Dimensional Analysis of Matrices
State-Space Models and Dimensionless Units

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Physical dimensions and units, such as mass (kg), length (m), time (s), and charge (C), provide the link between mathematics and the physical world. It is well known that careful attention to physical dimensions can provide valuable insight into relationships among physical quantities. In this regard, the Buckingham Pi theorem (see "The Buckingham Pi Theorem in a Nutshell"), which is essentially an application of the fundamental theorem of linear algebra on the sum of the rank and defect of a matrix, has been extensively applied [1]–[10]. Interesting historical remarks on the development of dimensional analysis are given in [11], while detailed discussions are given in [12, Chapter 10] and [13].

In the control literature, with its historically strong mathematical influence, it is not unusual to see expressions such as

$$V(x, \dot{x}) = x^2 + \dot{x}^2,$$

where x and $\dot{x}$ denote position and velocity states, respectively. Although this expression appears to be dimensionally incorrect, the reader usually assumes that unlabeled coefficients are present to convert units from squared position to squared velocity or vice versa.

A related issue concerns the appearance of dimensionless units. For example, for a stiffness k and a mass m, the expression $\sqrt{k/m}$ has the dimensions of reciprocal time. However, when used within the context of harmonic solutions of an oscillator, the same expression has the interpretation of rad/s, where the dimensionless unit "rad" is inserted to facilitate the use of trigonometric functions. Although this insertion is ad hoc, the recognition that radians are dimensionless provides reasonable justification.

A publication of special note is the book [6], which takes an in-depth look at the role of dimensions including matrices populated with dimensioned quantities. Although this text provides no situations in which the "usual" rules of dimensional analysis lead to incorrect answers, the careful reexamination in [6] of the treatment of dimensions, especially for matrices, motivates the present article.

The main objective of this article is to examine the dimensional structure of the dynamics matrix A that arises in the linear state-space system $\dot{x} = Ax$. To do this, we extend results of [6] and provide a self-contained treatment of the dimensional structure of A and its exponential. Our investigation of the physical dimensions of A motivates us to look at the algebraic structure of dimensioned quantities. This development forces us to define multiple, distinct, group identity elements, which are the dimensionless units. One such dimensionless unit is the radian. However, to complete the analysis, we introduce an additional dimensionless quantity for each physical dimension and each product of dimensions.

This approach immediately clarifies the mysterious appearance of radians in the example above. Specifically, $[\sqrt{k/m}] = ([k]/[m])^{1/2} = ((\mathrm{N/m})/\mathrm{kg})^{1/2} = [\,]_{\mathrm{m}}[\,]_{\mathrm{kg}}/\mathrm{s}$, where $[a]$ denotes the physical dimensions of a, $[\,]_{\mathrm{kg}} \triangleq \mathrm{kg}^0$ is the identity element in the group of mass dimensions, and $[\,]_{\mathrm{m}} \triangleq \mathrm{m}^0$ is the identity element in the group of length dimensions. In fact, $[\,]_{\mathrm{m}}$ is the traditional radian, whose appearance is natural and need not be inserted with the justification that "radians are dimensionless." Rather, $[\,]_{\mathrm{m}}$ appears because the mathematical structure of physical dimensions requires that it be present. By the same reasoning, the *massian* $[\,]_{\mathrm{kg}}$ is also present in $[\sqrt{k/m}]$.

As an additional example, consider the expression $\omega = v/r$, where ω is angular velocity, v is translational velocity, and r is radius. Then $[\omega] = [v]/[r] = (\mathrm{m/s})/\mathrm{m} = \mathrm{m}^0/s = [\,]_{\mathrm{m}}/\mathrm{s} = \mathrm{rad/s}$. Again, there is no need to artificially insert the dimensionless unit "rad" in order to obtain the angular velocity in the expected units. We also note that, for an angle θ in radians, the fact that $[\,]_{\mathrm{m}}^{\alpha} = (\mathrm{m}^0)^{\alpha} = \mathrm{m}^0 = [\,]_{\mathrm{m}}$ for all real numbers α implies that

$$[\sin\theta] = \left[\theta - \frac{\theta^3}{3!} + \cdots\right] = [\,]_{\mathrm{m}},$$

which is consistent with the fact that both θ and $\sin\theta$ are ratios of lengths.

In real computations involving physical quantities, that is, aside from pure theory, it is necessary to keep track of physical dimensions and their associated units. Elucidation of the physical dimension structure of state space models can thus be useful for verifying the model structure and ensuring that the units are consistent within the context of state-space computations.

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ALGEBRAIC STRUCTURE OF UNITS

For simplicity, we consider the fundamental dimensions mass (kg), length (m), and time (s) only. For convenience, we use kg, m, and s to represent the respective physical dimension as well as the associated unit. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. Define $G_{\rm kg} \triangleq \{{\rm kg}^\alpha : \alpha \in \mathbb{R}\}$, $G_{\rm m} \triangleq \{{\rm m}^\beta : \beta \in \mathbb{R}\}$, and $G_{\rm s} \triangleq \{{\rm s}^\gamma : \gamma \in \mathbb{R}\}$. Note that $G_{\rm kg}$, $G_{\rm m}$, and $G_{\rm s}$ are Abelian (commutative) groups (see "What Is a Group?") with the identity elements $[\,]_{\rm kg}$, $[\,]_{\rm m}$, and $[\,]_{\rm s}$, respectively, which are dimensionless units referred to as the *massian*, *lengthian*, and *timian*. The lengthian $[\,]_{\rm m}$ in $G_{\rm m}$, when interpreted within the context of a circle, is the radian. Next, define the set G of all mixed units

$$G \triangleq \{{\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma : \alpha,\ \beta,\ \gamma \in \mathbb{R}\}. \tag{1}$$

Since, for all $\alpha, \beta, \gamma \in \mathbb{R}$, ${\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma = {\rm kg}^\alpha {\rm s}^\gamma {\rm m}^\beta = {\rm m}^\beta {\rm kg}^\alpha {\rm s}^\gamma = {\rm m}^\beta {\rm s}^\gamma {\rm kg}^\alpha = {\rm s}^\gamma {\rm m}^\beta {\rm kg}^\alpha = {\rm s}^\gamma {\rm kg}^\alpha {\rm m}^\beta$, we have the following result.

Fact 1

G is an Abelian group with the identity element $[\,]_{\rm kg}[\,]_{\rm m}[\,]_{\rm s}$.

The four products of the identity elements are represented by $[\,]_{\rm kg,m} \triangleq [\,]_{\rm kg}[\,]_{\rm m}$, $[\,]_{\rm kg,s} \triangleq [\,]_{\rm kg}[\,]_{\rm s}$, $[\,]_{\rm m,s} \triangleq [\,]_{\rm m}[\,]_{\rm s}$, and $[\,]_{\rm kg,m,s} \triangleq [\,]_{\rm kg}[\,]_{\rm m}[\,]_{\rm s}$, of which only the last is an element of G. Note that the dimensionless Reynolds number in fluid dynamics defined by

$${\rm Re} \triangleq \frac{v_s L}{\nu},$$

where v_s is the mean fluid velocity, L is the characteristic length of the flow, and ν is the kinematic fluid viscosity, has the units

$$[{\rm Re}] = [\,]_{\rm kg,m,s}.$$

Similarly, the dimensionless Froude number in fluid mechanics defined by

$${\rm Fr} \triangleq \frac{v_s}{Lg},$$

where g is acceleration due to gravity, has the units

$$[{\rm Fr}] = [\,]_{\rm m,s}.$$

Table 1 classifies several dimensionless quantities based on their units.

