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On the Accuracy of Least Squares Algorithms for Estimating Zeros

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***Abstract*— We investigate the accuracy of least-squares-based algorithms for estimating system zeros in the presence of known or unknown order and known or unknown relative degree. Specifically, we use least-squares to estimate the parameters of ARX and μ-Markov models from which zero estimates are calculated directly using the numerator polynomial as well as indirectly using the truncated Laurent expansion or the eigensystem realization algorithm (ERA). To employ the truncated Laurent expansion or ERA, we consider the Markov parameters estimated from the μ-Markov model. Lastly, we investigate the spurious zeros of the μ-Markov model and truncated Laurent expansion to determine to what extent these zeros behave in a predictable manner.**

I. INTRODUCTION

The role of nonminimum-phase (NMP) zeros in systems and control theory [1] motivates the need to develop identification methods for estimating the presence and location of these zeros. In this paper we compare the accuracy of several identification techniques when used specifically to estimate NMP zeros. Although some of these techniques also provide estimates of system poles, the accuracy of the pole estimates is not of interest in this study.

The techniques that we compare numerically for SISO systems include least squares estimation of the numerator coefficients as well as Markov-parameter-based techniques that use either the truncated Laurent expansion or Ho-Kalman realization theory [2], [3]. The problem of estimating Markov parameters is considered in [4] and the references given therein.

For model structures, least squares estimation is used with either the standard IIR model or the μ-Markov model [5], [6], which provides an overparameterized model structure that explicitly displays μ Markov parameters in the numerator polynomial. As shown in [5], [6] these Markov parameters can be estimated consistently (probability-1 convergence under asymptotically large data sets) under certain types of noise. However, the remaining coefficient estimates are not generally consistent, and thus their accuracy can adversely impact the accuracy of the zero estimates.

To reflect practical application, we apply these methods within the context of four cases, namely, known system order and known relative degree; known system order and unknown relative degree; unknown system order and known relative degree; and unknown system order and unknown relative degree. Of particular interest, especially within the context of the adaptive control method developed in [7], [1] is the location of the spurious zeros due to underestimated relative degree. In addition, the μ-Markov model yields spurious zeros due to overparameterization.

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II. PROBLEM FORMULATION

Consider the SISO ARX model

$$A(\mathbf{q})y_0(k) = B(\mathbf{q})u_0(k), \tag{1}$$

where $\mathbf{q}$ is the forward shift operator, u_0 is the true input, and y_0 is the true output. $A(\mathbf{q})$ and $B(\mathbf{q})$ are given by

$$A(\mathbf{q}) = 1 + a_1\mathbf{q}^{-1} + \cdots + a_n\mathbf{q}^{-n} \tag{2}$$

$$B(\mathbf{q}) = b_d\mathbf{q}^{-d} + \cdots + b_n\mathbf{q}^{-n}, \tag{3}$$

where $0 \le d \le n$, $b_d \ne 0$, and d represents the relative degree. We also assume that the true input is known exactly but the output is measured with additive Gaussian white noise w of unknown variance σ_w^2, that is, $y(k) = y_0(k) + w(k)$.

Here we present a numerical comparison of the accuracy of the estimated zeros of $B(\mathbf{q})$. To this end, we consider several least-squares-based techniques when the order n and relative degree d are either known or unknown.

III. MODEL STRUCTURES

A. ARX Model

The ARX model structure is given by

$$\begin{aligned} y(k) = &-\alpha_1 y(k-1) - \cdots - \alpha_{n_{\rm mod}} y(k - n_{\rm mod}) \\ &+ \beta_{d_{\rm mod}} u(k - d_{\rm mod}) + \cdots + \beta_{n_{\rm mod}} u(k - n_{\rm mod}), \end{aligned} \tag{4}$$

where $n_{\rm mod}$ is the model order, $d_{\rm mod}$ is the model relative degree, and $\beta_{d_{\rm mod}}$ is the first nonzero Markov parameter $H_{d_{\rm mod}}$. We assume that an upper bound on n and a lower bound on d are known so that $n_{\rm mod} \ge n$ and $d_{\rm mod} \le d$. The number of spurious zeros is thus $n_{\rm mod} - n + d - d_{\rm mod}$.

B. μ-Markov Model

The μ-Markov model structure is an ARX model with μ-step prediction given by

$$y(k) = -\sum_{i=\mu}^{n_{\rm mod}+\mu-1} \alpha_i' y(k-i) + \sum_{i=d_{\rm mod}}^{d_{\rm mod}+\mu-1} H_i u(k-i) + \sum_{i=d_{\rm mod}+\mu}^{n_{\rm mod}+\mu-1} \beta_i' u(k-i). \tag{5}$$

Note that (5) is an overparameterized ARX model with $\mu \ge 1$ explicitly displayed Markov parameters $H_{d_{\rm mod}}, \ldots, H_{d_{\rm mod}+\mu-1}$. The μ-Markov model contains $n_{\rm mod} - n + d - d_{\rm mod} + \mu$ spurious zeros. As shown in [5], when the input is persistently exciting, the first μ Markov parameters of the μ-Markov model can be estimated consistently using least-squares in the presence of colored process noise of which white output measurement noise is a special case. Note that (4) is a special case of (5) with $\mu = 1$.

