Dimensional Analysis of Matrices
State-Space Models and Dimensionless Units
HARISH J. PALANTHANDALAM-MADAPUSI, DENNIS S. BERNSTEIN, and RAVINDER VENUGOPAL

Physical dimensions and units, such as mass (kg), length (m), time (s), and charge (C), provide the link between mathematics and the physical world. It is well known that careful attention to physical dimensions can provide valuable insight into relationships among physical quantities. In this regard, the Buckingham Pi theorem (see “The Buckingham Pi Theorem in a Nutshell”), which is essentially an application of the fundamental theorem of linear algebra on the sum of the rank and defect of a matrix, has been extensively applied [1]–[10]. Interesting remarks on the development of dimensional analysis are given in [11], while detailed discussions are given in [12, Chapter 10] and [13].

In the control literature, with its historically strong mathematical influence, it is not unusual to see expressions such as

\[ V(x, \dot{x}) = x^2 + \dot{x}^2, \]

where \( x \) and \( \dot{x} \) denote position and velocity states, respectively. Although this expression appears to be dimensionally incorrect, the reader usually assumes that unlabeled coefficients are present to convert units from squared position to squared velocity or vice versa.

A related issue concerns the appearance of dimensionless units. For example, for a stiffness \( k \) and a mass \( m \), the expression \( \sqrt{k/m} \) has the dimensions of reciprocal time. However, when used within the context of harmonic solutions of an oscillator, the same expression has the interpretation of radians. Although this insertion is ad hoc, the recognition that radians are dimensionless provides reasonable justification.

A publication of special note is the book [6], which takes an in-depth look at the role of dimensions including matrices populated with dimensioned quantities. Although this text provides no situations in which the “usual” rules of dimensional analysis lead to incorrect answers, the careful reexamination in [6] of the treatment of dimensions, especially for matrices, motivates the present article.

The main objective of this article is to examine the dimensional structure of the dynamics matrix \( A \) that arises in the linear state-space system \( \dot{x} = Ax \). To do this, we extend results of [6] and provide a self-contained treatment of the dimensional structure of \( A \) and its exponential. Our investigation of the physical dimensions of \( A \) motivates us to look at the algebraic structure of dimensioned quantities. This development forces us to define multiple, distinct, group identity elements, which are the dimensionless units. One such dimensionless unit is the radian. However, to complete the analysis, we introduce an additional dimensionless quantity for each physical dimension and each product of dimensions.

This approach immediately clarifies the mysterious appearance of radians in the example above. Specifically, \( \sqrt{k/m} = (k/L^2) \), where \([a] \) denotes the physical dimensions of \( a \), \([1]_k \triangleq \text{kg}^0 \) is the identity element in the group of mass dimensions, and \([1]_m \triangleq m^0 \) is the identity element in the group of length dimensions. In fact, \([1]_m \) is the traditional radian, whose appearance is natural and need not be inserted with the justification that “radians are dimensionless.” Rather, \([1]_m \) appears because the mathematical structure of physical dimensions requires that it be present. By the same reasoning, the mass unit \([1]_k \) is also present in \( \sqrt{k/m} \).

An additional example concerns the angular velocity \( \omega = \dot{v}/r \), where \( \omega \) is angular velocity, \( v \) is translational velocity, and \( r \) is radius. Then \([\omega] = [v]/[r] = (\text{m/s})/\text{m} = \text{m}^0/\text{s} = [1]_m/\text{s} = \text{rad}/\text{s} \). Again, there is no need to artificially insert the dimensionless unit “rad” in order to obtain the angular velocity in the expected units. We also note that, for an angle \( \theta \) in radians, the fact that \( [\sin \theta] = (\text{m}^0)^0 = \text{m}^0 = [1]_m \) for all real numbers \( \theta \) implies that

\[ [\sin \theta] = \left[ \theta - \frac{\theta^3}{3!} + \cdots \right] = [1]_m, \]

which is consistent with the fact that both \( \theta \) and \( \sin \theta \) are ratios of lengths.

In real computations involving physical quantities, that is, aside from pure theory, it is necessary to keep track of physical dimensions and their associated units. Elucidation of the physical dimension structure of state space models can thus be useful for verifying the model structure and ensuring that the units are consistent within the context of state-space computations.
ALGEBRAIC STRUCTURE OF UNITS

For simplicity, we consider the fundamental dimensions mass (kg), length (m), and time (s) only. For convenience, we use kg, m, and s to represent the respective physical dimension as well as the associated unit. Let \( \mathbb{R} \) and \( \mathbb{C} \) denote the real and complex numbers, respectively. Define \( G_{kg} \triangleq [kg^a : a \in \mathbb{R}] \), \( G_m \triangleq [m^b : b \in \mathbb{R}] \), and \( G_s \triangleq [s^c : c \in \mathbb{R}] \). Note that \( G_{kg}, G_m, \) and \( G_s \) are Abelian (commutative) groups (see “What Is a Group?”) with the identity elements \([1]_{kg}, [1]_m, \) and \([1]_s\), respectively, which are dimensionless units referred to as the massian, lengthian, and timian. The lengthian \([1]_m \) in \( G_m \), when interpreted within the context of a circle, is the radian. Next, define the set \( G \) of all mixed units

\[
G \triangleq [kg^a m^b s^c : \alpha, \beta, \gamma \in \mathbb{R}].
\]

(1)

Since, for all \( \alpha, \beta, \gamma \in \mathbb{R} \), \( kg^a m^b s^c = kg^a s^c m^b = m^b kg^a s^c = s^c kg^a m^b = s^c kg^a m^b \), we have the following result.

**Fact 1**

G is an Abelian group with the identity element \([1]_{kg}[1]_m[1]_s\).

The four products of the identity elements are represented by \([1]_{kg,m} \triangleq [1]_{kg}[1]_m[1]_s, [1]_{kg,s} \triangleq [1]_{kg}[1]_s[1]_m, [1]_{m,s} \triangleq [1]_m[1]_s[1]_m, \) and \([1]_{kg,m,s} \triangleq [1]_{kg}[1]_m[1]_s\), of which only the last is an element of \( G \). Note that the dimensionless Reynolds number in fluid dynamics defined by

\[
Re \triangleq \frac{v_2 L}{v},
\]

where \( v_2 \) is the mean fluid velocity, \( L \) is the characteristic length of the flow, and \( v \) is the kinematic fluid viscosity, has the units

\[
[Re] = [1]_{kg,m,s}.
\]

Similarly, the dimensionless Froude number in fluid mechanics defined by

\[
Fr \triangleq \frac{v_2 g}{L},
\]

where \( g \) is acceleration due to gravity, has the units

\[
[Fr] = [1]_{m,s}.
\]

Table 1 classifies several dimensionless quantities based on their units.