The set $\mathcal{D}$ of *dimensioned scalars* consists of elements of the form $a{\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma$, where $a \in \mathbb{C}$ and $\alpha,\ \beta,\ \gamma \in \mathbb{R}$. We allow $a \in \mathbb{C}$ to accommodate complex eigenvalues and eigenvectors. We define the units operator $[\,]$ as

$$[a{\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma] \triangleq {\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma.$$

Note that $[0\,{\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma] \triangleq {\rm kg}^\alpha {\rm m}^\beta {\rm s}^\gamma$. Let $a_1{\rm kg}^{\alpha_1}{\rm m}^{\beta_1}{\rm s}^{\gamma_1}$ and $a_2{\rm kg}^{\alpha_2}{\rm m}^{\beta_2}{\rm s}^{\gamma_2}$ be dimensioned scalars. Then the product of two dimensioned scalars always exists and is defined to be $a_1{\rm kg}^{\alpha_1}{\rm m}^{\beta_1}{\rm s}^{\gamma_1} a_2{\rm kg}^{\alpha_2}{\rm m}^{\beta_2}{\rm s}^{\gamma_2} = a_1 a_2 {\rm kg}^{\alpha_1+\alpha_2}{\rm m}^{\beta_1+\beta_2}{\rm s}^{\gamma_1+\gamma_2}$. However, the sum $a_1{\rm kg}^{\alpha_1}{\rm m}^{\beta_1}{\rm s}^{\gamma_1} + a_2{\rm kg}^{\alpha_2}{\rm m}^{\beta_2}{\rm s}^{\gamma_2}$ is defined only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$, in which case $a_1{\rm kg}^{\alpha_1}{\rm m}^{\beta_1}{\rm s}^{\gamma_1} + a_2{\rm kg}^{\alpha_2}{\rm m}^{\beta_2}{\rm s}^{\gamma_2} = (a_1 + a_2){\rm kg}^{\alpha_1}{\rm m}^{\beta_1}{\rm s}^{\gamma_1}$. Furthermore, although quantities such as $a{\rm kg}^\alpha$ and $b{\rm s}^\gamma$ are not elements of $\mathcal{D}$, we assume that all operations occur after these quantities are embedded in the appropriate group containing all of the common units. For example, $(a{\rm kg}^\alpha)(b{\rm s}^\gamma) \triangleq (a{\rm kg}^\alpha [\,]_{\rm s})(b{\rm s}^\gamma [\,]_{\rm kg}) = ab{\rm kg}^\alpha {\rm s}^\gamma$.

Dimensioned vectors and *dimensioned matrices* are denoted by $\mathcal{D}^n$ and $\mathcal{D}^{n\times m}$, respectively, all of whose entries are dimensioned scalars (see "Energy Versus Moment" for an example of the difference between dimensioned scalars and dimensioned vectors). Let $P \in \mathcal{D}^{n\times m}$ and define

$$[P] \triangleq \begin{bmatrix} [P_{1,1}] & \cdots & [P_{1,m}] \\ \vdots & \ddots & \vdots \\ [P_{n,1}] & \cdots & [P_{n,m}] \end{bmatrix} \in G^{n\times m}, \tag{2}$$

where $P_{i,j}$ is the $(i,\ j)$ entry of P and $G^{n\times m}$ denotes the set of $n \times m$ matrices with entries in G. Note that $[P^{\rm T}] = [P]^{\rm T}$. If $P \in \mathcal{D}^{n\times m}$ and $Q \in \mathcal{D}^{m\times p}$, then PQ *exists* if all addition operations required to form the product are defined.

Fact 2

Let $P \in \mathcal{D}^{n\times m}$ and $Q \in \mathcal{D}^{m\times p}$. Then PQ exists if and only if, for all $i = 1, \ldots, n$ and $j = 1, \ldots, p$,

$$[P_{i,1}][Q_{1,j}] = [P_{i,2}][Q_{2,j}] = \cdots = [P_{i,n}][Q_{n,j}]. \tag{3}$$

Furthermore, if PQ exists, then

$$[PQ] = [P][Q]. \tag{4}$$

Fact 3

Let $P \in \mathcal{D}^{n\times n}$. If P^2 exists, then

$$[P_{1,1}] = [P_{2,2}] = \cdots = [P_{n,n}]. \tag{5}$$

Proof

Since P^2 exists, it follows that, for all $i,\ j = 1, \ldots, n$,

$$[(P^2)_{i,i}] = [P_{i,1}][P_{1,i}] = [P_{i,2}][P_{2,i}] = \cdots = [P_{i,n}][P_{n,i}].$$

Now, let $i,\ j \in \{1, \ldots, n\}$. Then $[P_{i,i}][P_{i,i}] = [P_{i,j}][P_{j,i}] = [P_{j,i}][P_{i,j}] = [P_{j,j}][P_{j,j}]$. Hence $[P_{i,i}] = [P_{j,j}]$. □

Fact 4

Let $P \in \mathcal{D}^{n\times n}$. If P^2 exists, then, for all positive integers k, P^k exists and $[P^k] = [P]^k$. Furthermore, for all $i = 1, \ldots, n$ and for all positive integers k,

$$[P^k] = [(P_{i,i})^{k-1}][P]. \tag{6}$$

Proof

Since, for all $i,\ j = 1, \ldots, n$,

$$[(P^2)_{i,j}] = [P_{i,1}][P_{1,j}] = [P_{i,2}][P_{2,j}] = \cdots = [P_{i,n}][P_{n,j}],$$

it follows that

$$[(P^2)_{i,j}] = [P_{i,i}][P_{i,j}].$$

Hence $[P^2] = [P_{i,i}][P]$. Induction yields (6). □

Fact 5

Let $P \in \mathcal{D}^{n\times n}$. Then P^2 exists if and only if there exist $z_1,\ z_2 \in G^n$ such that $z_2^{\mathrm{T}} z_1$ exists and

$$[P] = z_1 z_2^{\mathrm{T}}. \tag{7}$$

Proof

Sufficiency is immediate. To prove necessity, define

$$z_1 \triangleq \begin{bmatrix} [P_{1,1}] \\ [P_{2,1}] \\ \vdots \\ [P_{n,1}] \end{bmatrix}, \quad z_2 \triangleq \begin{bmatrix} [P_{1,1}]/[P_{1,1}] \\ [P_{1,2}]/[P_{1,1}] \\ \vdots \\ [P_{1,n}]/[P_{1,1}] \end{bmatrix}.$$

The Buckingham Pi Theorem in a Nutshell

Let $u_1, \ldots, u_p$ be fundamental dimensions and let $G \triangleq \{\prod_{i=1}^{p} u_i^{\alpha_i} : \alpha_1, \ldots, \alpha_p \in \mathbb{R}$ be the corresponding Abelian group. Then the set $\mathcal{D}$ of dimensioned scalars consists of elements of the form $a\prod_{i=1}^{p} u_i^{\alpha_i}$, where $a \in \mathbb{C}$ and $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$. The following theorem, called the Buckingham Pi theorem [S1], shows that a relationship between q dimensioned quantities induces a collection of dimensionless quantities.