IV. Zero-Estimation Methods

We are concerned with least squares estimation of (1) using ARX and μ-Markov models. From the estimated models, zero estimates are obtained in several ways.

A. Numerator Polynomial

When the system order n and relative degree d are known, the zero estimates are given by the roots of the estimated ARX numerator polynomial. However, when $\mu > 1$, the μ-Markov model contains spurious poles and zeros. Letting $B_\mu(\mathbf{q})$ and $A_\mu(\mathbf{q})$ denote the numerator and denominator of the μ-Markov model, respectively, we have that

$$B_\mu(\mathbf{q}) = B(\mathbf{q})p_\mu(\mathbf{q}), \quad A_\mu(\mathbf{q}) = A(\mathbf{q})p_\mu(\mathbf{q}), \qquad (6)$$

where the polynomial

$$p_\mu(\mathbf{q}) \triangleq \mathbf{q}^{-(\mu-1)} + p_1\mathbf{q}^{-(\mu-2)} + \cdots + p_{\mu-2}\mathbf{q}^{-1} + p_{\mu-1}, \quad (7)$$

is uniquely determined by

$$\begin{bmatrix} p_{\mu-1} \\ \vdots \\ p_1 \end{bmatrix} = - \begin{bmatrix} a_0 & & 0 \\ \vdots & \ddots & \\ a_{\mu-2} & \cdots & a_0 \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_{\mu-1} \end{bmatrix}, \qquad (8)$$

with $a_i = 0$ for $i > n$ [8]. In contrast to (6), let $B_{\rm ARX}(\mathbf{q})$ and $A_{\rm ARX}(\mathbf{q})$ denote the numerator and denominator polynomials of the ARX model, respectively, with $n_{\rm mod} > n$. In this case, $B_{\rm ARX}(\mathbf{q}) = B(\mathbf{q})r(\mathbf{q})$ and $A_{\rm ARX}(\mathbf{q}) = A(\mathbf{q})r(\mathbf{q})$, where $r(\mathbf{q})$ is an arbitrary polynomial of order $n_{\rm mod} - n$.

B. Truncated Laurent Expansion

Let H_i denote the i^{th} Markov parameter of the discrete transfer function $G(\mathbf{z})$. Then the Laurent expansion of $G(\mathbf{z})$ about $\mathbf{z} = \infty$ is given by

$$G(\mathbf{z}) = \sum_{i=0}^{\infty} \mathbf{z}^{-i} H_i, \qquad (9)$$

which converges uniformly on all compact subsets of $\{\mathbf{z} : |\mathbf{z}| > \rho_{\max}\}$, where $\rho_{\max}$ denotes the radius of the largest pole of $G(\mathbf{z})$ [9], [1]. Truncating the Laurent expansion (9), we obtain the approximate transfer function $G_N(\mathbf{z}) = \sum_{i=0}^{N} \mathbf{z}^{-i} H_i$, whose numerator is the N^{th}*-order Markov parameter polynomial* (MPP). Since $G_N(\mathbf{z})$ approximates $G(\mathbf{z})$ for $|\mathbf{z}| > \rho_{\max}$, we examine how well the roots of the N^{th}-order MPP with radius greater than $\rho_{\max}$ approximate the true system zeros. Consider the ARX model

$$G(\mathbf{q}) = (\mathbf{q} - 0.9) / (\mathbf{q} - 0.8). \qquad (10)$$

Letting z denote the largest root of the N^{th}-order MPP and z_0 denote the true zero of $G(\mathbf{q})$, Figure 1 shows that, as the order of the MPP increases, the absolute value of the error between z and z_0 tends toward zero. This is a result of $G(\mathbf{z})$ being analytic outside the disk of radius $\rho_{\max}$.

C. Realization

The eigensystem realization algorithm (ERA) requires at least $2n_{\rm mod} + 1$ Markov parameters to produce a state space model of order $n_{\rm mod}$ [2]. System zeros can then be calculated using the reconstructed state space model. Here we use $2n_{\rm mod} + 1$ Markov parameters estimated with a μ-Markov model. Note that when more than $2n_{\rm mod} + 1$ Markov parameters are used in ERA, alternative methods are available [2], [3], although these are not considered here.