The set \( D \) of dimensioned scalars consists of elements of the form \( ak^a m^b s^c \), where \( a \in \mathbb{C} \) and \( \alpha, \beta, \gamma \in \mathbb{R} \). We allow \( a \in \mathbb{C} \) to accommodate complex eigenvalues and eigenvectors. We define the units operator \([\_]\) as

\[
[ak^a m^b s^c] \triangleq kg^a m^b s^c.
\]

Note that \([0 kg^a m^b s^c] \triangleq kg^a m^b s^c\). Let \( a_1 kg^{a_1} m^{b_1} s^{c_1} \) and \( a_2 kg^{a_2} m^{b_2} s^{c_2} \) be dimensioned scalars. Then the product of two dimensioned scalars always exists and is defined to be \( a_1 kg^{a_1} m^{b_1} s^{c_1} a_2 kg^{a_2} m^{b_2} s^{c_2} = a_1 a_2 kg^{a_1+a_2} m^{b_1+b_2} s^{c_1+c_2} \). However, the sum \( a_1 kg^{a_1} m^{b_1} s^{c_1} + a_2 kg^{a_2} m^{b_2} s^{c_2} \) is defined only if \( \alpha_1 = \alpha_2, \beta_1 = \beta_2, \) and \( \gamma_1 = \gamma_2, \) in which case \( a_1 kg^{a_1} m^{b_1} s^{c_1} + a_2 kg^{a_2} m^{b_2} s^{c_2} = (a_1 + a_2) kg^{a_1} m^{b_1} s^{c_1} \). Furthermore, although quantities such as \( ak^a m^b s^c \) and \( bs^c \) are not elements of \( D \), we assume that all operations occur after these quantities are embedded in the appropriate group containing all of the common units. For example, \((ak^a m^b s^c)(bs^c) \triangleq (ak^a[1]_s)(bs^c[1]_{kg}) = abk^a m^b s^c\).

Dimensioned vectors and dimensioned matrices are denoted by \( D^n \) and \( D^{n \times m} \), respectively, all of whose entries are dimensioned scalars (see “Energy Versus Moment” for an example of the difference between dimensioned scalars and dimensioned vectors). Let \( P \in D^{n \times m} \) and define

\[
[P] \triangleq \begin{bmatrix}
[P_{1,1}] & \cdots & [P_{1,m}]
\vdots & \ddots & \vdots
\cdots & \cdots & \cdots & \cdots & \cdots

[P_{n,1}] & \cdots & [P_{n,m}]
\end{bmatrix} \in G^{n \times m},
\]

(2)

where \( P_{ij} \) is the \((i, j)\) entry of \( P \) and \( G^{n \times m} \) denotes the set of \( n \times m \) matrices with entries in \( G \). Note that \([P^T] = [P]^T\). If \( P \in D^{m \times n} \) and \( Q \in D^{m \times p} \), then \( PQ \) exists if all addition operations required to form the product are defined.

**Fact 2**

Let \( P \in D^{n \times m} \) and \( Q \in D^{m \times p} \). Then \( PQ \) exists if and only if, for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, p \),

\[
[P_{i,1}][Q_{1,}] = [P_{i,2}][Q_{2,}] = \cdots = [P_{i,n}][Q_{n,}].
\]

(3)

Furthermore, if \( PQ \) exists, then

\[
[PQ] = [P][Q].
\]

(4)

**Fact 3**

Let \( P \in D^{n \times n} \). If \( P^2 \) exists, then

\[
[P_{1,1}] = [P_{2,2}] = \cdots = [P_{n,n}].
\]

(5)

**Proof**

Since \( P^2 \) exists, it follows that, for all \( i, j = 1, \ldots, n \),

\[
([P^2]_{i,j}) = [P_{1,1}][P_{1,i}] = [P_{2,1}][P_{2,i}] = \cdots = [P_{n,n}][P_{n,i}].
\]

Now, let \( i, j = 1, \ldots, n \). Then \( [P_{i,1}][P_{1,}] = [P_{i,2}][P_{2,}] = [P_{i,3}][P_{3,}] = \cdots = [P_{i,n}][P_{n,}]. \) Hence \([P_{i,}] = [P_{i,}]\).

**Fact 4**

Let \( P \in D^{n \times n} \). If \( P^2 \) exists, then, for all positive integers \( k \), \( P^k \) exists and \([P^k] = [P]^k\). Furthermore, for all \( i = 1, \ldots, n \) and for all positive integers \( k \),

\[
[ak^a m^b s^c]^k \triangleq kg^a m^b s^c.
\]
Proof
Since, for all \( i, j = 1, \ldots, n \),
\[
[(P^2)_{i,j}] = [P_{i,1}] [P_{1,j}] = [P_{i,2}] [P_{2,j}] = \cdots = [P_{i,n}] [P_{n,j}].
\]
it follows that
\[
[(P^2)_{i,j}] = [P_{i,j}] [P_{j,j}] \].
Hence \([P^2] = [P_i, j] [P_j, j] \). Induction yields (6). \( \square \)

The Buckingham Pi Theorem in a Nutshell

Let \( u_1, \ldots, u_p \) be fundamental dimensions and let \( G = \prod_{i=1}^{p} u_i^{a_i} : a_1, \ldots, a_p \in \mathbb{R} \) be the corresponding Abelian group. Then the set \( D \) of dimensioned scalars consists of elements of the form \( a \prod_{i=1}^{p} u_i^{a_i} \), where \( a \in C \) and \( a_1, \ldots, a_p \in \mathbb{R} \).

The following theorem, called the Buckingham Pi theorem [S1], shows that a relationship between \( q \) dimensioned quantities induces a collection of dimensionless quantities.

