 \infty} P_{\infty} G^T S_\infty. \] (7.11), (7.12)

Then \((\hat{A}, \hat{D})\) is stabilizable if and only if \(\hat{A}\) is asymptotically stable. In this case, the closed-loop transfer function \(H(s)\) satisfies the \(H_\infty\) disturbance attenuation constraint (2.20).

\textbf{Proof.} The proof follows from Theorem 6.1 by using the relation \(G^T \hat{S} \Gamma = S \tau\), where \(\hat{S} = (\alpha^2 I_n + \gamma^{-2} \beta^2 \Gamma \hat{Q} \hat{P} G^T)^{-1}\) and letting \(R_1, \ R_{12}, \ \alpha \to 0. \)

\textbf{Remark 7.1.} Theorem 7.1 presents sufficient conditions for designing reduced-order controllers with a prespecified \(H_\infty\) constraint on the closed-loop transfer function with no \(H_2\) contribution. Thus, Theorem 7.1 addresses the pure reduced-order \(H_\infty\)-standard problem. Note that considerable simplification can be achieved in the design equations by setting \(R_{12 \infty}, \ E_\infty, \) and \(D\) to zero.

\section{8. Numerical solution of the design equations}

Although the design equations appearing in Theorems 4.1, 6.1 and 7.1 appear formidable, they are, in fact, quite numerically tractable. One of the principal motivations of the Riccati equation approach to the mixed norm problem is the opportunity it provides for developing efficient computational algorithms for control design. In particular, the goal is to develop numerical methods that exploit the structure of these modified Riccati equations. It should be noted, however, that existing methods for solving standard Riccati equations cannot account for the additional terms that appear in the modified equations such as (6.5)–(6.8). Therefore, a new class of numerical algorithms has been developed based upon homotopic continuation methods. These methods operate by first replacing the original problem by a simpler problem with a known solution. The desired solution is then reached by integrating along a path (homotopy path) that connects the starting problem to the original problem. The advantage of such algorithms is that they are based on theories which are global in nature. In particular, homotopy methods facilitate the finding of (multiple) solutions to a problem, and the convergence of the homotopy algorithms is generally not dependent upon having initial conditions which are in some sense close to the actual solution. These ideas have been illustrated for the \(H_2\) reduced-order problem in [9] and the \(H_\infty\) constrained problem in [1] where the additional coupling terms preclude standard solution techniques. A complete description of the homotopy algorithm is given in [10].

\textbf{References}


