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Geometric Insight and Structure Algorithms for Unknown-State, Unknown-Input Reconstruction in Linear Multivariable Systems

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Abstract: An algebraic approach to the synthesis of a dynamic system that reconstructs the generic inaccessible input of a discrete-time linear multivariable system with unknown initial state is discussed. The method devised exploits geometric properties of key subspaces for the original system and algebraic properties of the Moore-Penrose inverse of Toeplitz matrices related to the algorithms for computing those subspaces. Nonminimum-phase invariant zeros are taken into account implicitly with the proposed techniques, while minimum-phase invariant zeros require that a filter be inserted between the original system and the reconstructor. The procedure applies to either strictly-proper or non-strictly-proper systems.

Keywords: state and input reconstruction; invariant zeros; discrete-time linear systems.

1. INTRODUCTION

The problem of reconstructing the inaccessible inputs to linear multivariable systems with unknown initial states has recently received renewed attention, mainly due to its connections with fault detection, despite its intrinsic limitation related to the presence of invariant zeros. As is shown, e.g., in (Basile and Marro, 1992, Sect. 4.4), for any invariant zero structure, there exists at least one initial state such that, when the modes of the invariant zero are suitably injected into the system, the output is zero. Necessary and sufficient conditions for a strictly-proper continuous-time system to be completely unknown-state, unknown-input reconstructible by measurement differentiation were also given in Basile and Marro (1992). Further contributions can be found in Hou and Patton (1998); Corless and Tu (1998); Xiong and Saif (2003), where specific applications to fault detection were considered.

The discrete-time case was first examined in Floquet and Barbot (2006), where an algorithm for verifying whether a left-invertible system without invariant zeros is state and input reconstructible, possibly with some delay, was proposed. Algorithms for possibly-delayed state and input reconstruction in discrete-time systems were formulated in terms of Markov parameters in Kirtikar et al. (2011). Those algorithms are based on least-square techniques and operate so that the error in the reconstruction of a generic input converges to zero in minimum-phase systems, is constant in systems with simple invariant zeros on the unit circle, and diverges in systems either with multiple invariant zeros on the unit circle or with invariant zeros in the open set outside the unit disc. In Marro and Zattoni (2010), unknown-state, unknown-input reconstruction in strictly-proper, discrete-time systems was tackled by means of the geometric approach. The algorithms in Marro and Zattoni (2010), which presuppose cancellation of the zeros in the open unit disc, guarantee that the error in the reconstruction of a generic input to a strictly-proper, discrete-time system which also satisfies the conditions of being controllable, observable and left-invertible converges to zero not only if the invariant zeros are in the open unit disc, but also if they lie outside the closed unit disc. The results of Marro and Zattoni (2010) are derived with duality arguments from the reasoning that yields the control policies steering the state of the system along trajectories in the maximal controlled invariant subspace contained in the kernel of the output and the minimal conditioned invariant subspace containing the image of the input. Extending those techniques to systems with feedthrough terms requires control strategies be devised so as to drive the state along trajectories in the maximal output-nulling controlled invariant subspace and the minimal input-containing conditioned invariant subspace.

The approach adopted in this work for synthesizing those control strategies is based on some new algorithms for computing the abovementioned subspaces, inspired by those of Silverman (1976). In such algorithms, possible feedthrough matrices are directly included in Toeplitz matrices of Markov parameters, so that the intermediate step which consists in reducing non-strictly-proper systems to equivalent strictly-proper systems and which plays a key role in Marro et al. (2010), is avoided. Hence, the procedure for the synthesis of the unknown-state, unknown-input reconstructor directly applies to systems either with or without feedthrough terms.