**Theorem S1**

Let \( Q_1, Q_2, \ldots, Q_q \in D \) be dimensioned scalars such that, for \( i = 1, \ldots, q \), \( Q_i \triangleq a_i \prod_{j=1}^{p} u_j^{a_j} \), and assume that
\[
\sum_{k=1}^{K} c_k Q_k^{a_1} \cdots Q_k^{a_q} = 0, \quad (S1)
\]
where \( c_1, \ldots, c_K \in \mathbb{R} \) are nonzero. Let \( \Gamma \triangleq [a_l] \in \mathbb{R}^{q \times (q-r)} \), and let \( r = \text{rank} \( \Gamma \) \). Then there exists \( \Pi \in \mathbb{R}^{r \times (q-r)} \) such that \( \text{rank} \( \Pi \) = q - r \), \( \Pi \Gamma = 0 \), and, for \( i = 1, \ldots, q - r \),
\[
\Pi_i \triangleq Q_i^{a_1} \cdots Q_i^{a_q} \quad (S2)
\]
are dimensionless.

**Proof**

It follows from the fundamental theorem of linear algebra [S2, p. 33] that
\[
\text{rank} \( \mathcal{A} \) + \text{def} \( \mathcal{A} \) = q
\]
and thus
\[
\text{def} \( \mathcal{A} \) = q - r,
\]
where def \( \mathcal{A} \) is the dimension of the nullspace of \( \mathcal{A} \). Next, let \( \Gamma \triangleq [\gamma_i] \in \mathbb{R}^{r \times (q-r)} \) be such that the columns of \( \Gamma \) form a basis for the nullspace of \( \mathcal{A} \). Then it follows that \( \text{rank} \( \Gamma \) = q - r \) and \( \mathcal{A} \Gamma = 0 \). Next, since the \( (j, i) \) entry of \( \mathcal{A} \Gamma \) is \( \sum_{k=1}^{q} a_{k,j} \gamma_{k,i} = 0 \), it follows that, for all \( i = 1, \ldots, q - r \),
\[
[\mathcal{A}] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & -1 & -1 & -1 & -1
\end{bmatrix}
\]
and \( r = 2 \). Therefore, in accordance with Theorem S1, there exist \( q - r = 4 \) dimensionless quantities. These dimensionless quantities can be computed by determining a basis for the null space of \( \mathcal{A} \). For example, choosing
\[
\Gamma = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
yields the dimensionless quantities.

**Fact 5**

Let \( P \in \mathbb{D}^{m \times n} \). Then \( P^2 \) exists if and only if there exist \( z_1, z_2 \in \mathbb{G}^n \) such that \( z_1^T z_1 \) exists and
\[
[P] = z_1 z_2^T. \quad (7)
\]

**Proof**

 Sufficiency is immediate. To prove necessity, define
\[
z_1 \triangleq \begin{bmatrix}
[P_{1,1}] \\
[P_{2,1}] \\
\vdots \\
[P_{n,1}]
\end{bmatrix}, \quad z_2 \triangleq \begin{bmatrix}
[P_{1,1}] / [P_{1,1}] \\
[P_{2,1}] / [P_{1,1}] \\
\vdots \\
[P_{n,1}] / [P_{1,1}]
\end{bmatrix}.
\]
Since \( P^2 \) exists it follows that \( [(P^2)_{i,j}]/[P_{i,j}] = z_2^2 z_3 \) exists. Furthermore, let \( k \in \{1, \ldots, n\} \) and define \( z_3 \in \mathbb{C}^n \) by

\[
\left[ \begin{array}{c}
\Pi_1 \\
\Pi_2 \\
\Pi_3 \\
\Pi_4 \\
\end{array} \right] = \left[ \begin{array}{c}
\frac{m_1}{m_2} \\
\frac{v_3}{v_4} \\
\frac{v_1}{v_2} \\
\frac{v_1}{v_2} \\
\end{array} \right].
\]

Then, \( z_2^2 z_3 \) exists and thus the rows of \( [P] \) are dimensioned scalar multiples of each other. Hence

\[
\Pi_1 = \frac{m_1}{m_2}, \quad \Pi_2 = \frac{v_3}{v_4}, \quad \Pi_3 = \frac{v_1}{v_2}, \quad \Pi_4 = \frac{v_1}{v_2}.
\]

Note that these dimensionless quantities are not unique.

An application of the Buckingham Pi Theorem is to derive physical relationships between dimensioned quantities. For example, consider the problem of deriving an expression for the time period of oscillations of a pendulum. We expect the time period \( T \) to depend on the length \( l \) of the pendulum, the acceleration due to gravity \( g \) and perhaps the mass \( m \) of the pendulum. Since \([T] = s, [l] = m, [g] = m/s^2\), and \([m] = kg\), we choose \( u_1 = kg, u_2 = m, u_3 = s, \) \( Q_1 = T, Q_2 = l, Q_3 = g, \) and \( Q_4 = m. \) Noting that \( p = 3, q = 3, \)

\[
A = \left[ \begin{array}{cccc}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -1 & 0 \\
0 & 1 & 1 & 0 & 0 & -1 \\
\end{array} \right],
\]

and \( r = 3. \) Thus we have \( q - r = 6 - 3 = 3 \) dimensionless quantities. Choosing \( \Gamma \) to be

\[
\Gamma = \left[ \begin{array}{ccc}
0 & 0 & 1 \\
1 & 1 & -2 \\
1 & 0 & -1 \\
0 & 1 & 0 \\
-1 & 0 & 0 \\
\end{array} \right].
\]

it follows that

\[
\Pi_1 = \frac{T \sqrt{g}}{\sqrt{l}}, \quad \Pi_2 = \frac{dN}{\nu}, \quad \Pi_3 = \frac{F}{\nu^3}. \]

Therefore,

\[
T = \Pi_1 \sqrt{\frac{l}{g}}.
\]

where the dimensionless constant \( \Pi_1 \) can be determined experimentally to be \( 2\pi. \) Note that the time period does not depend on the mass of the pendulum, a result due to Galileo.