THEOREM S1

Let $Q_1, Q_2, \ldots, Q_q \in \mathcal{D}$ be dimensioned scalars such that, for $i = 1, \ldots, q$, $Q_i \triangleq a_i \prod_{j=1}^{p} u_j^{\alpha_{ij}}$, and assume that

$$\sum_{k=1}^{K} c_k Q_1^{\beta_{1k}} \cdots Q_q^{\beta_{qk}} = 0, \tag{S1}$$

where $c_1, \ldots, c_K \in \mathbb{R}$ are nonzero. Let $\mathcal{A} \triangleq [\alpha_{ij}]^{\mathrm{T}} \in \mathbb{R}^{p\times q}$, and let $r \triangleq \operatorname{rank} \mathcal{A}$. Then there exists $\Gamma \triangleq [\gamma_{ij}] \in \mathbb{R}^{q\times(q-r)}$ such that $\operatorname{rank}\Gamma = q - r$, $\mathcal{A}\Gamma = 0$, and, for $i = 1, \ldots, q-r$,

$$\Pi_i \triangleq Q_1^{\gamma_{1i}} \cdots Q_q^{\gamma_{qi}} \tag{S2}$$

are dimensionless.

PROOF

It follows from the fundamental theorem of linear algebra [S2, p. 33] that

$$\operatorname{rank} \mathcal{A} + \operatorname{def} \mathcal{A} = q$$

and thus

$$\operatorname{def} \mathcal{A} = q - r,$$

where def A is the dimension of the nullspace of $\mathcal{A}$. Next, let $\Gamma \triangleq [\gamma_{ij}] \in \mathbb{R}^{q\times(q-r)}$ be such that the columns of Γ form a basis for the nullspace of $\mathcal{A}$. Then it follows that $\operatorname{rank}\Gamma = q - r$ and $\mathcal{A}\Gamma = 0$. Next, since the (j, i) entry of $\mathcal{A}\Gamma$ is $\sum_{k=1}^{q} \alpha_{kj}\gamma_{ki} = 0$, it follows that, for all $i = 1, \ldots, q-r$,

$$\begin{aligned}\Pi_i &= Q_1^{\gamma_{1i}} \cdots Q_q^{\gamma_{qi}} \\ &= a_1^{\gamma_{1i}} \cdots a_q^{\gamma_{qi}} \prod_{j=1}^{p} u_j^{\sum_{k=1}^{q} \alpha_{kj}\gamma_{ki}} \\ &= a_1^{\gamma_{1i}} \cdots a_q^{\gamma_{qi}} \prod_{j=1}^{p} u_j^{0}\end{aligned}$$

is dimensionless. □

As an example, consider the law of conservation of momentum in a collision between two rigid bodies given by

$$m_1 v_1^- + m_2 v_2^- = m_1 v_1^+ + m_2 v_2^+, \tag{S3}$$

where m_1 and m_2 are the masses of the bodies, v_1^- and v_2^- are the velocities of the bodies before collision, and v_1^+ and v_2^+ are the velocities of the bodies after collision, respectively. Note that $[m_1] = [m_2] = \mathrm{kg}$ and $[v_1^-] = [v_2^-] = [v_1^+] = [v_2^+] = \mathrm{m/s}$. Furthermore, choosing $u_1 = \mathrm{kg}$, $u_2 = \mathrm{m}$, $u_3 = \mathrm{s}$, $Q_1 = m_1$, $Q_2 = m_2$, $Q_3 = v_1^-$, $Q_4 = v_2^-$, $Q_5 = v_1^+$, and $Q_6 = v_2^+$, it follows that $p = 3$, $q = 6$,

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 \end{bmatrix}, \tag{S4}$$

and $r = 2$. Therefore, in accordance with Theorem S1, there exist $q - r = 4$ dimensionless quantities. These dimensionless quantities can be computed by determining a basis for the null space of $\mathcal{A}$. For example, choosing

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{S5}$$

yields the dimensionless quantities

Since P^2 exists it follows that $[(P^2)_{1,1}]/[P_{1,1}] = z_2^{\mathrm{T}} z_1$ exists. Furthermore, let $k \in \{1, \ldots, n\}$ and define $z_3 \in G^n$ by

$$z_3 \triangleq \begin{bmatrix} [P_{1,k}] \\ [P_{2,k}] \\ \vdots \\ [P_{n,k}] \end{bmatrix}.$$

Then, $z_2^{\mathrm{T}} z_3$ exists and thus the rows of $[P]$ are dimensioned scalar multiples of each other. Hence

$$[P] = \begin{bmatrix} [P_{1,1}] & [P_{1,2}] & \cdots & [P_{1,n}] \\ [P_{2,1}] & [P_{1,2}][P_{2,1}]/[P_{1,1}] & \cdots & [P_{1,n}][P_{2,1}]/[P_{1,1}] \\ \vdots & \vdots & \ddots & \vdots \\ [P_{n,1}] & [P_{1,2}][P_{n,1}]/[P_{1,1}] & \cdots & [P_{1,n}][P_{n,1}]/[P_{1,1}] \end{bmatrix}$$
$$= z_1 z_2^{\mathrm{T}}. \qquad \square$$

Fact 6

Let $P \in \mathcal{D}^{n\times n}$. Then $e^P \in \mathcal{D}^{n\times n}$ exists if and only if P^2 exists and $[P] = [P^2]$. Furthermore, if e^P exists then

$$[e^P] = [P]. \tag{8}$$

$$\Pi_1 = \frac{m_1}{m_2}, \quad \Pi_2 = \frac{v_1^-}{v_2^-},$$
$$\Pi_3 = \frac{v_1^-}{v_1^+}, \quad \Pi_4 = \frac{v_1^+}{v_2^+}.$$

Note that these dimensionless quantities are not unique.

An application of the Buckingham Pi Theorem is to derive physical relationships between dimensioned quantities. For example, consider the problem of deriving an expression for the time period of oscillations of a pendulum. We expect the time period T to depend on the length l of the pendulum, the acceleration g due to gravity, and perhaps the mass m of the pendulum. Since $[T] = \mathrm{s}$, $[l] = \mathrm{m}$, $[g] = \mathrm{m/s^2}$, and $[m] = \mathrm{kg}$, we choose $u_1 = \mathrm{kg}$, $u_2 = \mathrm{m}$, $u_3 = \mathrm{s}$, $Q_1 = T$, $Q_2 = l$, $Q_3 = g$, and $Q_4 = m$. Noting that $p = 3$, $q = 3$,

$$\mathcal{A} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix}, \tag{S6}$$

and $r = 3$, it follows that there exists $q - r = 1$ dimensionless quantity given by

$$\Pi_1 = \frac{T\sqrt{g}}{\sqrt{l}}. \tag{S7}$$

Therefore,

$$T = \Pi_1 \sqrt{\frac{l}{g}}, \tag{S8}$$

where the dimensionless constant Π_1 can be determined experimentally to be 2π. Note that the time period does not depend on the mass of the pendulum, a result due to Galilieo.

As a final example, consider the force generated by a propeller on an aircraft. Presumably, the force F depends on the diameter d of the propeller, the velocity v of the airplane, the density ρ of the air, the rotational speed N of the propeller, and the dynamic viscosity ν of the air. Noting that $[F] = \mathrm{kgm/s^2}$, $[d] = \mathrm{m}$, $[v] = \mathrm{m/s}$, $[\rho] = \mathrm{kg/m^3}$, $[N] = [\,]_{\mathrm{m}}/\mathrm{s}$, $[\nu] = \mathrm{m^2/s}$, we choose $u_1 = \mathrm{kg}$, $u_2 = \mathrm{m}$, $u_3 = \mathrm{s}$, $Q_1 = F$, $Q_2 = d$, $Q_3 = v$, $Q_4 = \rho$, $Q_5 = N$, and $Q_6 = \nu$. Therefore, we have $p = 3$, $q = 6$,

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & -3 & 0 & 2 \\ -2 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}, \tag{S9}$$

and $r = 3$. Thus we have $q - r = 6 - 3 = 3$ dimensionless quantities. Choosing Γ to be

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & -2 \\ 1 & -1 & -2 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \tag{S10}$$

it follows that

$$\Pi_1 = \frac{dv}{\nu}, \quad \Pi_2 = \frac{dN}{v}, \quad \Pi_3 = \frac{F}{d^2 v^2 \rho},$$

where Π_1 is the Reynolds number, Π_2 is the top-speed ratio, and Π_3 is the dynamic-force ratio.