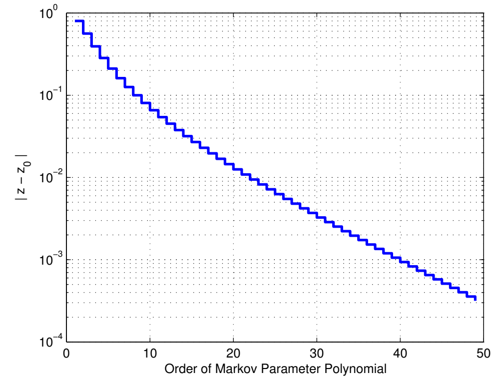

Fig. 1. Accuracy of the approximate zero of (10) using the MPP.

V. Model Uncertainty

We consider cases in which the model order and relative degree are either known or unknown. For the following examples we consider the ARX model

$$G_{\rm ARX}(\mathbf{q}) = \frac{(\mathbf{q} - 1.2)(\mathbf{q} - 0.8)(\mathbf{q} - 0.1)}{(\mathbf{q} - 0.4)(\mathbf{q} - 0.6)(\mathbf{q}^2 + 0.36)}, \qquad (11)$$

which has the NMP zero $z_1 = 1.2$, two minimum-phase zeros $z_2 = 0.8$ and $z_3 = 0.1$, and a relative degree $d = 1$. We consider a white, Gaussian input signal with zero mean and variance $\sigma_u^2 = 1$. We also consider white, Gaussian output measurement noise with zero mean and variance $\sigma_w^2 = 0.04$, which is uncorrelated with the input. The results are averaged over N trials, each with a new realization of w, to obtain statistical regularity. Except where noted, we set $N = 2,000$.

We choose the performance metric

$$\varepsilon_i = \frac{1}{N} \sum_{j=1}^{N} \min_{z \in \hat{Z}(j)} |z - z_i|, \qquad (12)$$

where $i = \{1, 2, 3\}$, ε_i represents the average estimation error of z_i, z_i represents the true zero of (11), and $\hat{Z}(j)$ represents the set of estimated zeros for the j^{th} data set. For instance, ε_1 represents the average estimation error of the zero at 1.2. Note that the performance metric ε_i is not affected by spurious zeros since it considers only the zero estimate closest to the true zero.

A. Known Order, Known Relative Degree

Assume that the system order and relative degree of (11) are known, that is, $n_{\rm mod} = n$ and $d_{\rm mod} = d$. We consider the accuracy of the zero estimates according to the performance metric (12) for the ARX and ERA reconstructed models as well as the effect of increasing μ on the accuracy of the NMP zero estimate for both the μ-Markov and truncated Laurent approach. For all of these tests we consider the effect of increasing data to examine the consistency of the estimates.

We begin by comparing the accuracy of the zero estimates using an ARX model as more data are made available. Figure 2 shows that the zeros of the estimated ARX model are not consistent and that the accuracy of the estimates are inversely proportional to the magnitude of the zeros being estimated.

Next, we consider the NMP zero estimate extracted from the MPP. Figure 3 suggests that, for each μ, the NMP zero estimate is not consistent, although the error decreases with increasing μ and the number of samples.

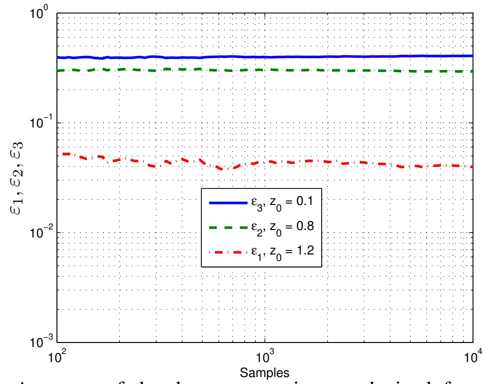

Fig. 2. Accuracy of the three zero estimates obtained from the ARX numerator when the relative degree and order are both known.

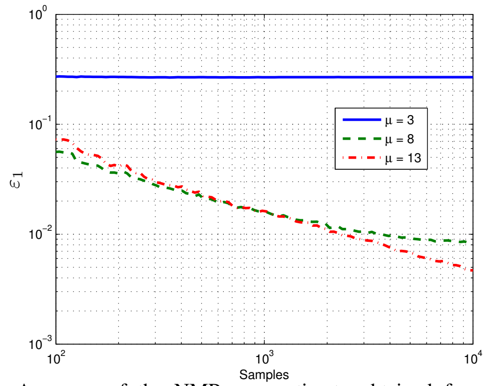

Fig. 3. Accuracy of the NMP zero estimate obtained from the MPP for various polynomial orders when both the relative degree and order are known.

Third, we consider the NMP zero estimate obtained from the estimated μ-Markov numerator. Figure 4 suggests consistency with improved estimation accuracy for the NMP zero compared to the ARX and truncated Laurent approaches. The results also appear to be moderately insensitive to the choice of μ.