As a final example, consider the force generated by a propeller on an aircraft. Presumably, the force \( F \) depends on the diameter \( d \) of the propeller, the velocity \( v \) of the airplane, the density \( \rho \) of the air, the rotational speed \( N \) of the propeller, and the dynamic viscosity \( \nu \) of the air. Noting that \([F] = kgm/s^2, [d] = m, [v] = m/s, [\rho] = kg/m^3, [N] = [m/s, [\nu] = m^2/s,\) we choose \( u_1 = kg, u_2 = m, u_3 = s, Q_1 = F, Q_2 = d, Q_3 = v, Q_4 = \rho, Q_5 = N, \) and \( Q_6 = \nu. \) Therefore, we have \( p = 3, q = 6, \)

\[
A = \left[ \begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & 1 & -3 & 0 \\
1 & -2 & 0 & -1 \\
\end{array} \right],
\]

\[
\Pi_1 = \frac{dN}{\nu}, \quad \Pi_2 = \frac{dN}{\nu}, \quad \Pi_3 = \frac{F}{\nu^3}, \quad \Pi_4 = \frac{F}{\nu^3}, \quad \Pi_5 = \frac{F}{\nu^3}, \quad \Pi_6 = \frac{F}{\nu^3}.
\]

where \( \Pi_1 \) is the Reynolds number, \( \Pi_2 \) is the top-speed ratio, and \( \Pi_3 \) is the dynamic-force ratio.

REFERENCES
Proof
By definition, the matrix exponential $e^P \in \mathcal{D}^{n \times n}$ is given by

$$e^P = I + \frac{1}{1!}P + \frac{1}{2!}P^2 + \ldots.$$ (9)

Necessity is immediate. To prove sufficiency, note that, since $P^2$ exists and $[P] = [P^2]$, it follows from Fact 4 that $[P] = [P^k]$ for all positive integers $k$. Thus $e^P$ exists. Next, it follows from (9) that (8) holds. □

**Fact 7**
Let $P \in \mathcal{D}^{n \times n}$ and assume that $e^P$ exists. Then, for all $i = 1, \ldots, n$,

$$[P_{1i}] = [1]_{kg, m, s}.$$ (10)

**Proof**
The result follows immediately from facts 6 and 4. □

For a real scalar $q$ and $P \in \mathcal{D}^{n \times m}$, the Schur power $P^{q} \in \mathcal{D}^{n \times m}$ is defined by

$$(P^q)_{ij} \triangleq (P_{ij})^q,$$ (11)

assuming the right hand side exists. The notation $[P]_{C} \in \mathbb{C}^{n \times m}$ denotes the numerical part of the dimensioned matrix $P \in \mathcal{D}^{n \times m}$. Note that $P = [P]_{C} \circ [P]$, where $\circ$ is the Schur (entry-wise) product. We write $[P]_{C}$ as $[P]_{R}$ if $[P]_{C}$ is real. Let $I_{R}$ denote the identity matrix in $\mathbb{R}^{n \times n}$. Furthermore, let $Q \in \mathcal{D}^{m \times p}$ and assume that $PQ$ exists. Then 

$$PQ = ([P]_{C} \circ [P])([Q]_{C} \circ [Q]) = ([P]_{C}[Q]_{C}) \circ ([P][Q]) = [PQ]_{C} \circ [PQ].$$ (13)

**Fact 8**
Let $P \in \mathcal{D}^{n \times m}$, and let $y \in \mathcal{D}^{n}$ and $u \in \mathcal{D}^{m}$ be such that

$$y = Pu.$$ (14)

Then

$$[P] = [y][u^T]^{-1}.$$ (15)

**Proof**
The $i$th component equation of (15) is

$$[P_{i1}][u_1] + [P_{i2}][u_2] + \cdots + [P_{im}][u_m] = [y_i].$$

Therefore,

$$[P_{i1}][u_1] = [P_{i2}][u_2] = \cdots = [P_{im}][u_m] = [y_i],$$

and thus $[P_{i,j}] = [y_i]/[u_i]$. Hence (15) holds. □

### TABLE 1 Classification of dimensionless units and examples.
These seven dimensionless units are defined in terms of ratios of the basic physical dimensions.

<table>
<thead>
<tr>
<th>Dimensionless Unit</th>
<th>Name</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\text{s}]_{kg}$</td>
<td>Massian</td>
<td>Air-fuel ratio, Stoichiometric mass ratio</td>
</tr>
<tr>
<td>$[\text{s}]_{m}$</td>
<td>Lengthian</td>
<td>Radian, Strain, Poisson’s ratio, Fresnel number, Aspect ratio</td>
</tr>
<tr>
<td>$[\text{s}]_{s}$</td>
<td>Timian</td>
<td>Courant-Friedrichs-Lewy (CFL) number, Damkohler numbers</td>
</tr>
<tr>
<td>$[\text{s}]_{kg, m}$</td>
<td>Densian</td>
<td>Density ratio, Moment-of-inertia ratio</td>
</tr>
<tr>
<td>$[\text{s}]_{kg, s}$</td>
<td>Flowian</td>
<td>Mass-flow ratio</td>
</tr>
<tr>
<td>$[\text{s}]_{m, s}$</td>
<td>Velocian</td>
<td>Froude number, Fourier number, Mach number, Stokes number</td>
</tr>
<tr>
<td>$[\text{s}]_{kg, m, s}$</td>
<td>Forcian</td>
<td>Reynolds number, Weber number, Coefficient of friction, Drag coefficient</td>
</tr>
</tbody>
</table>
Next, let \( P \in D^{n \times n} \). Then, the determinant \( \det P \) of \( P \) is defined to be

\[
\det P = \sum_{p \in \mathcal{P}_n} \sigma(p) P_{p_1,1} P_{p_2,2} \cdots P_{p_n,n},
\]

(16)

where \( \mathcal{P}_n \) is the set of all permutations \( p = (p_1, \ldots, p_n) \) of \( (1, 2, \ldots, n) \), and \( \sigma(p) \) is the signature of the permutation \( p \), which is 1 if \( p \) is achieved by applying an even number of transpositions to \( (1, 2, \ldots, n) \) and \(-1\) if \( p \) is reached by applying an odd number of transpositions to \( (1, 2, \ldots, n) \). Note that if \( P \in D^{n \times n} \) then \( \det P \) exists if and only if \([P_{p_1,1} P_{p_2,2} \cdots P_{p_n,n}]\) is the same for all \( p \in \mathcal{P}_n \). Hence, if \( \det P \) exists, we have

\[
[\det P] = [P_{p_1,1} P_{p_2,2} \cdots P_{p_n,n}]
\]

(17)

for all \( p \in \mathcal{P}_n \). Note that

\[
\det [P]_C = [\det P]_C
\]

(18)

and

\[
\det P = (\det [P]_C)[\det P].
\]

(19)

The following result presents necessary and sufficient conditions for the existence of \( \det P \).