2. GEOMETRIC APPROACH BACKGROUND

Consider the discrete-time linear time-invariant system

$$x_{t+1} = A\,x_t + B\,u_t, \tag{1}$$

$$y_t = C\,x_t + D\,u_t, \tag{2}$$

where $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, with $p \leq n$ and $q \leq n$, are the state, the input, the output, respectively. Du is the feedthrough term. A, B, C, D are constant real matrices of appropriate dimensions. $[\begin{smallmatrix} B \\ D \end{smallmatrix}]$ and $[C\,D]$ are full-rank matrices. The set $\mathcal{U}_f$ of the admissible input sequences is the set of all sequences with bounded values in $\mathbb{R}^p$. Geometric objects extensively used in this work are $\mathcal{B}$, the image of B, $\mathcal{C}$, the kernel of C, $\mathcal{R} = \min \mathcal{J}(A, \mathcal{B})$, the minimal A-invariant subspace containing $\mathcal{B}$ or the reachable subspace of (A, B), $\mathcal{Q} = \max \mathcal{J}(A, \mathcal{C})$, the maximal A-invariant subspace contained in $\mathcal{C}$ or the unobservable subspace of (A, C), $\mathcal{V}^*$ or $\max \mathcal{V}(A, B, C, D)$, the maximal output-nulling controlled invariant subspace of (1), (2), $\mathcal{S}^*$ or $\min \mathcal{S}(A, B, C, D)$, the minimal input-containing conditioned invariant subspace of (1), (2), $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$, the reachability subspace on $\mathcal{V}^*$. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is an output-nulling controlled invariant subspace of (1), (2) if and only if at least one linear map F exists, such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker(C+DF)$. A subspace $\mathcal{S} \subseteq \mathbb{R}^n$ is an input-containing conditioned invariant subspace of (1), (2) if and only if at least one linear map G exists, such that $(A+GC)\mathcal{S} \subseteq \mathcal{S}$ and $\mathcal{S} \supseteq \operatorname{im}(B+GD)$. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C+DF)$, then $(A+BF)\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{R}_{\mathcal{V}^*}$ and $\mathcal{R}_{\mathcal{V}^*} \subseteq \ker(C+DF)$ hold. The spectrum of $(A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}$ is assignable. The spectrum of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$ is fixed. The set $\sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}})$ is also known as the set of the internal unassignable eigenvalues of $\mathcal{V}^*$ or as the set $\mathcal{Z}(A, B, C, D)$ of the invariant zeros of (1), (2). The invariant zeros of (1), (2) in $\mathbb{C}^{\odot}$ are called the minimum-phase invariant zeros of (1), (2) and their set is denoted by $\mathcal{Z}_{MP}(A, B, C, D)$. Similarly, the invariant zeros of (1), (2) in $\mathbb{C}^{\otimes}$ are called the nonminimum-phase invariant zeros of (1), (2) and their set is denoted by $\mathcal{Z}_{NMP}(A, B, C, D)$. A geometric condition equivalent to the property of system (1), (2) of being right-invertible is $\mathcal{V}^* + \mathcal{S}^* = \mathbb{R}^n$. A geometric condition equivalent to the property of system (1), (2) of being left-invertible is $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$.

3. PROBLEM STATEMENT

Consider system (1), (2), where u is the unknown input and y the measured output. The initial state x_0 is unknown. Our goal is to provide an algebraic procedure for the synthesis of a discrete-time linear time-invariant system, henceforth called the unknown-state, unknown-input reconstructor, which, having the measured output of the original system as input, produces as output both the state and the input of the original system, with an admissible delay of a certain number of steps. The solution is presented under the following assumptions:

$\mathcal{A}1$. $\mathcal{R} = \mathcal{X} = \mathbb{R}^n$ and $\mathcal{Q} = \{0\}$;
$\mathcal{A}2$. $\mathcal{V}^* \cap \mathcal{S}^* = \{0\}$;
$\mathcal{A}3$. $\mathcal{Z}(A, B, C, D) \subset \mathbb{C}^{\odot} \cup \mathbb{C}^{\otimes}$, where $\mathbb{C}^{\odot}$ and $\mathbb{C}^{\otimes}$ are the open unit disc and the open set outside the unit disc of $\mathbb{C}$.

A discrete-time linear time-invariant system which, as will be shown, solves the unknown-state, unknown-input reconstruction problem for a system like (1), (2), satisfying Assumptions $\mathcal{A}1$–$\mathcal{A}3$, has the following structure. If the original system has any minimum-phase invariant zeros, these are cancelled by a filter which is permanently connected in cascade to the original system, so as to guarantee the synchronization of the respective unknown states. A finite impulse response (FIR) system processes the filter output to provide the initial state and the subsequent state trajectory of the original system. The reconstructed state trajectory will be delayed with respect to the original one by a number of steps equal to the length of the FIR system window. The latter will be chosen in connection with some properties of the original system. If the original system does not have any minimum-phase invariant zeros, the FIR system is directly fed with the measured output. A dynamic system processes the reconstructed state trajectory and the measured output, consistently delayed, in order to reproduce the unknown input. The reconstructed input will be delayed by one further step.

4. SYSTEMS WITH NO INVARIANT ZEROS OR NONMINIMUM-PHASE INVARIANT ZEROS ONLY

This section is focused on the synthesis of the FIR system and the input reconstructor, on the assumption that the system either does not have any invariant zeros or has nonminimum-phase invariant zeros only. The FIR system convolution profiles will be determined in the dual context of control, since this approach lends itself to an intuitive intepretation in terms of subspaces and trajectories.

4.1 The FIR System for the Dual Control Problem

Consider a system like (1), (2) and assume $\mathcal{A}1$,
$\mathcal{A}2'$. $\mathcal{V}^* + \mathcal{S}^* = \mathcal{X} = \mathbb{R}^n$,
$\mathcal{A}3'$. $\mathcal{Z}(A, B, C, D) \subset \mathbb{C}^{\otimes}$.