REFERENCES

[S1] E. Buckingham, "On physically similar systems: Illustration of the use of dimensional equations," *Phys. Rev.*, vol. 4, no. 4, pp. 345–376, 1914.

[S2] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory*. Princeton, NJ: Princeton University Press, 2005.

Proof

By definition, the matrix exponential $e^P \in \mathcal{D}^{n\times n}$ is given by

$$e^P = I + \frac{1}{1!}P + \frac{1}{2!}P^2 + \cdots. \tag{9}$$

Necessity is immediate. To prove sufficiency, note that, since P^2 exists and $[P] = [P^2]$, it follows from Fact 4 that $[P] = [P^2]$ implies that $[P] = [P^k]$ for all positive integers k. Thus e^P exists. Next, it follows from (9) that (8) holds. □

Fact 7

Let $P \in \mathcal{D}^{n\times n}$ and assume that e^P exists. Then, for all $i = 1, \ldots, n$,

$$[P_{i,i}] = [\,]_{\mathrm{kg,m,s}}. \tag{10}$$

Proof

The result follows immediately from facts 6 and 4. □

For a real scalar q and $P \in \mathcal{D}^{n\times m}$, the *Schur power* $P^{\{q\}} \in \mathcal{D}^{n\times m}$ is defined by

$$(P^{\{q\}})_{i,j} \triangleq (P_{i,j})^q, \tag{11}$$

assuming the right hand side exists. The notation $[P]_{\mathbb{C}} \in \mathbb{C}^{n\times m}$ denotes the numerical part of the dimensioned matrix $P \in \mathcal{D}^{n\times m}$. Note that

$$P = [P]_{\mathbb{C}} \circ [P], \tag{12}$$

where $\circ$ is the Schur (entry-wise) product. We write $[P]_{\mathbb{C}}$ as $[P]_{\mathbb{R}}$ if $[P]_{\mathbb{C}}$ is real. Let $I_{\mathbb{R}}$ denote the identity matrix in $\mathbb{R}^{n\times n}$. Furthermore, let $Q \in \mathcal{D}^{m\times p}$ and assume that PQ exists. Then $[PQ]_{\mathbb{C}} = [P]_{\mathbb{C}}[Q]_{\mathbb{C}}$ and

$$\begin{aligned} PQ &= ([P]_{\mathbb{C}} \circ [P])([Q]_{\mathbb{C}} \circ [Q]) \\ &= ([P]_{\mathbb{C}}[Q]_{\mathbb{C}}) \circ ([P][Q]) = [PQ]_{\mathbb{C}} \circ [PQ]. \end{aligned} \tag{13}$$

Fact 8

Let $P \in \mathcal{D}^{n\times m}$, and let $y \in \mathcal{D}^n$ and $u \in \mathcal{D}^m$ be such that

$$y = Pu. \tag{14}$$

Then

$$[P] = [y][u^{\mathrm{T}}]^{\{-1\}}. \tag{15}$$

Proof

The ith component equation of (15) is

$$[P_{i,1}][u_1] + [P_{i,2}][u_2] + \cdots + [P_{i,m}][u_m] = [y_i].$$

Therefore,

$$[P_{i,1}][u_1] = [P_{i,2}][u_2] = \cdots = [P_{i,m}][u_m] = [y_i],$$

and thus $[P_{i,j}] = [y_i]/[u_j]$. Hence (15) holds. □

What Is a Group?

A *group* $(G, *)$ is a set G with a binary operation $* : G \times G \to G$ that satisfies the following axioms:

A1) For all $a, b \in G$, $a * b \in G$.

A2) For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.

A3) There exists an identity element $e \in G$ such that, for all $a \in G$, $e * a = a * e = a$.

A4) For all $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$, where e is the identity element in G.

Note that, A1–A4 do not imply that, for all $a, b \in G$, $a * b = b * a$. However, if, for all $a, b \in G$, $a * b = b * a$, then the group G is an *Abelian group*.

The set of real numbers with $e = 0$ and the binary operation of addition is a group. However, the set of real numbers with $e = 1$ and the binary operation of multiplication is not a group since A4 is not satisfied for $a = 0$. Furthermore, since addition is commutative, the set of real numbers with the addition operation is an Abelian group.

The set of units $G = \{\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma : \alpha, \beta, \gamma \in \mathbb{R}\}$ with $e = [\,]_{\mathrm{kg,m,s}}$ and the binary operation of multiplication is an Abelian group.

TABLE 1 Classification of dimensionless units and examples. These seven dimensionless units are defined in terms of ratios of the basic physical dimensions.

Dimensionless Unit	Name	Examples
$[\,]_{\mathrm{kg}}$	Massian	Air-fuel ratio Stoichiometric mass ratio
$[\,]_{\mathrm{m}}$	Lengthian	Radian Strain Poisson's ratio Fresnel number Aspect ratio
$[\,]_{\mathrm{s}}$	Timian	Courant-Friedrichs-Lewy (CFL) number Damkohler numbers
$[\,]_{\mathrm{kg,m}}$	Densian	Density ratio Moment-of-inertia ratio
$[\,]_{\mathrm{kg,s}}$	Flowian	Mass-flow ratio Stiffness ratio
$[\,]_{\mathrm{m,s}}$	Velocian	Froude number Fourier number Mach number Stokes number
$[\,]_{\mathrm{kg,m,s}}$	Forcian	Reynolds number Weber number Coefficient of friction Lift coefficient Drag coefficient

Next, let $P \in \mathcal{D}^{n\times n}$. Then, the determinant det P of P is defined to be

$$\det P = \sum_{p\in\mathcal{P}_n} \sigma(p) P_{1,p_1} P_{2,p_2} \cdots P_{n,p_n}, \tag{16}$$

where $\mathcal{P}_n$ is the set of all permutations $p = (p_1, \ldots, p_n)$ of $(1, 2, \ldots, n)$, and $\sigma(p)$ is the signature of the permutation p, which is 1 if p is achieved by applying an even number of transpositions to $(1, 2, \ldots, n)$ and -1 if p is reached by applying an odd number of transpositions to $(1, 2, \ldots, n)$. Note that if $P \in \mathcal{D}^{n\times n}$ then det P exists if and only if $[P_{1,p_1} P_{2,p_2} \cdots P_{n,p_n}]$ is the same for all $p \in \mathcal{P}_n$. Hence, if det P exists, we have

$$[\det P] = [P_{1,p_1} P_{2,p_2} \cdots P_{n,p_n}] \tag{17}$$

for all $p \in \mathcal{P}_n$. Note that

$$\det [P]_{\mathbb{C}} = [\det P]_{\mathbb{C}} \tag{18}$$

and

$$\det P = (\det [P]_{\mathbb{C}})[\det P]. \tag{19}$$

The following result presents necessary and sufficient conditions for the existence of det P.