**Fact 9**

Let \( P \in D^{n \times n} \). Then \( \det P \) exists if and only if there exist \( z_1, z_2 \in \mathbb{C}^n \) such that

\[
[P] = z_1 z_2^T.
\]

(20)

**Proof**

Sufficiency is immediate. To prove necessity, first let \( n = 2 \). Then, since \( \det P \) exists, it follows that

\[
\begin{bmatrix}
P_{1,1} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
P_{2,1} & \frac{1}{2}
\end{bmatrix}
\end{bmatrix}
\]

(21)

Thus the columns of \([P]\) are dimensioned scalar multiples of each other. Next, let \( n = 3 \) and assume that \( \det P \) exists. Then it follows from the cofactor expansion of \( \det P \) that the determinant of every \( 2 \times 2 \) submatrix of \( P \) exists. Hence (21) holds. Next, it follows that \([P_{1,1} P_{2,2} P_{3,3}] = [P_{1,2} P_{2,3} P_{3,1}]\) and hence

\[
\begin{bmatrix}
P_{1,1} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
P_{2,1} & \frac{1}{2}
\end{bmatrix}
\]

(22)

Furthermore, using \([P_{1,2} P_{2,3} P_{3,1}] = [P_{1,2} P_{2,3} P_{3,1}]\) and \([P_{1,2} P_{2,1} P_{3,2}] = [P_{1,2} P_{2,1} P_{3,2}]\), it follows that \([P_{1,2} P_{1,3}] = [P_{2,1} P_{3,2}] \) and \([P_{1,2}] P_{3,1}] = [P_{2,1}] P_{3,2}] \). Thus the columns of \([P]\) are dimensioned scalar multiples of each other. Likewise, for all \( n \geq 1 \), it can be seen that, since \( \det P \) exists, the columns of \([P]\) are dimensioned scalar multiples of each other. Thus, defining

\[
[z_1] \triangleq \begin{bmatrix}
P_{1,1} \\
P_{2,1}
\end{bmatrix}
\begin{bmatrix}
P_{1,1}
\end{bmatrix}
\begin{bmatrix}
P_{1,2}
\end{bmatrix}
\end{bmatrix}
\]

(23)

it follows that (20) holds. \( \square \)

Note that if \( P^2 \) exists then \( \det P \) exists. However, the following example shows that the converse does not hold.

**Example 1**

Let \( P \in D^{2 \times 2} \) be such that

\[
[P] = \begin{bmatrix}
m & m^2
\end{bmatrix}
\begin{bmatrix}
s
ms
\end{bmatrix}
\]

(23)

Then \( \det P \) exists, but \( P^2 \) does not exist.

Let \( P \in D^{n \times n} \). Then \( \lambda \in \mathbb{D} \) and \( \nu \in \mathbb{D}^n \) are an **eigenvalue-eigenvector pair** of \( P \) if \([\nu]_C \) is not zero and \( \lambda \) and \( \nu \) satisfy

\[
P \nu = \lambda \nu.
\]

(24)

**Fact 10**

Let \( P \in D^{n \times n} \). Then \( P \) has an eigenvalue-eigenvector pair \( \lambda \in \mathbb{D} \), \( \nu \in \mathbb{D}^n \) if and only if \( \det P \) exists and, for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, n \),

\[
[P_{i,j}] = [P_{j,i}]
\]

(25)
In this case,
\[ [P] = [\lambda v][v^T]^{-1} \]  
(26)
and, for all \( i = 1, \ldots, n \),
\[ [P_{ii}] = [\lambda]. \]  
(27)

Proof
To prove necessity, note that it follows from Fact 8 that (24) implies (26). It thus follows from Fact 9 that \( \det P \) exists. Furthermore, it follows from (24) that, for all \( i = 1, \ldots, n \),
\[ [P_{ii}] = [\lambda]. \]  
(27)

Thus
\[ [P_{ii}] = [\lambda]. \]

Hence, for \( i = 1, \ldots, n, \ j = 1, \ldots, n \), it follows that \( [P_{ii}] = [P_{jj}] \).

To prove sufficiency, from (20) and (25) it follows that
\[ \det [P_{ii}] = \det [P_{jj}]. \]

where \( (z_1)_i \) denotes the \( i \)th component of \( z_1 \). Thus, \( \lambda_G \triangleq z_2^T z_1 \) exists. Note that \( \lambda_G z_1 = z_1 z_2^T z_1 = [P] z_1 \). Next, let \( \lambda_G \in \mathbb{C} \) and \( v_G \in \mathbb{C}^n \) be such that
\[ [P] v_G = \lambda_G v_G. \]  
(29)

Then defining \( \lambda \in \mathbb{D} \) and \( v \in \mathbb{D}^n \) by \( \lambda \triangleq \lambda_G \lambda_C \) and \( v \triangleq v_G z_1 \) it follows that
\[ P v = ([P] C \circ [P]) (v_G \circ [v]) \]
\[ = ([P] C v_G) \circ z_1 z_2^T z_1 \]
\[ = (\lambda_C v_G) \circ \lambda_G z_1 \]
\[ = \lambda_C \lambda_G (v_G \circ z_1) \]
\[ = \lambda v. \]

Next, let \( P \in \mathbb{D}^{n \times n} \). Then, if \( \det [P] \neq 0 \), we define the inverse \( P^{-1} \) of \( P \) by
\[ P^{-1} \triangleq \frac{1}{\det P} P^A, \]  
(30)
where the adjugate \( P^A \) is defined by \( (P^A)_{ij} \triangleq (-1)^{i+j} \det P_{ij} \), where \( P_{ij} \) denotes the \((n-1) \times (n-1)\) cofactor of \( P_{ii} \). Hence
\[ [P^{-1}] = \frac{1}{\det P} [P^A]. \]  
(31)

The following example shows that, for \( P \in \mathbb{D}^{n \times n} \) such that \( P^{-1} \) exists, in general \( [P^{-1}] = [P][P^{-1}] \).