Consider the problem of steering the state from the origin to an assigned final state, while maintaining zero output until the last step but one. As will be shown in the following, a feedforward FIR system having the desired final state as pulse input and the system input as output can solve this problem either exactly, if the given system does not have any invariant zeros, or up to an arbitrary degree of accuracy, if the given system has nonminimum-phase invariant zeros only. Let the FIR system be described by

$$u_t = \sum_{\ell=0}^{N-1} \Phi_\ell\, h_{t-\ell}, \quad t \in \mathbb{Z}, \tag{3}$$

where $N \in \mathbb{Z}^+$ is the FIR system time window and Φ_ℓ, $\ell = 0, 1, \ldots, N-1$, is the matrix of the convolution profiles. Let $h_t = x_f \delta_t$, $t \in \mathbb{Z}$, where $x_f \in \mathbb{R}^n$ is the desired final state for the original system and δ is the discrete unit pulse. The equivalent ISO-equations are

$$x_{fir,t+1} = A_{fir}\, x_{fir,t} + B_{fir}\, h_t, \tag{4}$$

$$u_t = C_{fir}\, x_{fir,t} + D_{fir}\, h_t, \tag{5}$$

where

$$A_{fir} = \begin{bmatrix} O & I_n & O & \ldots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & O \\ \vdots & & & \ddots & I_n \\ O & \ldots & \ldots & \ldots & O \end{bmatrix}, \quad B_{fir} = \begin{bmatrix} O \\ \vdots \\ \vdots \\ O \\ I_n \end{bmatrix}, \tag{6}$$

$$C_{fir} = [\,\Phi_{N-1}\ \Phi_{N-2}\ \ldots\ \Phi_1\,], \quad D_{fir} = \Phi_0. \tag{7}$$

The FIR system convolution profiles will be obtained through the solution of a simple algebraic problem, discussed in the light of the geometric properties of the basic subspaces of the original system revisited in Appendix A. With the notation introduced in (A.1), the problem of finding a control sequence u_t, with $t = 0, 1, \ldots, N-1$, driving the state of (1), (2) from $x_0 = 0$ to $x_N = x_f$, with $y_t = 0$ for $t = 0, 1, \ldots, N-1$, reduces to the problem of finding U_N that solves the algebraic equation

$$\begin{bmatrix} 0 \\ x_f \end{bmatrix} = M_N U_N, \qquad (8)$$

where $M_N = \left[\begin{smallmatrix} B_N \\ L_N \end{smallmatrix} \right]$, with L_N and B_N respectively defined by (A.2) and (A.4). In fact, (8) is a compact writing for (A.5) and (A.6) with $Y_N = 0$ and $x_N = x_f$. Systems without invariant zeros and systems with nonminimum-phase zeros only will henceforth be considered separately.

Theorem 1. Consider system (1), (2). Let $\mathcal{A}1$, $\mathcal{A}2'$, $\mathcal{A}3'$ hold with $\mathcal{Z}(A,B,C,D) = \emptyset$. Let $N = \rho$, with ρ defined by Algorithm 20 in Appendix A. Then, (8) is solvable and $U_N = M_N^\dagger \left[\begin{smallmatrix} 0 \\ x_f \end{smallmatrix} \right]$ is the solution with the least Euclidean norm.

Proof. The relations $\mathcal{Z}(A,B,C,D) = \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}^*_{\mathcal{V}}})$, where F is such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$, reviewed in Section 2, along with assumption $\mathcal{Z}(A,B,C,D) = \emptyset$ imply $\mathcal{V}^* = \mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{S}^*$. Moreover, in light of $\mathcal{A}2'$, one gets $\mathcal{V}^* + \mathcal{S}^* = \mathcal{S}^* = \mathcal{X} = \mathbb{R}^n$. Hence, $x_f \in \mathcal{S}^*$. As reviewed in Appendix A, $\mathcal{S}^*$ is the maximum set of states reachable from the origin along trajectories that give rise to zero output until the last step but one. Since N is set equal to ρ, the number of steps for computing $\mathcal{S}^*$, $\left[\begin{smallmatrix} 0 \\ x_f \end{smallmatrix} \right] \in \operatorname{im} M_N$. Therefore, (8) is solvable and the thesis is proved by virtue of the properties of the Moore-Penrose inverse. □

Remark 2. Under the assumptions of Theorem 1, (8) is solvable for any $N \in \mathbb{Z}^+$ such that $N \geq \rho$. However, as will be shown in Section 4.2, in systems without invariant zeros, N is also the reconstruction delay in the dual problem. Therefore, the least value is the most interesting.