Fact 9

Let $P \in \mathcal{D}^{n\times n}$. Then det P exists if and only if there exist $z_1, z_2 \in G^n$ such that

$$[P] = z_1 z_2^{\mathrm{T}}. \tag{20}$$

Proof

Sufficiency is immediate. To prove necessity, first let $n = 2$. Then, since det P exists, it follows that

$$\frac{[P_{1,1}]}{[P_{1,2}]} = \frac{[P_{2,1}]}{[P_{2,2}]}. \tag{21}$$

Thus the columns of $[P]$ are dimensioned scalar multiples of each other. Next, let $n = 3$ and assume that det P exists. Then it follows from the cofactor expansion of det P that the determinant of every 2×2 submatrix of P exists. Hence (21) holds. Next, it follows that $[P_{1,1}P_{2,3}P_{3,2}] = [P_{1,2}P_{2,3}P_{3,1}]$ and hence

$$\frac{[P_{1,1}]}{[P_{1,2}]} = \frac{[P_{3,1}]}{[P_{3,2}]}. \tag{22}$$

Furthermore, using $[P_{1,2}P_{2,3}P_{3,1}] = [P_{1,3}P_{2,2}P_{3,1}]$ and $[P_{1,2}P_{2,1}P_{3,2}] = [P_{1,3}P_{2,1}P_{3,2}]$, it follows that $[P_{1,2}]/[P_{1,3}] = [P_{2,2}]/[P_{2,3}]$ and $[P_{1,2}]/[P_{1,3}] = [P_{3,2}]/[P_{3,3}]$. Thus the columns of $[P]$ are dimensioned scalar multiples of each other. Likewise, for all $n \geq 1$, it can be seen that, since det P exists, the columns of $[P]$ are dimensioned scalar multiples of each other. Thus, defining

$$z_1 \triangleq \begin{bmatrix} [P_{1,1}] \\ [P_{2,1}] \\ \vdots \\ [P_{n,1}] \end{bmatrix}, \quad z_2 \triangleq \begin{bmatrix} [P_{1,1}]/[P_{1,1}] \\ [P_{1,2}]/[P_{1,1}] \\ \vdots \\ [P_{1,n}]/[P_{1,1}] \end{bmatrix},$$

it follows that (20) holds. □

Note that if P^2 exists then det P exists. However, the following example shows that the converse does not hold.

Example 1

Let $P \in \mathcal{D}^{2\times 2}$ be such that

$$[P] = \begin{bmatrix} \mathrm{m} & \mathrm{m}^2 \\ \mathrm{s} & \mathrm{ms} \end{bmatrix}. \tag{23}$$

Then det P exists, but P^2 does not exist.

Let $P \in \mathcal{D}^{n\times n}$. Then $\lambda \in \mathcal{D}$ and $v \in \mathcal{D}^n$ are an *eigenvalue-eigenvector pair* of P if $[v]_{\mathbb{C}}$ is not zero and λ and v satisfy

$$Pv = \lambda v. \tag{24}$$

Fact 10

Let $P \in \mathcal{D}^{n\times n}$. Then P has an eigenvalue-eigenvector pair $\lambda \in \mathcal{D}$, $v \in \mathcal{D}^n$ if and only if det P exists and, for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$,

$$[P_{i,i}] = [P_{j,j}]. \tag{25}$$

Energy Versus Moment

Since energy is force times displacement, it follows that the units of energy are $\mathrm{J} = \mathrm{Nm} = \mathrm{kgm}^2/\mathrm{s}^2$. On the other hand, since moment times angular displacement is energy, it follows that the units of moment are $\mathrm{J/rad} = \mathrm{Nm/rad} = \mathrm{kgm}^2/\mathrm{s}^2\mathrm{rad}$. Furthermore, since $\mathrm{rad} = [\,]_{\mathrm{m}} = \mathrm{m}^0$, it follows that $\mathrm{J/rad} = \mathrm{kgm}^2/\mathrm{s}^2\mathrm{rad} = \mathrm{kgm}^2/\mathrm{s}^2 = \mathrm{J}$, and hence moment has the same units as energy.

Although the above analysis suggests that energy and moment are indistinguishable, we know intuitively that they are different. This apparent contradiction is resolved by the fact that energy is a dimensioned scalar in $\mathcal{D}$, while moment is a dimensioned vector in $\mathcal{D}^3$. In fact, the work done by a moment through an angle is the dot product of the moment and a dimensionless *angle vector*, which is a dimensionless vector perpendicular to the plane containing the angle. The direction of the angle vector is determined by the right-hand rule, and its dimensionless magnitude is given by the radian measure of the angle.

In this case,

$$[P] = [\lambda v][v^{\mathrm{T}}]^{\{-1\}} \tag{26}$$

and, for all $i = 1, \ldots, n$,

$$[P_{i,i}] = [\lambda]. \tag{27}$$

Proof

To prove necessity, note that it follows from Fact 8 that (24) implies (26). It thus follows from Fact 9 that det P exists. Furthermore, it follows from (24) that, for all $i = 1, \ldots, n$,

$$[P_{i,i}][v_i] = [\lambda][v_i].$$

Thus

$$[P_{i,i}] = [\lambda].$$

Hence, for $i = 1, \ldots, n$, $j = 1, \ldots, n$, it follows that $[P_{i,i}] = [P_{j,j}]$.

To prove sufficiency, from (20) and (25) it follows that

$$[(z_1)_1(z_2)_1] = [(z_1)_2(z_2)_2] = \cdots = [(z_1)_n(z_2)_n], \tag{28}$$

where $(z_1)_i$ denotes the ith component of z_1. Thus, $\lambda_G \triangleq z_2^{\mathrm{T}} z_1$ exists. Note that $\lambda_G z_1 = z_1 z_2^{\mathrm{T}} z_1 = [P] z_1$. Next, let $\lambda_{\mathbb{C}} \in \mathbb{C}$ and $v_{\mathbb{C}} \in \mathbb{C}^n$ be such that

$$[P]_{\mathbb{C}} v_{\mathbb{C}} = \lambda_{\mathbb{C}} v_{\mathbb{C}}. \tag{29}$$

Then defining $\lambda \in \mathcal{D}$ and $v \in \mathcal{D}^n$ by $\lambda \triangleq \lambda_{\mathbb{C}}\lambda_G$ and $v \triangleq v_{\mathbb{C}} \circ z_1$ it follows that

$$\begin{aligned} Pv &= ([P]_{\mathbb{C}} \circ [P])(v_{\mathbb{C}} \circ [v]) \\ &= ([P]_{\mathbb{C}} v_{\mathbb{C}}) \circ z_1 z_2^{\mathrm{T}} z_1 \\ &= (\lambda_{\mathbb{C}} v_{\mathbb{C}}) \circ \lambda_G z_1 \\ &= \lambda_{\mathbb{C}}\lambda_G (v_{\mathbb{C}} \circ z_1) \\ &= \lambda v. \end{aligned}$$

□

Next, let $P \in \mathcal{D}^{n\times n}$. Then, if det $[P]_{\mathbb{C}} \neq 0$, we define the inverse P^{-1} of P by

$$P^{-1} \triangleq \frac{1}{\det P} P^{\mathrm{A}}, \tag{30}$$

where the adjugate P^{A} is defined by $(P^{\mathrm{A}})_{i,j} \triangleq (-1)^{i+j} \det P_{[j,i]}$, where $P_{[j,i]}$ denotes the $(n-1)\times(n-1)$ cofactor of $P_{i,i}$. Hence

$$[P^{-1}] = \frac{1}{[\det P]}[P^{\mathrm{A}}] \tag{31}$$

and

$$[P^{-1}]_{\mathbb{C}} = \frac{1}{[\det P]_{\mathbb{C}}}[P^{\mathrm{A}}]_{\mathbb{C}}. \tag{32}$$

The following example shows that, for $P \in \mathcal{D}^{n\times n}$ such that P^{-1} exists, in general $[P^{-1}][P] \neq [P][P^{-1}]$.