Example 2
Let \( P \in \mathbb{D}^{n \times n} \) be such that
\[ [P] = \begin{bmatrix} m & 1/s^2 & 1/s \end{bmatrix} \]
and assume that \( P^{-1} \) exists. Then
\[ [P^{-1}] = \begin{bmatrix} 1/m & 1/s & 1/s^2 \\ s & 1 \\ 1/m s^2 & 1 \end{bmatrix}. \]  
(32)

Thus \( [P^{-1}] \neq [P][P^{-1}] \).

DIMENSIONS OF MATRICES
IN STATE-SPACE MODELS
Consider the system
\[ \dot{x}(t) = Ax(t) + Bu(t), \]  
(33)
\[ y(t) = Cx(t) + Du(t), \]  
(34)

where \( [A], [B], [C], [D] \in \mathbb{D}^{m \times n}, \ A \in \mathbb{D}^{m \times n}, \ B \in \mathbb{D}^{n \times m}, \ C \in \mathbb{D}^{l \times n}, \) and \( D \in \mathbb{D}^{l \times m} \). Every component of \( x(t), \ y(t), \ u(t) \), and thus every entry of \( A, B, C, D \), is a dimensioned scalar. Taking units on both sides of (33) yields
\[ \dot{x}(t) = [A][x(t)] = [B][u(t)], \]  
(35)
\[ y(t) = [C][x(t)] = [D][u(t)]. \]  
(36)

The following result is given on page 150 of [6].

Fact 11

\[ [A] = \frac{1}{s}[x(t)][x^T(t)]^{-1}, \]  
(37)
\[ [B] = \frac{1}{s}[x(t)][u^T(t)]^{-1}, \]  
(38)
\[ [C] = [y(t)][x^T(t)]^{-1}, \]  
(39)

and
\[ [D] = [y(t)][u^T(t)]^{-1}. \]  
(40)
Proof
The result follows from \([\dot{x}(t) = (1/s)x(t)]\) and Fact 8.

Next, define the transfer function matrix \(H(s) \in \mathbb{D}^{\times m}\) by

\[
H(s) \triangleq C(sI_n - A)^{-1}B + D,
\]

where \(s \in \mathbb{D}\) is the Laplace variable, \([s] = 1/s\), and \(I_n \triangleq I_n \circ s[A]\).

Fact 12

\[
[H(s)] = [y(t)][u^T(t)][s].
\]

Proof
Note that

\[
[C(sI_n - A)^{-1}B] = [y(t)][x^T(t)][s] = [x(t)][u^T(t)][s],
\]

\[
\det A = [s].
\]

Fact 13

For all \(i = 1, \ldots, n\),

\[
[A_{i,i}] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^{-1}.
\]

Furthermore, \(\det A\) exists and satisfies

\[
[\det A] = [1]\begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^{-n}.
\]

Proof. It follows from (37) that

\[
[A_{i,i}] = \frac{1}{s}[x_i(t)] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s.
\]

Next, note that

\[
[A_{i,p}] = \frac{1}{s}[x_i(t)] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s.
\]

Thus, for all \(p \in \mathcal{P}_n\),

\[
[A_{1,p}A_{2,p} \cdots A_{n,p}] = \frac{1}{s^n}[x_1(t)][x_2(t)] \cdots [x_n(t)]
\]

\[
= \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Since \([A_{1,p}A_{2,p} \cdots A_{n,p}]\) is the same for all \(p \in \mathcal{P}_n\), \(\det A\) exists. Finally, since \([\det A] = \prod_{i=1}^n[A_{i,i}]\) for all \(p \in \mathcal{P}_n\), it follows that

\[
[\det A] = [A_{1,p}A_{2,p} \cdots A_{n,p}] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Fact 14

Let \(t \in \mathbb{D}\) be such that \([t] = s\). Then

\[
\det [A(t)] = [1]_{kg,m,s}.
\]

MATRICES EXPONENTIAL

Lemma 1

Let \(t \in \mathbb{D}\) be such that \([t] = s\). Then the following statements hold:

i) For all positive integers \(k, A^k\) exists.

ii) For all \(k \geq 1, [A^k] = (1/s^k)[A]\).

iii) For all \(k \geq 1, [A^k] = (1/s)[A^{k-1}]\).

iv) For all \(k \geq 1, [A^{k+1}] = [A][A^k]\).

Proof.

Statements i–iv follow from Fact 4. Next, we prove vi).

Since \((A^{-1})_{i,i} = \det A_{i,i}/\det A\), it follows that \([A^{-1}]_{i,i} = \det A_{i,i}/\det A\) is such that \([A^{-1}]_{i,i} = \det A_{i,i}/\det A\) and \([A^{-1}]_{i,j} = [A_{i,j}][A_{i,i}]^{-1}[A_{i,i}]^{-1}\) for all \(i, j = 1, \ldots, n\). Thus, the diagonal entries of \([A][A^{-1}]\) satisfy

\[
[(A)[A^{-1}]]_{i,i} = [1]_{kg,m,s}, \quad i = 1, \ldots, n.
\]

Therefore,

\[
[(A)[A^{-1}]]_{i,i} = [A_{i,i}][A^{-1}_{i,i}] + [A_{i,j}][A^{-1}_{i,j}]
\]

\[
+ \cdots + [A_{n,i}][A^{-1}_{n,i}][A_{j,j}]
\]

which implies that

\[
[(A^{-1})_{i,i}] = [A_{i,j}]^{-1}.
\]

Thus, vi) is satisfied.

To prove vii), note that

\[
[(A^T)]_{i,j} = \frac{1}{s}[x_j(t)] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Thus, from (46) it follows that

\[
[(A^{-1})_{i,j}] = [A_{i,j}]^{-1} = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Next, from (46) it follows that

\[
[(A^T)]_{i,j} = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Thus from (47) and (48), it follows that

\[
[A^T] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

To prove viii), using vi) in (49), we have

\[
[A^T] = \begin{bmatrix} 1 \end{bmatrix}_{kg,m} s^n.
\]

Taking transposes yields vii).

\[
\square
\]
Fact 15
\[ [A^{-1}] = s[x(t)][x^T(t)]^{-1}. \] (51)

Furthermore,
\[ [A^{-1}] = [A][A^{-1}] = [A][A^{-1}][x(t)][x^T(t)]^{-1}. \] (52)

Proof
Note that
\[ [A^{-1}] = [A^T]^{-1} = s[x(t)][x^T(t)]^{-1}. \]

Hence
\[ [A^{-1}] = [A][A^{-1}] = [x(t)][x^T(t)]^{-1}[x(t)]^{-1}. \] (53)