Remark 3. Under the assumptions of Theorem 1, if $\ker M_N \neq \{0\}$, the solution of (8) is nonunique. The set of all solutions is $U_N = M_N^\dagger \left[\begin{smallmatrix} 0 \\ x_f \end{smallmatrix} \right] + \Omega\,\gamma$, where Ω is a basis matrix of $\ker M_N$ and γ is a parameter vector.

Theorem 4. Consider system (1), (2). Let $\mathcal{A}1$, $\mathcal{A}2'$, $\mathcal{A}3'$ hold with $\mathcal{Z}(A,B,C,D) \neq \emptyset$. Let $N \in \mathbb{Z}^+$ be such that $\rho \leq N < \infty$, with ρ defined by Algorithm 20 in Appendix A. Then, (8) is not solvable and $U_N = M_N^\dagger \left[\begin{smallmatrix} 0 \\ x_f \end{smallmatrix} \right]$ is the vector with the least Euclidean norm such that

$$\left\| \begin{bmatrix} 0 \\ x_f \end{bmatrix} - M_N U_N \right\|_2 \neq 0 \qquad (9)$$

is minimal.

Proof. The relations $\mathcal{Z}(A,B,C,D) = \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}^*_{\mathcal{V}}})$, where F is such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$, along with assumptions $\mathcal{A}3'$ and $\mathcal{Z}(A,B,C,D) \neq \emptyset$ imply $\mathcal{V}^* = \mathcal{R}_{\mathcal{V}^*} \oplus \mathcal{V}_U$, where $\oplus$ denotes the direct sum of subspaces and $\mathcal{V}_U$ an output-nulling controlled invariant subspace of (1), (2) such that $\sigma((A+BF)|_{\mathcal{V}_U}) = \mathcal{Z}(A,B,C,D) \subset \mathbb{C}^{\otimes}$. Moreover, in light of $\mathcal{A}2'$, one gets $\mathcal{V}_U + \mathcal{S}^* = \mathcal{X} = \mathbb{R}^n$. Hence, $x_f = x_S + x_{V_U}$, where $x_S \in \mathcal{S}^*$ and $x_{V_U} \in \mathcal{V}_U$. Owing to the properties of $\mathcal{S}^*$ reviewed in Appendix A, the component x_S is reachable from the origin in $N \geq \rho$ steps along trajectories that give rise to zero output until the last step but one. Since $\mathcal{V}_U$ is an internally antistable output-nulling controlled invariant subspace, in view of the properties of $\mathcal{V}^* \supseteq \mathcal{V}_U$ reviewed in Appendix A, the component x_{V_U} could only be reached from the origin along trajectories giving rise to zero output by applying a control action computed backward in time, starting at the time $-\infty$, and such that the state evolves according to the dynamics of $(A+BF)|_{\mathcal{V}_U}$. Therefore, for any finite $N \in \mathbb{Z}^+$ $\left[\begin{smallmatrix} 0 \\ x_f \end{smallmatrix} \right] \notin \operatorname{im} M_N$, which implies that (8) is not solvable. Nonetheless, the Euclidean norm in (9) is minimal owing to the properties of the Moore-Penrose inverse and, in light of the considerations above, its value goes to zero as N approaches infinity. □

Theorems 1 and 4 have shown how to compute a control sequence u_t, $t = 0, 1, \ldots, N-1$, that solves the problem of driving the state of (1), (2) from $x_0 = 0$ to $x_N = x_f$, with $y_t = 0$ for $t = 0, 1, \ldots, N-1$, either exactly or up to an arbitrary degree of accuracy, depending on whether the system has any nonminimum-phase invariant zeros or not. In the following these arguments are generalized so that the gain matrix Φ_ℓ, $\ell = 0, 1, \ldots, N-1$, of the FIR system be computed. Since any $x_f \in \mathbb{R}^n$ is a linear combination of the column vectors of I_n, let (8) be replaced by the algebraic matrix equation $\left[\begin{smallmatrix} O \\ I_n \end{smallmatrix} \right] = M_N U_N$, where $U_N = \left[\, \Phi_0^\top \; \Phi_1^\top \; \ldots \; \Phi_{N-1}^\top \, \right]^\top$. With this notation, the j-th column of the i-th matrix of U_N is the control input to be applied at $t = i-1$ in order for the state to reach e_j, the j-th vector of the main basis of $\mathbb{R}^n$, at $t = N$ with zero output until $t = N-1$. Then, by linearity and time invariance of system (1), (2), the matrices Φ_ℓ, $\ell = 0, 1, \ldots, N-1$, of the FIR system can be obtained by means of the Moore-Penrose inverse of M_N, according to the abovementioned generalization of the results of Theorems 1 and 4. Note that in the presence of nonminimum-phase zeros, the FIR system generate an approximation of the control input which would steer the state along the trajectories associated with the zeros, because of the truncation implicit in the computation of the convolution profiles. In order for the effects of the truncation to vanish as time increases the system must be asymptotically stable.