Example 2

Let $P \in \mathcal{D}^{n\times}$ be such that

$$[P] \ = \ [\,]_{\mathrm{m,s}} \begin{bmatrix} \mathrm{m} & 1/\mathrm{s} \\ \mathrm{ms}^2 & \mathrm{s} \end{bmatrix}$$

and assume that P^{-1} exists. Then

$$\begin{aligned} [P^{-1}] \ &= \ [\,]_{\mathrm{m,s}} \begin{bmatrix} 1/\mathrm{m} & 1/\mathrm{ms}^2 \\ \mathrm{s} & 1/\mathrm{s} \end{bmatrix}, \\ [P][P^{-1}] \ &= \ [\,]_{\mathrm{m,s}} \begin{bmatrix} 1 & 1/\mathrm{s}^2 \\ \mathrm{s}^2 & 1 \end{bmatrix}, \end{aligned}$$

and

$$[P^{-1}][P] \ = \ [\,]_{\mathrm{m,s}} \begin{bmatrix} 1 & 1/\mathrm{ms} \\ \mathrm{ms} & 1 \end{bmatrix}$$

Thus $[P^{-1}][P] \neq [P][P^{-1}]$.

DIMENSIONS OF MATRICES IN STATE-SPACE MODELS

Consider the system

$$\begin{aligned} \dot{x}(t) \ &= \ Ax(t) + Bu(t), & (33) \\ y(t) \ &= \ Cx(t) + Du(t), & (34) \end{aligned}$$

where $[t] = \mathrm{s}$, $x(t) \in \mathcal{D}^n$, $y(t) \in \mathcal{D}^l$, $u(t) \in \mathcal{D}^m$, $A \in \mathcal{D}^{n\times n}$, $B \in \mathcal{D}^{n\times m}$, $C \in \mathcal{D}^{l\times n}$, and $D \in \mathcal{D}^{l\times m}$. Every component of $x(t)$, $y(t)$, $u(t)$, and thus every entry of A, B, C, D, is a dimensioned scalar. Taking units on both sides of (33) yields

$$\begin{aligned} [\dot{x}(t)] &= [A][x(t)] = [B][u(t)], & (35) \\ [y(t)] &= [C][x(t)] = [D][u(t)]. & (36) \end{aligned}$$

The following result is given on page 150 of [6].

Fact 11

$$\begin{aligned} [A] \ &= \ \frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}, & (37) \\ [B] \ &= \ \frac{1}{\mathrm{s}}[x(t)][u^{\mathrm{T}}(t)]^{\{-1\}}, & (38) \\ [C] \ &= \ [y(t)][x^{\mathrm{T}}(t)]^{\{-1\}}, & (39) \end{aligned}$$

and

$$[D] \ = \ [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}}. \tag{40}$$

Proof

The result follows from $[\dot{x}(t)] = (1/\mathrm{s})[x(t)]$ and Fact 8. □

Next, define the transfer function matrix $H(s) \in \mathcal{D}^{l\times m}$ by

$$H(s) \triangleq C(sI_s - A)^{-1}B + D, \tag{41}$$

where $s \in \mathcal{D}$ is the Laplace variable, $[s] = 1/\mathrm{s}$, and $I_s \triangleq I_{\mathbb{R}} \circ \mathrm{s}[A]$.

Fact 12

$$[H(s)] = [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}}. \tag{42}$$

Proof

Note that

$$\begin{aligned}[C(sI - A)^{-1}B] &= [y(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}} \\ &\quad \times [x(t)][u^{\mathrm{T}}(t)]^{\{-1\}}, \\ &= [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}} \\ &= [D].\end{aligned}$$

□

Fact 13

For all $i = 1, \ldots, n$,

$$[A_{i,i}] = [\,]_{\mathrm{kg,m}}\mathrm{s}^{-1}. \tag{43}$$

Furthermore, $\det A$ exists and satisfies

$$[\det A] = [\,]_{\mathrm{kg,m}}\mathrm{s}^{-n}. \tag{44}$$

Proof. It follows from (37) that

$$[A_{i,i}] = \frac{1}{\mathrm{s}}\frac{[x_i(t)]}{[x_i(t)]} = \frac{[\,]_{\mathrm{kg,m}}}{\mathrm{s}}.$$

Next, note that

$$[A_{i,p_i}] = \frac{1}{\mathrm{s}}\frac{[x_i(t)]}{[x_{p_i}(t)]}.$$

Thus, for all $p \in \mathcal{P}_n$,

$$\begin{aligned}[A_{1,p_1}A_{2,p_2}\cdots A_{n,p_n}] &= \frac{1}{\mathrm{s}^n}\frac{[x_1(t)][x_2(t)]\cdots[x_n(t)]}{[x_{p_1}(t)][x_{p_2}(t)]\cdots[x_{p_n}(t)]} \\ &= \frac{[\,]_{\mathrm{kg,m}}}{\mathrm{s}^n}.\end{aligned}$$

Since $[A_{1,p_1}A_{2,p_2}\cdots A_{n,p_n}]$ is the same for all $p \in \mathcal{P}_n$, $\det A$ exists. Finally, since $[\det A] = \prod_{i=1}^{n}[A_{1,p_i}]$ for all $p \in \mathcal{P}_n$, it follows that

$$[\det A] = [A_{1,p_1}A_{2,p_2}\cdots A_{n,p_n}] = \frac{[\,]_{\mathrm{kg,m}}}{\mathrm{s}^n}.$$

□

Fact 14

Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then

$$\det [At] = [\,]_{\mathrm{kg,m,s}}. \tag{45}$$

MATRIX EXPONENTIAL

Lemma 1

Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then the following statements hold:

i) For all positive integers k, A^k exists.
ii) For all $k \geq 1$, $[A^k] = (1/\mathrm{s}^{k-1})[A]$.
iii) For all $k \geq 1$, $[A^k] = (1/\mathrm{s})[A^{k-1}]$.
iv) For all $k \geq 1$, $[A^k t^k] = [At]$.
v) $[A]^{\{-1\}} = (1/\mathrm{s}^2)[A]$.

If, in addition, A^{-1} exists, then

vi) $[A^{-1}] = [A^{\mathrm{T}}]^{\{-1\}}$.
vii) $[A^{-1}] = \mathrm{s}^2[A]^{\mathrm{T}}$.

Proof

Statements i)–iv) follow from Fact 4. Next, we prove vi). Since $(A^{-1})_{i,i} = \det A_{[i,i]}/\det A$, it follows that $[(A^{-1})_{i,i}] = = \det\,[A_{[i,i]}]/\det\,[A] = [A_{1,1}]\cdots[A_{i-1,i-1}]\,[A_{i+1,i+1}]\cdots [A_{n,n}]/([A_{1,1}]\cdots[A_{n,n}]) = 1/[A_{i,i}]$. Thus, the diagonal entries of $[A][A^{-1}]$ satisfy

$$([A][A^{-1}])_{i,i} = [\,]_{\mathrm{kg,m,s}}, \quad i = 1, \ldots, n.$$

Therefore,

$$\begin{aligned}([A][A^{-1}])_{i,i} &= [A_{i,1}][A^{-1}_{1,i}] + [A_{i,2}][A^{-1}_{2,i}] \\ &\quad + \cdots + [A_{i,n}][A^{-1}_{n,i}][A_{n,n}] \\ &= [\,]_{\mathrm{kg,m,s}},\end{aligned}$$

which implies that

$$[(A^{-1})_{i,j}] = [A_{j,i}]^{-1}. \tag{46}$$

Thus, vi) is satisfied.