Fact 16
Let \( t \in \mathcal{D} \) be such that \( [t] = s \). Then
\[ [\hat{e}^A] = [At] = [x(t)][x^T(t)]^{-1}. \] (54)

EIGENVALUES AND EIGENVECTORS OF A

Fact 17
Let \( \lambda \in \mathcal{D} \) be an eigenvalue of \( A \), and let \( \nu \in \mathcal{D}^n \) be an associated eigenvector. Then, for all \( i = 1, \ldots, n \),
\[ \lambda = [A_{i,i}] \] (55)

and
\[ [\nu] = [x^T(t)]^{-1}[\nu][x(t)]. \] (56)

Proof
Since \( A\nu = \lambda \nu \), it follows that, for all \( i = 1, \ldots, n \),
\[ [A_{i,i}] = [A_{i,i}] \] (57)

and thus \( \lambda = [A_{i,i}] \). Next, since \( A\nu = \lambda \nu \), it follows that
\[ \frac{1}{s}[x(t)][x^T(t)]^{-1}[\nu] = \frac{1}{s}[\nu], \]

which implies (55).

DC MOTOR EXAMPLE

Consider a dc motor with constant armature current \( I_0 \). Defining the state vector to be
\[ x = [i \quad \omega]^T, \]
where \( i \) is the armature current and \( \omega \) is the motor angular velocity, we have
\[ A = \begin{bmatrix} \frac{R_t}{I} & 0 \\ \frac{L_t}{I} & -\frac{c}{J} \end{bmatrix}. \] (58)

where \( R_t \) and \( L_t \) are the field resistance and inductance, respectively, \( B \) is the electromagnetic constant of the motor, \( J \) is the inertia of the motor shaft and external load, and \( c \) is the angular damping coefficient. The units of \( R_t, L_t, I, B, J, \) and \( c \) are \( \text{m}^2 \text{kg}/\text{C}^2 \cdot \text{s}, \ \text{m}^2 \text{kg}/\text{C}^2, \ \text{C}/\text{s}, \ \text{kgm}^2/\text{C}^2, \ \text{kgm}^2, \) and \( \text{kgm}^2/\text{s} \), respectively.

Taking units yields
\[ [\dot{x}(t)] = \begin{bmatrix} \frac{C}{s} \\ \frac{1}{\text{Im/s}} \end{bmatrix}. \] (59)

Thus
\[ [A] = \frac{1}{s}[x(t)][x^T(t)]^{-1} = \begin{bmatrix} \frac{cR_f}{J} \\ \frac{\text{ImC}/s}{\text{Im/s}} \end{bmatrix}. \] (60)

where \( [\text{ImC}/s] \) denotes the Coulombic. Hence
\[ [\text{det } A] = \frac{\text{ImC}/s}{\text{ImC}/s^2}. \]

Next, if \( [cR_f] \neq 0 \) then \( [\text{det } A] \neq 0 \) and \( [A^{-1}] \) is given by
\[ [A^{-1}] = \begin{bmatrix} \frac{c}{s} \\ \frac{1}{\text{ImC}/s} \end{bmatrix}. \] (61)

Finally,
\[ [\hat{e}^A] = \begin{bmatrix} \frac{cR_f}{J} \\ \frac{\text{ImC}/s}{\text{ImC}/s} \end{bmatrix}. \] (62)

SPRING-DAMPER SYSTEM EXAMPLE

Consider the spring-mass system shown in Figure 1. By defining the state \( x(t) = [q_1 \ q_1 \ q_2 \ q_2]^T \), where \( q_1 \) and \( q_2 \) are the displacement and velocity of the \( i \)th mass, respectively, we have
\[ A = \begin{bmatrix} \frac{k_1 c_1}{m_1} & \frac{c_1 c_2}{m_1} & \frac{0}{m_1} & \frac{0}{m_1} \\ \frac{k_2 c_1}{m_2} & \frac{c_1 c_2}{m_2} & \frac{0}{m_2} & \frac{0}{m_2} \\ \frac{k_1 c_2}{m_2} & \frac{c_2 c_2}{m_2} & \frac{0}{m_2} & \frac{0}{m_2} \\ \end{bmatrix}. \] (63)

Taking units yields
\[ [x(t)] = \begin{bmatrix} \frac{\text{m}}{\text{m/s}} \\ \frac{\text{m}}{\text{m/s}} \end{bmatrix}. \] (64)
Thus,

\[ [A] = \frac{1}{s} [x(t)]^{1/3}(t)^{-1} \]

\[ = \begin{bmatrix} 1/s & 1/s & 1/s & 1 \\ 1/s^2 & 1/s^2 & 1/s & 1 \\ 1/s & 1/s & 1/s & 1 \\ 1/s^2 & 1/s^2 & 1/s & 1 \end{bmatrix}_{\text{m}}. \]  \tag{65}

Hence \( \det A = [1/m^4] \). Furthermore,

\[ \det A_C = \det [A]_R = \begin{bmatrix} k_2 \\ m_1 m_2 \end{bmatrix}_R. \]  \tag{66}

Thus,

\[ \det A = \begin{bmatrix} k_2 \\ m_1 m_2 \end{bmatrix}_R [1/m]_{s^4}. \]  \tag{67}

Next, if \( [k_2]_R \neq 0 \) then \( \det [A]_R \neq 0 \) and \( [A^{-1}] \) is given by

\[ [A^{-1}] = [1/m] \begin{bmatrix} s & s^2 & s & s^2 \\ 1 & s & 1 & s \\ s & s^2 & s & s^2 \\ 1 & s & 1 & s \end{bmatrix}. \]  \tag{68}

Finally,

\[ [e^{A t}] = [1/m] \begin{bmatrix} 1 & s & 1 & s \\ 1/s & 1/s & 1 & s \\ 1/s & 1/s & 1/s & 1/s \\ 1/s & 1/s & 1/s & 1/s \end{bmatrix}. \]  \tag{69}

**CONCLUSIONS**

Physical dimensions are the link between mathematical models and the real world. In this article we extended results of [6] by determining the dimensional structure of a matrix under which standard operations involving the inverse, powers, exponential, and eigenvalues are valid. These results were applied to state space models. We also distinguished between different types of dimensionless units, namely, the massian, lengthian, timian, densian, flowian, velocian, and forcian. These dimensionless units arise naturally from the structure of the groups of units, and appear throughout science and engineering.