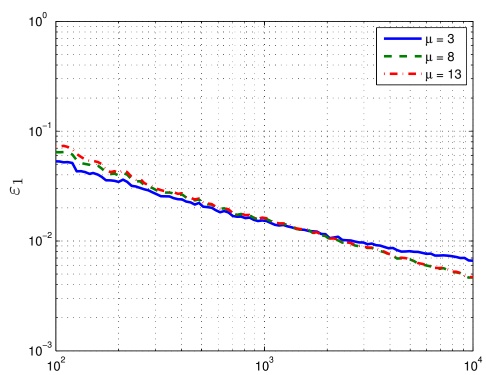

Fig. 4. Accuracy of the NMP zero estimate obtained from the μ-Markov numerator for various μ when both the relative degree and order are known.

Lastly, we consider the accuracy of the zero estimates obtained from the reconstructed ERA model. Figure 5 indicates consistency of the estimates as well as comparable or better accuracy than the previous approaches.

Since the ARX model fails to provide consistent zero estimates and is the least accurate method when the order and relative degree are known, we do not consider it for the remaining cases.

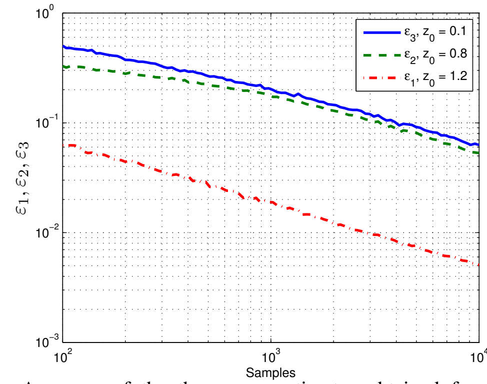

Fig. 5. Accuracy of the three zero estimates obtained from the ERA reconstructed numerator when both the relative degree and order are known.

B. Known Order, Unknown Relative Degree

We now assume that the order is known but the relative degree is unknown. Hence $n_{\rm mod} = n$ and $d_{\rm mod} = 0 < d$.

First, we consider the NMP zero estimate extracted from the MPP. Figure 6 indicates that, for each μ, the NMP zero estimate is not consistent, although the error appears to decrease with increasing μ and the number of samples.

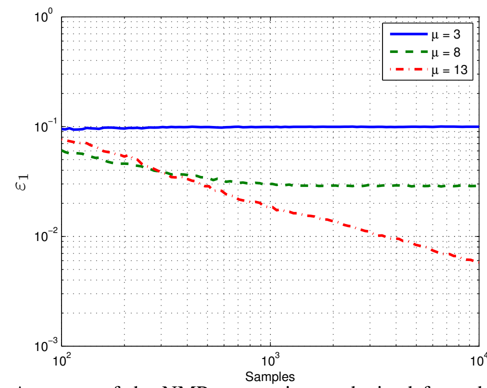

Fig. 6. Accuracy of the NMP zero estimate obtained from the MPP for various polynomial orders when the relative degree is unknown and the order is known.

Second, we consider the NMP zero estimate from the estimated μ-Markov numerator. Figure 7 suggests that the estimate is consistent for all μ and more accurate than the MPP approach for small μ. The results appear to be insensitive to the choice of μ.

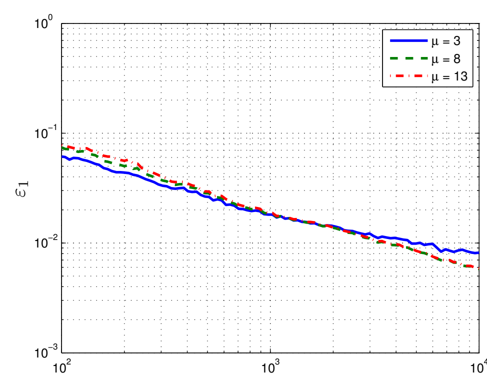

Fig. 7. Accuracy of the NMP zero estimate obtained from the μ-Markov numerator for various μ with unknown relative degree and known order.

Lastly, we consider the three zero estimates obtained from the reconstructed ERA model. Figure 8 suggests that the zero estimates are consistent although with less accuracy than the known relative degree case shown in Figure 5.

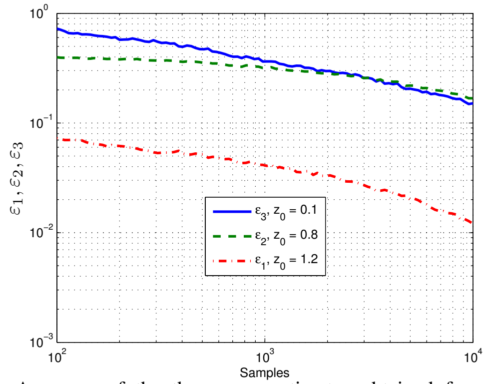

Fig. 8. Accuracy of the three zero estimates obtained from the ERA reconstructed numerator when the relative degree is unknown and the order is known.