To prove vii), note that

$$([A]^{\mathrm{T}})_{i,j} = \frac{1}{\mathrm{s}}\frac{[x_j(t)]}{[x_i(t)]}. \tag{47}$$

Next, from (46) it follows that

$$[(A^{-1})_{i,j}] = [A_{j,i}]^{-1} = \mathrm{s}\frac{[x_j(t)]}{[x_i(t)]}. \tag{48}$$

Thus from (47) and (48), it follows that

$$[A]^{\mathrm{T}} = \frac{1}{\mathrm{s}^2}[A^{-1}]. \tag{49}$$

To prove v), using vi) in (49), we have

$$[A]^{\mathrm{T}} = \frac{1}{\mathrm{s}^2}[A^{\mathrm{T}}]^{\{-1\}}. \tag{50}$$

Taking transposes yields v). □

Fact 15

$$[A^{-1}] = \mathrm{s}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}. \quad (51)$$

Furthermore,

$$[A^{-1}][A] = [A][A^{-1}]. \quad (52)$$

Proof

Note that

$$[A^{-1}] = [A^{\mathrm{T}}]^{\{-1\}} = \mathrm{s}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}.$$

Hence $[A^{-1}][A] = [A][A^{-1}] = [x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}$. □

Fact 16

Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then

$$[e^{At}] = [At] = [x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}. \quad (53)$$

EIGENVALUES AND EIGENVECTORS OF A

Fact 17

Let $\lambda \in \mathcal{D}$ be an eigenvalue of A, and let $v \in \mathcal{D}^n$ be an associated eigenvector. Then, for all $i = 1, \ldots, n$,

$$[\lambda] = [A_{i,i}] \quad (54)$$

and

$$[v] = [x^{\mathrm{T}}(t)]^{\{-1\}}[v][x(t)]. \quad (55)$$

Proof

Since $Av = \lambda v$, it follows that, for all $i = 1, \ldots, n$, $[A_{i,i}][v_i] = [\lambda][v_i]$, and thus $[\lambda] = [A_{i,i}]$. Next, since $Av = \lambda v$, it follows that

$$\frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[v] = \frac{1}{\mathrm{s}}[v],$$

which implies (55). □

DC MOTOR EXAMPLE

Consider a dc motor with constant armature current I_{a}. Defining the state vector to be $x \triangleq [\, i_{\mathrm{f}} \quad \omega \,]^{\mathrm{T}}$, where i_{f} is the field current and ω is the motor angular velocity, we have

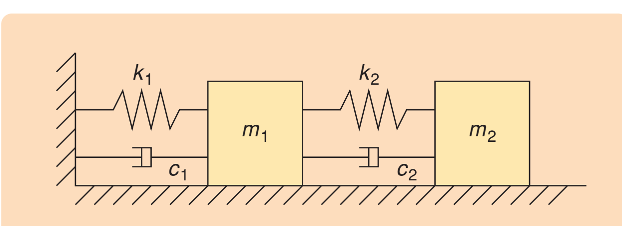

FIGURE 1 Two-mass spring-damper system.

$$A = \begin{bmatrix} -\frac{R_{\mathrm{f}}}{L_{\mathrm{f}}} & 0 \\ \frac{BI_{\mathrm{a}}}{J} & -\frac{c}{J} \end{bmatrix}, \quad (56)$$

where R_{f} and L_{f} are the field resistance and inductance, respectively, B is the electromagnetic constant of the motor, J is the inertia of the motor shaft and external load, and c is the angular damping coefficient. The units of R_{f}, L_{f}, I_{a}, B, J, and c are $\mathrm{m^2kg/C^2s}$, $\mathrm{m^2kg/C^2}$, $\mathrm{C/s}$, $\mathrm{kgm^2/C^2}$, $\mathrm{kgm^2}$, and $\mathrm{kgm^2/s}$, respectively.

Taking units yields

$$[x(t)] = \begin{bmatrix} \mathrm{C/s} \\ [\,]_{\mathrm{m}}/\mathrm{s} \end{bmatrix}. \quad (57)$$

Thus

$$[A] = \frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}} = \begin{bmatrix} [\,]_{\mathrm{C}}/\mathrm{s} & [\,]_{\mathrm{m}}\mathrm{C/s} \\ [\,]_{\mathrm{m}}/\mathrm{Cs} & [\,]_{\mathrm{m}}/\mathrm{s} \end{bmatrix}, \quad (58)$$

where $[\,]_{\mathrm{C}}$ denotes the *Coulombian*. Hence $[\det A] = [\,]_{\mathrm{m,C}}/\mathrm{s}^2$. Furthermore,

$$[\det A]_{\mathbb{C}} = \det\, [A]_{\mathbb{R}} = \left[\frac{cR_{\mathrm{f}}}{L_{\mathrm{f}}J}\right]_{\mathbb{R}}. \quad (59)$$

Thus,

$$\det A = \left[\frac{cR_{\mathrm{f}}}{JL_{\mathrm{f}}}\right]_{\mathbb{R}} \frac{[\,]_{\mathrm{m,C}}}{\mathrm{s}^2}. \quad (60)$$

Next, if $[cR_{\mathrm{f}}]_{\mathbb{R}} \neq 0$ then $\det\, [A]_{\mathbb{R}} \neq 0$ and $[A^{-1}]$ is given by

$$[A^{-1}] = \begin{bmatrix} [\,]_{\mathrm{C}}\mathrm{s} & [\,]_{\mathrm{m}}\mathrm{Cs} \\ [\,]_{\mathrm{m}}\mathrm{s/C} & [\,]_{\mathrm{m}}\mathrm{s} \end{bmatrix}. \quad (61)$$

Finally,

$$[e^{At}] = \begin{bmatrix} [\,]_{\mathrm{C,s}} & [\,]_{\mathrm{m,s}}\mathrm{C} \\ [\,]_{\mathrm{m,s}}/\mathrm{C} & [\,]_{\mathrm{m,s}} \end{bmatrix}. \quad (62)$$

SPRING-DAMPER SYSTEM EXAMPLE

Consider the spring-mass system shown in Figure 1. By defining the state $x(t) \triangleq [\, q_1 \quad \dot{q}_1 \quad q_2 \quad \dot{q}_2 \,]^{\mathrm{T}}$, where q_i and $\dot{q}_i$ are the displacement and velocity of the i^{th} mass, respectively, we have

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_1}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}. \quad (63)$$

Taking units yields

$$[x(t)] = \begin{bmatrix} \mathrm{m} \\ \mathrm{m/s} \\ \mathrm{m} \\ \mathrm{m/s} \end{bmatrix}. \quad (64)$$

Thus,

$$[A] = \frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}$$
$$= [\,]_{\mathrm{m}} \begin{bmatrix} 1/\mathrm{s} & 1 & 1/\mathrm{s} & 1 \\ 1/\mathrm{s}^2 & 1/\mathrm{s} & 1/\mathrm{s}^2 & 1/\mathrm{s} \\ 1/\mathrm{s} & 1 & 1/\mathrm{s} & 1 \\ 1/\mathrm{s}^2 & 1/\mathrm{s} & 1/\mathrm{s}^2 & 1/\mathrm{s} \end{bmatrix}. \tag{65}$$

Hence $[\det A] = [\,]_{\mathrm{m}}/\mathrm{s}^4$. Furthermore,

$$[\det A]_{\mathbb{C}} = \det\,[A]_{\mathbb{R}} = \left[\frac{k_2}{m_1 m_2}\right]_{\mathbb{R}}. \tag{66}$$