4.2 Duality Arguments

This section shows the relation between the control scheme devised in Section 4.1 and the scheme for state reconstruction in the presence of unknown inputs outlined in Section 3. Recall that (A,B,C,D) is controllable and observable if and only if $(A^\top, C^\top, B^\top, D^\top)$ is controllable and observable; (A,B,C,D) is right invertible if and only if $(A^\top, C^\top, B^\top, D^\top)$ is left invertible; $\mathcal{Z}(A,B,C,D) = \mathcal{Z}(A^\top, C^\top, B^\top, D^\top)$. As mentioned in Section 4.1, the input to the feedforward FIR system is a discrete pulse with amplitude x_f applied at $t = 0$, where x_f is the desired final state for the controlled system. The state trajectory x_t, $t \in \mathbb{Z}_0^+$, of the controlled system is regarded as a fictitious output sequence $\eta_t = x_t$, $t \in \mathbb{Z}_0^+$, in addition to the output sequence y_t, $t \in \mathbb{Z}_0^+$, which is required to be zero until $t = N-1$. Modelization in the dual context requires that the unknown initial state x_0 be injected into the system as a discrete pulse applied at $t = 0$ to a fictitious input μ and the inaccessible input u_t, $t \in \mathbb{Z}_0^+$, be represented by a sequence different from zero starting from $t = 1$. This scenario entails a one-step shift with respect to that assumed in the problem statement (Section 3). These are mere technicalities aimed at ensuring a smooth formulation of Theorem 5. The output y_t of the control system from $t = 0$ to $t = N-1$ will henceforth be considered exactly equal to zero also for nonminimum-phase systems. From a practical point of view, this can be obtained by choosing N big enough to make the value of the Euclidean norm in (9) negligible. The cascade of the FIR system (4), (5) and system (1), (2) with the further output η is described by

$$\hat{x}_{t+1} = \hat{A}\hat{x}_t + \hat{B}h_t, \tag{10}$$

$$y_t = \hat{C}\hat{x}_t + \hat{D}h_t, \tag{11}$$

$$\eta_t = \hat{E}\hat{x}_t, \tag{12}$$

where $\hat{A} = \begin{bmatrix} A & BC_{fir} \\ O & A_{fir} \end{bmatrix}$, $\hat{B} = \begin{bmatrix} BD_{fir} \\ B_{fir} \end{bmatrix}$, $\hat{C} = [\, C \;\, DC_{fir} \,]$, $\hat{D} = DD_{fir}$, $\hat{E} = [\, I \;\, O \,]$. The cascade of the dual system with the further input μ and the dual FIR system is

$$\hat{x}_{t+1} = \hat{A}^\top \hat{x}_t + \hat{C}^\top u_t + \hat{E}^\top \mu_t, \tag{13}$$

$$\tilde{x}_t = \hat{B}^\top \hat{x}_t + \hat{D}^\top u_t. \tag{14}$$

Hence, Theorem 5 can be stated as follows. Its proof consists of mere technicalities and therefore will be omitted.

Theorem 5. Consider system (10)–(12). Let the initial state $\hat{x}_0$ of (10)–(12) be zero. Let $N \in \mathbb{Z}^+$ be the number of steps of the FIR system time window. Let the desired state $x_f \in \mathbb{R}^n$ of the controlled system at $t = N$ be injected into (10)–(12) as the discrete pulse signal $h_t = x_f \delta_t$, $t \in \mathbb{Z}_0^+$. Consider system (13), (14). Let the initial state $\hat{x}_0$ of (13), (14) be zero. Let the unknown input be a sequence u_t, $t \in \mathbb{Z}_0^+$, with $u_0 = 0$. Let the unknown initial state $x_0 \in \mathbb{R}^n$ of the observed system be injected into (13), (14) as the discrete pulse signal $\mu_t = x_0 \delta_t$, $t \in \mathbb{Z}_0^+$. Then, the output sequences y_t and η_t, $t \in \mathbb{Z}^+$, of (10)–(12) are such that

$$y_t = \begin{cases} 0, & t = 0, 1, \ldots, N-1, \\ CA^{t-N}x_f, & t = N, N+1, \ldots, \end{cases}$$

$$\eta_t = A^{t-N}x_f, \qquad t = N, N+1, \ldots,$$

if and only if the output sequence $\tilde{x}_t$, $t \in \mathbb{Z}_0^+$, of (13), (14) is such that

$$\tilde{x}_t = (A^\top)^{t-N} x_0 + \sum_{\ell=0}^{t-N} (A^\top)^{t-N-\ell} C^\top u_\ell, \; t = N, N+1, \ldots$$