C. Unknown Order, Known Relative Degree

Now assume that the order is unknown but the relative degree is known. Hence $n_{\rm mod} \geq n$ and $d_{\rm mod} = d$.

First, we consider the NMP zero estimate obtained from the estimated μ-Markov numerator with $\mu = 5$ for various model orders. Figure 9 suggests that the estimate is consistent for all model orders with similar accuracy to the case of unknown relative degree shown in Figure 7. The results also appear to be insensitive to the model order.

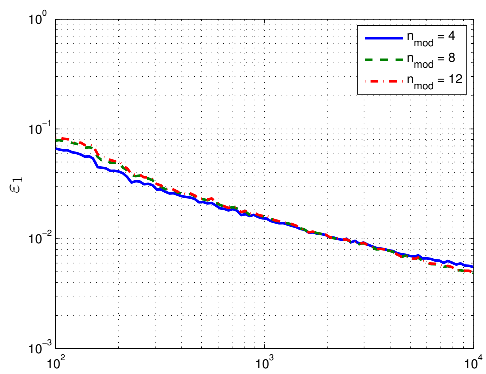

Fig. 9. Accuracy of the NMP zero estimate obtained from the μ-Markov numerator for various $n_{\rm mod}$ when $\mu = 5$, the relative degree is known, and the order is unknown.

Next, we consider the NMP zero estimate obtained from the ERA reconstructed model. To show the effect of increasing model order, $2n_{\rm mod}+1$ Markov parameters must be estimated, meaning that, for $n_{\rm mod} = 12$, approximately three times as many Markov parameters are estimated compared to the case $n_{\rm mod} = 4$. Figure 10 indicates consistency of the estimate with slightly increased accuracy compared to the unknown relative degree case shown in Figure 8.

D. Unknown Order, Unknown Relative Degree

Finally, we assume that both the order n and relative degree are unknown. Hence we choose $n_{\rm mod} \geq n$ and $d_{\rm mod} = 0$. We compare the accuracy of the NMP zero estimate for the 5-Markov, 10^{th}-order truncated Laurent, and ERA approaches. The results are averaged over 10,000 trials with 10,000 samples used in each data set. Figure 11

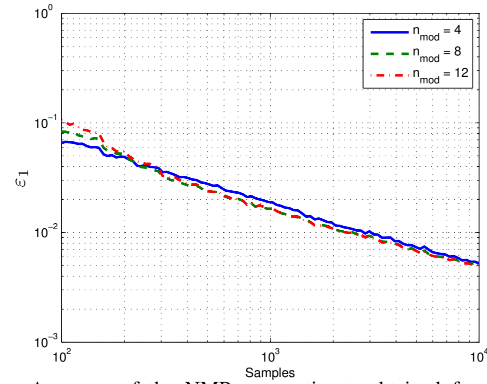

Fig. 10. Accuracy of the NMP zero estimate obtained from the ERA reconstructed numerator when the relative degree is known and the order is unknown.

indicates that the accuracy of the truncated Laurent and μ-Markov approaches are approximately independent of model order and that the ERA NMP zero estimate is comparable to the other approaches when $n_{\rm mod} > n$.

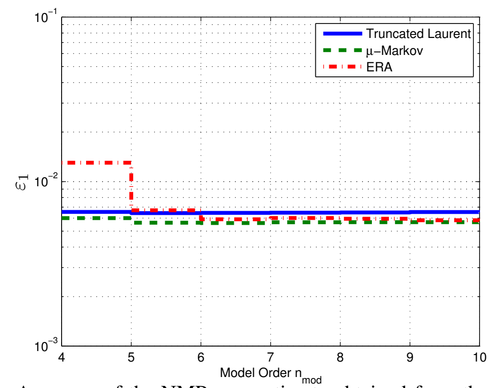

Fig. 11. Accuracy of the NMP zero estimate obtained from the μ-Markov numerator for various $n_{\rm mod}$ when $\mu = 5$, the relative degree is known, and the order is unknown.

VI. Spurious Zeros

We examine the spurious zeros of $B_\mu(\mathbf{q})$ and the MPP when either $\mu > 1$ or $d_{\rm mod} = 0 < d$. We first look at the true μ-Markov model with $\mu > 1$, which introduces spurious zeros. Next, we investigate the MPP with $\mu > 1$, which we test with the true Markov parameters, the Markov parameters corrupted by independent noise, and the estimated Markov parameters. To help us understand the locations of the spurious zeros when noise is introduced, we compare the MPP to a random-coefficient polynomial. Lastly, we look at the spurious zeros of the MPP when $d_{\rm mod} = 0 < d$.