Thus,

$$\det A = \left[\frac{k_2}{m_1 m_2}\right]_{\mathbb{R}} \frac{[\,]_{\mathrm{m}}}{\mathrm{s}^4}. \tag{67}$$

Next, if $[k_2]_{\mathbb{R}} \neq 0$ then $\det\,[A]_{\mathbb{R}} \neq 0$ and $[A^{-1}]$ is given by

$$[A^{-1}] = [\,]_{\mathrm{m}} \begin{bmatrix} \mathrm{s} & \mathrm{s}^2 & \mathrm{s} & \mathrm{s}^2 \\ 1 & \mathrm{s} & 1 & \mathrm{s} \\ \mathrm{s} & \mathrm{s}^2 & \mathrm{s} & \mathrm{s}^2 \\ 1 & \mathrm{s} & 1 & \mathrm{s} \end{bmatrix}. \tag{68}$$

Finally,

$$[e^{At}] = [\,]_{\mathrm{m}} \begin{bmatrix} [\,]_{\mathrm{s}} & \mathrm{s} & [\,]_{\mathrm{s}} & \mathrm{s} \\ 1/\mathrm{s} & [\,]_{\mathrm{s}} & 1/\mathrm{s} & [\,]_{\mathrm{s}} \\ [\,]_{\mathrm{s}} & \mathrm{s} & [\,]_{\mathrm{s}} & \mathrm{s} \\ 1/\mathrm{s} & [\,]_{\mathrm{s}} & 1/\mathrm{s} & [\,]_{\mathrm{s}} \end{bmatrix}. \tag{69}$$

CONCLUSIONS

Physical dimensions are the link between mathematical models and the real world. In this article we extended results of [6] by determining the dimensional structure of a matrix under which standard operations involving the inverse, powers, exponential, and eigenvalues are valid. These results were applied to state space models. We also distinguished between different types of dimensionless units, namely, the massian, lengthian, timian, densian, flowian, velocian, and forcian. These dimensionless units arise naturally from the structure of the groups of units, and appear throughout science and engineering.

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MISSING LINKS

I am contacting you concerning the article [1], which appeared in the December 2007 issue of *IEEE Control Systems Magazine*, coauthored by us and R. Venugopal. In this regard, I must bring to your attention the fact that I have discovered two additional articles on the subject. These articles present further insights into the properties of dimensions. The purpose of this letter is to briefly describe these insights.

The articles I am referring to are [2] and [3], which build on the book [4]. These works develop *orientation analysis*, which augments standard dimensional analysis in verifying candidate relationships and uncovering new relationships. Orientation analysis keeps track of the orientation of each quantity and ensures that, in each physical relationship, the quantities on both sides have the same orientation. Although this idea is intuitively clear, it has been widely ignored in textbooks.

From an orientation point of view, quantities such as length, velocity, acceleration, force, and moment have a direction in space, whereas quantities such as mass, time, energy, and power do not. Not surprisingly, the orientation of a position, velocity, or acceleration vector along the x-direction is different from that of a corresponding vector along the y-direction. From this point of view, two force vectors acting along different lines of action, although representing the same physical quantity, are fundamentally different because they have different directions. Although closely related, orientation is *not* the same as vector direction. For instance, area has orientation but is not a vector.

As an application of orientation analysis, consider the problem discussed in [2]–[4] of deriving an expression for the range R of a projectile fired horizontally from a height h, with a velocity v_0. Since $[h] = \mathrm{m}$, $[v_0] = \mathrm{m/s}$, $[R] = \mathrm{m}$, and $[g] = \mathrm{m/s^2}$, the problem involves four quantities and two dimensions. Therefore, it follows from the Buckingham Pi theorem given in [1] that there exist two dimensionless quantities, namely, $\Pi_1 = R/h$ and $\Pi_2 = v_0/\sqrt{hg}$. This result is not useful, however, for characterizing R in terms of v_0, h, and g. However, if we recognize that lengths along the vertical and horizontal directions have different orientations, and thus represent different physical dimensions, namely, m_x and m_y, we have $[h] = \mathrm{m}_y$, $[v_0] = \mathrm{m}_x/\mathrm{s}$, $[R] = \mathrm{m}_x$, and $[g] = \mathrm{m}_y/\mathrm{s}^2$. We then have four quantities and three dimensions, which yields the dimensionless quantity

$$\Pi_1 = \frac{v_0}{R}\sqrt{\frac{h}{g}},$$

and thus the useful expression $R = \Pi_1 v_0 \sqrt{h/g}$.

In addition to providing a tool for deriving and checking physical relationships, orientation analysis can be used to explain the nature of angles and dimensionless units such as radians. For example, an angle can be viewed as the ratio of two lengths along perpendicular directions (that is, radial and circumferential within the context of a circle), and thus angle has orientation, namely, along the direction perpendicular to the plane containing the two lengths, as noted in [2]. This point is consistent with the treatment of angle as a vector pointing out of the plane that contains the angle; this vector can be dotted with a moment vector to determine the work done by the moment vector when moving through an angle. Note that a ratio of identical dimensions, such as length/length, is dimensionless but can possess orientation.

Orientation analysis helps explain why radians are different from strain, which is also a ratio of lengths but does not have orientation. This distinction is not due to the fact that radians have a natural circle scale, namely, 2π, but rather the fact that strain is the ratio of lengths along the same direction. On the other hand, Poisson's ratio, which measures the ratio of the resulting bulge or shrinkage due to the strain in an orthogonal direction, *is* appropriately measured in radians. As another example, the distinguishing feature between moment and energy—which we explain in our article as being due to the distinction between a scalar and a vector—is that moment has orientation, whereas energy does not.

The rules and consequences of orientation analysis, which are explained in detail in [2] and [3], can be used to compute the orientations

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of various derived quantities. For example, as noted above, area has orientation, although volume is orientationless. Even more surprising is the consequence that $\cos\theta$ is orientationless, but $\sin\theta$ has orientation. Since the square of every orientation is orientationless, the identity $\cos^2\theta + \sin^2\theta = 1$ is orientationless. Along the same lines, $e^{\jmath\omega t} = \cos(\omega t) + \jmath\sin(\omega t)$ (where $\jmath \triangleq \sqrt{-1}$) is orientationless, which in turn implies that $\jmath$ has orientation. Since $\jmath$ represents a rotation about the z-axis, this observation seems reasonable. Although these observations are surprising, their consistency strikes me as beyond coincidental. Something deep and fundamental seems to be going on here.

Although orientation analysis offers insight into the nature of dimensionless units such as radians, there remain unanswered questions in the world of dimensions and units. For example, it occurs to me that the angle between a force vector and a displacement vector, whose dot product arises in computing work (that is, energy transferred), should not be measured in radians but rather in forcians. In other words, it seems that not all angles are created equal. Another mystery is the fact that orientation analysis does not offer a satisfactory explanation for the appearance of radians in the expression for natural frequency $\omega = \sqrt{k/m}$, where m is mass and k is the ratio of force to displacement. While we usually explain radians in terms of a ratio of lengths, the force and displacement in the simple harmonic oscillator are parallel, and thus orientation analysis suggests that the correct unit for natural frequency is not radian in the sense of orientation along the z-axis but rather the orientationless lengthian.

I suspect we haven't heard the last word on this subject. It often seems that definitively resolving fundamental issues is not a simple task.

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Editor's Note: Perhaps a rationale for orientational analysis can be found in geometric algebra, where area is treated as a bivector. See C. Doran and A. Lasenby, *Geometric Algebra for Physicists*, Cambridge University Press, 2005.