4.3 The Unknown-Input Reconstructor

Sections 4.1 and 4.2 have shown how to design a FIR system that provides a sequence $\tilde{x}_t$, $t \in \mathbb{Z}_0^+$, which is a replica of the state trajectory x_t, $t \in \mathbb{Z}_0^+$, with a delay of N steps. Consequently, the unknown input u_t, $t \in \mathbb{Z}_0^+$, can be derived from the reconstructed state trajectory and the delayed measured output with a further delay of one step by using the system equations. Namely, the sequence $\tilde{u}_t$, $t \in \mathbb{Z}_0^+$, of the reconstructed input is relevant starting from $t = N$ and is given by

$$\tilde{u}_t = \begin{bmatrix} B \\ D \end{bmatrix}^\dagger \left(\begin{bmatrix} \tilde{x}_{t+1} \\ y_{t-N} \end{bmatrix} - \begin{bmatrix} A \\ C \end{bmatrix} \tilde{x}_t \right), \; t = N, N+1, \ldots.$$

5. SYSTEMS WITH MINIMUM-PHASE ZEROS ALSO

This section shows how to extend reconstruction of unknown state and input to systems whose invariant zeros lie anywhere in the complex plane with the sole exception of the unit circle. As will be shown in the following, the system is permanently connected in cascade to filter that cancels its minimum-phase invariant zeros. A special state-space representation of the minimal form of the cascade preserves the dynamic matrix and the unknown-input distribution matrix of the original system. Hence, state reconstruction can be obtained with the FIR system designed on the basis of that minimal form. The design of the filter will be developed in the dual context of control, like that of the convolution profiles of the FIR system. Since dual systems have the same set of invariant zeros, the cascaded filter cancelling the minimum-phase zeros of the original system is the dual counterpart of a feedforward compensator cancelling the minimum-phase invariant zeros of the system considered in the control problem. Therefore, this section will directly refer to a system like (1), (2), that satisfies Assumptions $\mathcal{A}1$, $\mathcal{A}2'$, and $\mathcal{A}3$. The following statements are aimed at pointing out the key subspace, that will be denoted by $\mathcal{V}_M^*$, for the synthesis of the feedforward compensator. Proofs will be omitted for the sake of brevity.

Lemma 6. Consider (1), (2). Let $\mathcal{A}2'$ hold. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C+DF)$. Perform the similarity transformation $T = [T_1 \; T_2 \; T_3]$, where $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[T_1 \; T_2] = \mathcal{V}^*$, $\operatorname{im}\,[T_1 \; T_3] = \mathcal{S}^*$. Then,

$$A'_F = T^{-1}(A+BF)T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}, \tag{15}$$

$$C'_F = (C+DF)T = \begin{bmatrix} O & O & C'_3 \end{bmatrix}. \tag{16}$$

Remark 7. The set of the internal eigenvalues of $\mathcal{R}_{\mathcal{V}^*}$ is the set of the eigenvalues of A'_{11}: i.e., $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) = \sigma(A'_{11})$. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$ (or, equivalently the set of the system invariant zeros) is the set of the eigenvalues of A'_{22}: i.e., $\sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \mathcal{Z}(A,B,C,D) = \sigma(A'_{22})$.

Let $n_R = \dim \mathcal{R}_{\mathcal{V}^*}$, $n_V = \dim \mathcal{V}^* - n_R$, $n_S = \dim \mathcal{S}^* - n_R$,

$$T'_1 = \begin{bmatrix} I_{n_R} \\ O \\ O \end{bmatrix} \quad T'_3 = \begin{bmatrix} O \\ O \\ I_{n_S} \end{bmatrix} \tag{17}$$

Lemma 8. Consider (1), (2). Let $\mathcal{A}2'$ hold. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Consider (15), (16) and perform the similarity transformation $T' = [T'_1 \; T'_2 \; T'_3]$, where T'_1, T'_3 are defined by (17) and $T'_2 = [X^\top \; I_{n_V} \; O]^\top$, where X is the solution of the Sylvester equation

$$A'_{11}X - XA'_{22} = -A'_{12}. \tag{18}$$

Then,

$$A''_F = T'^{-1} A'_F T' = \begin{bmatrix} A'_{11} & O & A''_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}, \tag{19}$$

$$C''_F = C'_F T' = C'_F. \tag{20}$$

Lemma 9. Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3$ hold. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Consider (19), (20) and perform the similarity transformation $T'' = [T_1' \; T_2'' \; T_3']$, where T_1', T_3' are defined by (17) and $T_2'' = [O \;\; V^\top \;\; O]^\top$, where $V = [V_S \;\; V_U]$, with V_S and V_U basis matrices of the subspaces $\mathcal{V}_S$ and $\mathcal{V}_U$ of the stable and the unstable modes of A_{22}'. Then, $A_F''' = T''^{-1} A_F'' T'' = \left[\begin{smallmatrix} A_{11}' & O & A_{13}'' \\ O & A_{22}'' & A_{23}'' \\ O & O & A_{33}' \end{smallmatrix}\right]$, $C_F''' = C_F'' T'' = C_F'$, with $A_{22}'' = V^{-1} A_{22}' V = \left[\begin{smallmatrix} A_{22s}'' & O \\ O & A_{22u}'' \end{smallmatrix}\right]$.