We begin by examining the effect of $p_\mu(\mathbf{q})$ on the roots of $B_\mu(\mathbf{q})$ in (6) with a high μ. Figure 12 shows the roots of $B_\mu(\mathbf{q})$ with $\mu = 50$ and indicates that the numerator of the μ-Markov model contains roots near $\rho_{\rm max}$.

Next, we investigate the roots of the true MPP for the nominal model (11). Letting z_0 denote the true zeros and $z_{\rm est}$ denote the roots of the MPP, Figure 13 shows the roots of the 50^{th}-order MPP. In Figure 13 we can see that only the roots whose magnitude are larger than $\rho_{\rm max}$ are approximated. The spurious zeros form a ring of radius $\rho_{\rm max}$.

Next, we note that the estimated Markov parameters are themselves random variables. As an intermediate step in

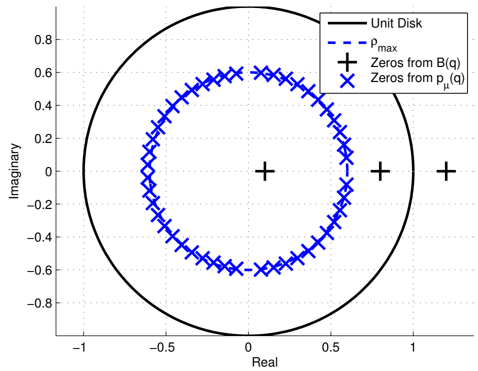

Fig. 12. Roots of $B_\mu(\mathbf{q})$ with $\mu = 50$. The additional coefficients in the μ-Markov numerator due to p_μ yield spurious zeros.

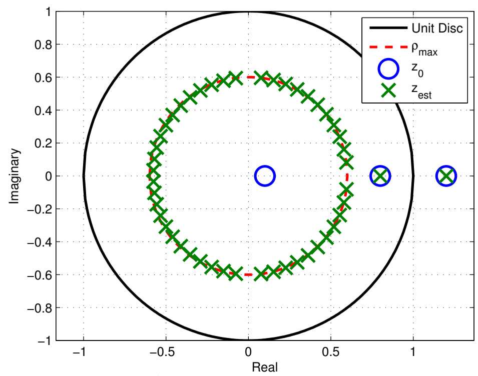

Fig. 13. Roots of the 50^{th}-order MPP compared to the true zeros of (11).

understanding what happens to the roots of the estimated MPP, we thus consider the mean radius of the roots of n^{th}-order random-coefficient polynomials whose coefficients are independent zero-mean Gaussian random variables with standard deviation $\sigma = 1$. Figure 14 shows a histogram of the mean radius over 10,000 trials, which indicates that, as the order of the polynomial increases, the roots of a random-coefficient polynomial approach the unit disk. Note that Figure 14 displays a numeric result similar to the analytic result of [10]. Furthermore, fixing the polynomial order n at 15 and comparing the mean radius of the roots for various standard deviations, Figure 15 indicates that the mean radius of the roots is independent of the standard deviation of the coefficients.

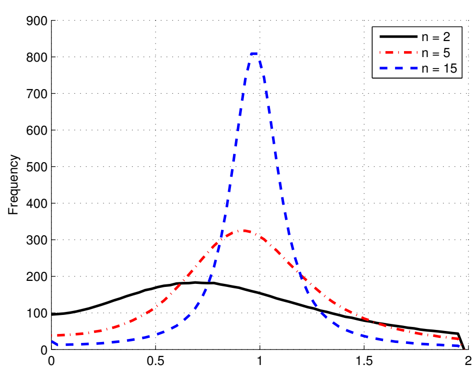

Fig. 14. Histogram of the mean radius of the roots of random-coefficient polynomials of varying order with standard deviation $\sigma = 1$.

Next, we consider the effect of additive Gaussian noise on the roots of the n^{th}-order MPP of (11). Specifically, we compare the radius of the ring of spurious roots due to noise of varying signal-to-noise ratios (SNRs), where the SNR is defined to be the ratio of RMS values. Taking the MPP to be the signal and w to be the noise sequence, the SNR is given by $\text{SNR} \triangleq \sqrt{(\sum_{i=0}^{N} H_i^2)/(\sum_{i=0}^{N} w_i^2)}$. Averaging the results over 1000 trials, Figure 16 indicates that the ring of spurious zeros moves from $\rho_{\max}$ to the unit disk as the polynomial order increases. Furthermore, the speed with which the ring moves toward the unit disk appears to be proportional to the amplitude of the noise superimposed on the MPP.