**ACKNOWLEDGMENTS**

We would like to thank Jan Willems for helpful suggestions and comments.

**REFERENCES**


**AUTHOR INFORMATION**

Harish J. Palanthandalam-Madapusi (hpalanth@umich.edu) received the B.E. degree from the University of Mumbai in mechanical engineering in 2001. In 2001 and 2002 he was a research engineer at the Indian Institute of Technology, Bombay. He received the Ph.D. degree from the Aerospace Engineering Department at the University of Michigan in 2007. He is currently an assistant professor in the Department of Mechanical and Aerospace Engineering at Syracuse University. His interests are in the areas of system identification, data assimilation, and estimation.

Dennis S. Bernstein is a professor in the Aerospace Engineering Department at the University of Michigan. He is editor-in-chief of the *IEEE Control Systems Magazine*, and he is the author of *Matrix Mathematics: Theory, Facts, and Formulas with Applications to Linear Systems* published by Princeton University Press in 2005. His interests are in system identification and adaptive control for aerospace applications.

Ravinder Venugopal received the B.Tech. degree in aerospace engineering from the Indian Institute of Technology, Madras, India, the M.S. degree in aerospace engineering from Texas A & M University, and the Ph.D. degree in aerospace engineering from the University of Michigan. From 1997 to 1999 he was a postdoctoral research fellow at the Aerospace Engineering Department of the University of Michigan. He is the founder and CEO of Sysendes, Inc. His research interests include discrete-time adaptive control, active noise and vibration control, and hydraulic control for industrial applications.
orientation of a position, velocity, or acceleration vector along the x-direction is different from that of a corresponding vector along the y-direction. From this point of view, two force vectors acting along different lines of action, although representing the same physical quantity, are fundamentally different because they have different directions. Although closely related, orientation is not the same as vector direction. For instance, area has orientation but is not a vector.

As an application of orientation analysis, consider the problem discussed in [2]–[4] of deriving an expression for the range $R$ of a projectile fired horizontally from a height $h$, with a velocity $v_0$. Since $[h] = m$, $[v_0] = m/s$, $[R] = m$, and $[g] = m/s^2$, the problem involves four quantities and two dimensions. Therefore, it follows from the Buckingham Pi theorem given in [1] that there exist two dimensionless quantities, namely, $\Pi_1 = R/h$ and $\Pi_2 = v_0/\sqrt{hg}$. This result is not useful, however, for characterizing $R$ in terms of $v_0$, $h$, and $g$. However, if we recognize that lengths along the vertical and horizontal directions have different orientations, and thus represent different physical dimensions, namely, $m_x$ and $m_y$, we have $[h] = m_y$, $[v_0] = m_x/s$, $[R] = m_x$, and $[g] = m_y/s^2$. We then have four quantities and three dimensions, which yields the dimensionless quantity

$$\Pi_1 = \frac{v_0}{R} \sqrt{\frac{h}{g}},$$

and thus the useful expression $R = \Pi_1 v_0 \sqrt{h/g}$.

In addition to providing a tool for deriving and checking physical relationships, orientation analysis can be used to explain the nature of angles and dimensionless units such as radians. For example, an angle can be viewed as the ratio of two lengths along perpendicular directions (that is, radial and circumferential within the context of a circle), and thus angle has orientation, namely, along the direction perpendicular to the plane containing the two lengths, as noted in [2]. This point is consistent with the treatment of angle as a vector pointing out of the plane that contains the angle; this vector can be dotted with a moment vector to determine the work done by the moment vector when moving through an angle. Note that a ratio of identical dimensions, such as length/length, is dimensionless but can possess orientation.

Orientation analysis helps explain why radians are different from strain, which is also a ratio of lengths but does not have orientation. This distinction is not due to the fact that strains have a natural circle scale, namely, $2\pi$, but rather the fact that strain is the ratio of lengths along the same direction. On the other hand, Poisson’s ratio, which measures the ratio of the resulting bulge or shrinkage due to the strain in an orthogonal direction, is appropriately measured in radians. As another example, the distinguishing feature between moment and energy—which we explain in our article as being due to the distinction between a scalar and a vector—is that moment has orientation, whereas energy does not.

The rules and consequences of orientation analysis, which are explained in detail in [2] and [3], can be used to compute the orientations...
of various derived quantities. For example, as noted above, area has orientation, although volume is orientationless. Even more surprising is the consequence that \( \cos \theta \) is orientationless, but \( \sin \theta \) has orientation. Since the square of every orientation is orientationless, the identity \( \cos^2 \theta + \sin^2 \theta = 1 \) is orientationless. Along the same lines, \( e^{\mathbf{i} \omega t} = \cos(\omega t) + j \sin(\omega t) \) (where \( j \equiv \sqrt{-1} \)) is orientationless, which in turn implies that \( j \) has orientation. Since \( j \) represents a rotation about the \( z \)-axis, this observation seems reasonable. Although these observations are surprising, their consistency strikes me as beyond coincidental. Something deep and fundamental seems to be going on here.

Although orientation analysis offers insight into the nature of dimensionless units such as radians, there remain unanswered questions in the world of dimensions and units. For example, it occurs to me that the angle between a force vector and a displacement vector, whose dot product arises in computing work (that is, energy transferred), should not be measured in radians but rather in forcians. In other words, it seems that not all angles are created equal. Another mystery is the fact that orientation analysis does not offer a satisfactory explanation for the appearance of radians in the expression for natural frequency \( \omega = \sqrt{k/m} \), where \( m \) is mass and \( k \) is the ratio of force to displacement. While we usually explain radians in terms of a ratio of lengths, the force and displacement in the simple harmonic oscillator are parallel, and thus orientation analysis suggests that the correct unit for natural frequency is not radian in the sense of orientation along the \( z \)-axis but rather the orientationless lengthian.

I suspect we haven’t heard the last word on this subject. It often seems that definitively resolving fundamental issues is not a simple task.

Harish J.
Palanthandalam-Madapusi
Syracuse University

References

Editor’s Note: Perhaps a rationale for orientational analysis can be found in geometric algebra, where area is treated as a bivector. See C. Doran and A. Lasenby, Geometric Algebra for Physicists, Cambridge University Press, 2005.