In light of Lemma 9, a representation of A_F''' and C_F''' is

$$A_F''' = \begin{bmatrix} A_{11}' & O & O & A_{13}'' \\ O & A_{22s}'' & O & A_{23s}'' \\ O & O & A_{22u}'' & A_{23u}'' \\ O & O & O & A_{33}' \end{bmatrix}, \tag{21}$$

$$C_F''' = \begin{bmatrix} O & O & O & C_3' \end{bmatrix}. \tag{22}$$

Theorem 10. Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3$ hold. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. Let the subspace $\mathcal{V}_M^*$ be defined by

$$\mathcal{V}_M^* = \operatorname{im} V_M^{*'''} = \operatorname{im} \begin{bmatrix} I_{n_R} & O \\ O & I_{n_{SS}} \\ O & O \\ O & O \end{bmatrix}, \tag{23}$$

where basis matrix $V_M^{*'''}$ refers to the coordinates introduced in Lemma 9 and is partitioned according to (21). Then: (i) $\mathcal{V}_M^*$ in the maximal internally-stable output-nulling controlled invariant subspace; (ii) $\sigma((A+BF)|_{\mathcal{V}_M^*}) = \sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cup \mathcal{Z}_{MP}(A,B,C,D)$.

Proof. Proposition (i): Equation $A_F''' V_M^{*'''} = V_M^{*'''} Y$ holds with $Y = \left[\begin{smallmatrix} A_{11}' & O \\ O & A_{22s}'' \end{smallmatrix}\right]$, due to (21) and (23). Therefore, $\mathcal{V}_M^*$ is an $(A+BF)$-invariant subspace with $\sigma((A+BF)|_{\mathcal{V}_M^*}) = \sigma(A_{11}') \cup \sigma(A_{22s}'')$, which implies that $\mathcal{V}_M^*$ is an $(A,\mathcal{B})$-controlled invariant subspace. Moreover, $C_F''' V_M^{*'''} = 0$, which implies that $\mathcal{V}_M^*$ is the maximal internally-stable output-nulling controlled invariant subspace. Proposition (ii) follows from Lemma 9, in light of Remark 7. □

Corollary 11. Consider (1), (2). Let $\mathcal{A}2'$, $\mathcal{A}3$ hold. Let F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. Let $\mathcal{V}_M^*$ be the maximal internally-stable output-nulling controlled invariant subspace defined by (23). Let $\bar{T} = T T' T''$, where T, T', T'' are the similarity transformations respectively considered in Lemmas 6, 8, 9, and let $V_M^* = \bar{T} V_M^{*'''}$ be the consistent basis matrix of $\mathcal{V}_M^*$ with respect to the original coordinates. Then, $A V_M^* - V_M^* W = -BL$ and $C V_M^* = -DL$ hold with $W = \left[\begin{smallmatrix} A_{11}' & O \\ O & A_{22s}'' \end{smallmatrix}\right]$ and $L = F V_M^*$.

The subspace $\mathcal{V}_M^*$ introduced in Theorem 10 is the key subspace for the synthesis of the feedforward compensator cancelling the minimum-phase invariant zeros of (1), (2). Matrices W, L defined in Corollary 11 will henceforth be used to that aim. The synthesis of the feedforward compensator and the properties of a specific ISO description of the cascade of the feedforward compensator and the system are the object of the remainder of this section. Let the feedforward compensator be

$$x_{f,t+1} = A_f \, x_{f,t} + B_f \, w_t, \tag{24}$$

$$u_t = C_f \, x_{f,t} + D_f \, w_t, \tag{25}$$

where

$$A_f = W, \quad B_f = \begin{bmatrix} I_{n_f} & O_{n_f \times p} \end{bmatrix}, \tag{26}$$

$$C_f = L, \quad D_f = \begin{bmatrix} O_{p \times n_f} & I_p \end{bmatrix}, \tag{27}$$

with $n_f = n_R + n_{SS} = \dim \mathcal{V}_M^*$. The cascade of (24), (25) and (1), (2) is

$$\bar{x}_{t+1} = \bar{A} \, \bar{x}_t + \bar{B} \, w_t, \tag{28}$$

$$y_t = \bar{C} \, \bar{x}_t + \bar{D} \, w_t, \tag{29}$$

where $\bar{A} = \left[\begin{smallmatrix} A & BC_f \\ O & A_f \end{smallmatrix}\right]$, $\bar{B} = \left[\begin{smallmatrix} BD_f \\ B_f \end{smallmatrix}\right]$, $\bar{C} = [\, C \;\; DC_f \,]$, $\bar{D} = DD_f$.