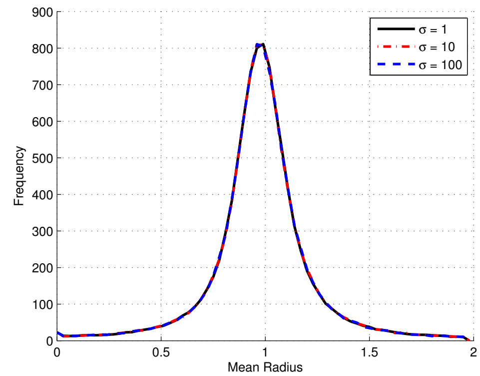

Fig. 15. Histogram of the mean radius of the roots of 15^{th}-order random-coefficient polynomials of varying standard deviation.

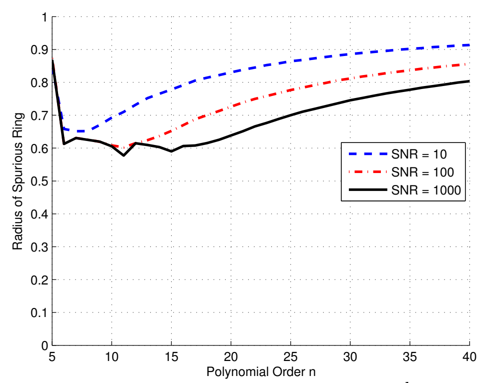

Fig. 16. Mean radius of the spurious roots of the n^{th}-order MPP when additive Gaussian noise of varying SNR is superimposed on the coefficients.

We consider the spurious roots of the identified MPP in the presence of varying levels of output measurement noise. Since the MPP was shown in Figure 13 to capture the zeros of (11) larger than $\rho_{\max}$, we compute the mean radius of the estimated MPP as $\bar{\rho} = \left(\sum_{i=1}^{n} |\hat{z}| - [1.2 + 0.8]\right)/\left(n - 2\right)$, where $\bar{\rho}$ denotes the average radius, n denotes the order of the polynomial, $\hat{z}$ denotes the roots of the identified MPP, and the adjustment terms account for the approximation of the two zeros of radius larger than $\rho_{\max}$. Averaging the results over 1000 trials and using 1000 samples per trial, Figure 17 indicates that the radius of the ring of spurious zeros approaches the unit disk as the order of the MPP increases. Furthermore, Figure 17 suggests that the amplitude of the spurious ring is proportional to the variance of the noise and the order of the MPP.

As one example of the roots of the identified 40^{th}-order MPP in the presence of output noise of variance $\sigma_w^2 = 1$, Figure 18 shows that the radius of the spurious ring is located at approximately 0.9. Additionally, Figure 18 suggests that zeros in the range $\rho_{\max} < \rho < 1$ are not approximated by the MPP when either the variance of the noise or the polynomial order is sufficiently high.

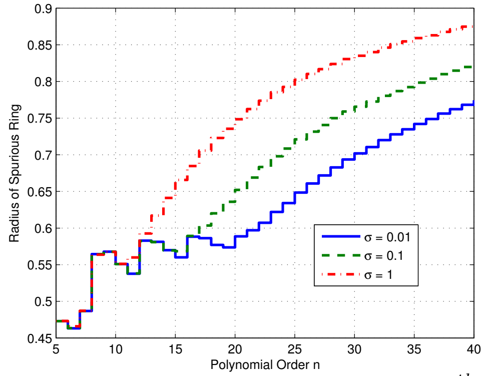

Fig. 17. Mean radius of the spurious roots of the identified n^{th}-order MPP in the presence of varying levels of output measurement noise.

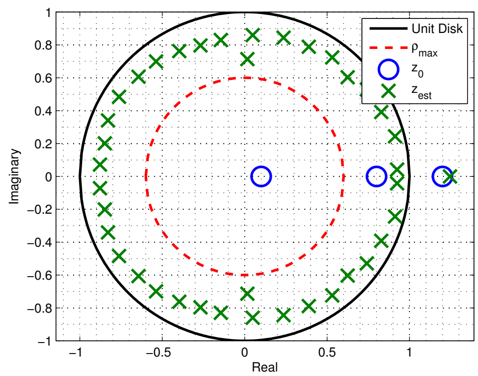

Fig. 18. Roots of an identified 40^{th}-order MPP and the true zeros of (11).

Lastly, we look at the effects of underestimating the relative degree. We consider the system given by

$$G(\mathbf{q}) = \frac{G_{\mathrm{ARX}}(\mathbf{q})}{(\mathbf{q}+0.1)\,(\mathbf{q}+0.4)\,(\mathbf{q}+0.6)\,(\mathbf{q}+0.8)}, \qquad (13)$$

which has a relative degree $d=5$, implying that $H_0 = \cdots = H_4 = 0$. We simulate the system with a white Gaussian input of zero mean and variance $\sigma^2 = 1$. Setting $d_{\mathrm{mod}} = 0$ and corrupting the output with white Gaussian noise of zero mean, the first 5 estimated Markov parameters are nonzero, producing 5 spurious zeros. We investigate the location of these spurious zeros by running trials at varying noise levels, with $\mu = 6$ to capture all 5 spurious zeros. The radii of all 5 spurious zeros from each trial are averaged, and Figure 19 shows a histogram of the mean radius from 20,000 trials.

From Figure 19, the radii of the spurious zeros are greater than 1 and tend to group together at the same radius, generally forming a ring. With increasing noise, the radius of the ring of spurious zeros approaches the unit disk.

VII. Conclusions

We investigated the accuracy of least-squares-based algorithms for estimating system zeros in the presence of known or unknown order and relative degree. Specifically, we used least-squares to estimate the parameters of ARX and μ-Markov models from which zero estimates could be calculated directly using the numerator polynomial as well

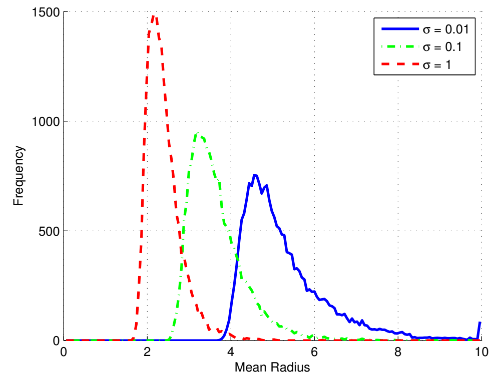

Fig. 19. Histogram of the mean radius of spurious zeros due to underestimating the relative degree of (13), with 20,000 trials, 10,000 samples, $n_{\mathrm{mod}} = n = 8$, and $\mu = 6$.

as indirectly using the truncated Laurent expansion or the eigensystem realization algorithm (ERA). To employ the truncated Laurent expansion or ERA, we considered the Markov parameters estimated from the μ-Markov model.

The numerical results suggest that both the μ-Markov and ERA techniques consistently estimate the NMP zero under all conditions, although the ERA technique requires estimating $2n_{\mathrm{mod}}+1$ Markov parameters and yields slightly less accurate estimates. The ARX and truncated Laurent expansion both seem to yield inconsistent estimates of the zeros, although the truncated Laurent approach can be made as accurate as the μ-Markov and ERA approaches by choosing a sufficiently high order MPP.

Finally, we investigated the spurious zeros of the μ-Markov model and truncated Laurent expansion. Numerical results suggest that when the Markov parameters are known, the spurious zeros form at ring at $\rho_{\max}$. However, as the order of the MPP or variance of the measurement noise increases, the ring of spurious zeros approaches the unit disk. Hence, for sufficiently large measurement noise or polynomial order, zeros in the range $\rho_{\max} < \rho < 1$ are no longer contained in the truncated Laurent expansion approximation.

References

[1] M. A. Santillo and D. S. Bernstein, "Adaptive Control Based on Retrospective Cost Optimization," *AIAA J. Guid. Contr. Dyn.*, vol. 33, pp. 289–304, 2010.

[2] J. N. Juang, *Applied System Identification*. Prentice-Hall, 1994.

[3] P. Vacher, "Three approaches for system identification by impulse response fitting," in 15^{th} *IFAC Symposium on System Identification*, Saint-Malo, France, July 2009, pp. 1644–1649.

[4] H. P.-M. R. J. F. M. S. Fledderjohn, M. S. Holzel and D. S. Bernstein, "A comparison of least squares algorithms for estimating markov parameters," in *submitted to Amer. Contr. Conf.*, Baltimore, MD, 2010.

[5] B. H. M. Kamrunnahar and D. G. Fisher, "Estimation of markov parameters and time-delay/interactor matrix," *Chemical Engineering Science*, September 2000.

[6] T. H. Van Pelt and D. S. Bernstein, "Least squares identification using μ-markov parameterizations," in *Conf. Dec. Contr.*, 1998, pp. 618–619.

[7] M. A. Santillo and D. S. Bernstein, "A retrospective correction filter for discrete-time adaptive control of nonminimum phase systems," in *Proc. Conf. Dec. Contr.*, 2008, pp. 690–695.

[8] T. H. Van Pelt, *Quadratically Constrained Least Squares Identification and Nonlinear System Identification using Hammerstein/Nonlinear Feedback Models*. Ph.D. thesis: University of Michigan, 2000.

[9] J. E. Marsden, *Basic Complex Analysis*. W.H. Freeman, 1973.

[10] C. P. Hughes and A. Nikeghbali, "The zeros of random polynomials cluster uniformly near the unit circle," *Compositio Mathematica,* **144**, pp. 734–746, 2008.