Lemma 12. Consider system (28), (29). Let

$$\mathcal{J} = \operatorname{im} J = \operatorname{im} \begin{bmatrix} I_n \\ O_{n_f \times n} \end{bmatrix}, \quad \mathcal{J}_c = \operatorname{im} J_c = \operatorname{im} \begin{bmatrix} V_M^* \\ I_{n_f} \end{bmatrix}.$$

Then: (i) $\mathcal{J}$ is an $\bar{A}$-invariant subspace; (ii) $\mathcal{J}_c$ is an $\bar{A}$-invariant subspace; (iii) $\mathcal{J} \oplus \mathcal{J}_c = \bar{\mathcal{X}}$, where $\bar{\mathcal{X}}$ is the state space of (28), (29).

Theorem 13. Consider (28), (29), where (A,B,C,D) satisfies $\mathcal{A}1'$ and (A_f,B_f,C_f,D_f) is defined according to (26), (27). System

$$x_{c,t+1} = A_c \, x_{c,t} + B_c \, w_t, \tag{30}$$

$$y_t = C_c \, x_{c,t} + D_c \, w_t, \tag{31}$$

with $A_c = A$, $B_c = [\, -V_M^* \;\; B \,]$, $C_c = C$, $D_c = [\, O \;\; D \,]$, is an IO-equivalent realization of (28), (29): i.e., its controllable and observable subsystem matches that of (28), (29).

Proof. Owing to Lemma 12, $T_c = [J \;\; J_c]$ is square and nonsingular. Hence, by applying the similarity transformation T_c to $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ one gets $\bar{A}' = T_c^{-1} \bar{A} T_c = \begin{bmatrix} A & O \\ O & A_f \end{bmatrix}$, $\bar{B}' = T_c^{-1} \bar{B} = \begin{bmatrix} -V_M^* & B \\ I & O \end{bmatrix}$, $\bar{C}' = \bar{C} T_c = [\, C \;\; O \,]$, $\bar{D}' = \bar{D}$. Then, the thesis follows by direct inspection of $\bar{A}'$, $\bar{B}'$, $\bar{C}'$, $\bar{D}'$: the dynamics of A_f, decoupled from that of A, is unobservable; controllability of (A,B) implies that of (A_c, B_c). □

The following statements are aimed at showing the main properties of system (30), (31). The proofs, that can be obtained by applying the algorithms reviewed in Appendix A, will be omitted.

Lemma 14. Consider systems (1), (2) and (30), (31). Let $\mathcal{V}^* = \max \mathcal{V}(A,B,C,D)$ and $\mathcal{V}_c^* = \max \mathcal{V}(A_c,B_c,C_c,D_c)$. Then, $\mathcal{V}_c^* = \mathcal{V}^*$.

Lemma 15. Consider systems (1), (2) and (30), (31). Let $\mathcal{S}^* = \min \mathcal{S}(A,B,C,D)$ and $\mathcal{S}_c^* = \min \mathcal{S}(A_c,B_c,C_c,D_c)$. Then, $\mathcal{S}_c^* = \mathcal{S}^* + \mathcal{V}_M^*$.

Theorem 16. Consider systems (1), (2) and (30), (31). Let (1), (2) satisfy $\mathcal{A}2'$. Then, (30), (31) satisfies $\mathcal{A}2'$.

Lemma 17. Consider systems (1), (2) and (30), (31). Let $\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^*$ and $\mathcal{R}_{\mathcal{V}_c^*} = \mathcal{V}_c^* \cap \mathcal{S}_c^*$. Then, $\mathcal{R}_{\mathcal{V}_c^*} = \mathcal{R}_{\mathcal{V}^*} + \mathcal{V}_M^*$.

Theorem 18. Consider systems (1), (2) and (30), (31). Let $\mathcal{Z}_{NMP}(A,B,C,D)$ be the set of the nonminimum-phase invariant zeros of (1), (2). Let $\mathcal{Z}(A_c,B_c,C_c,D_c)$ be the set of the invariant zeros of (30), (31). Then, $\mathcal{Z}(A_c,B_c,C_c,D_c)=\mathcal{Z}_{NMP}(A,B,C,D)$.

6. CONCLUSION

A complete scheme for the reconstruction of generic inaccessible inputs in linear multivariable discrete-time systems with unknown initial state was developed by exploiting the properties of output-nulling controlled invariant subspaces and input-containing conditioned invariant subspaces. Although the insight into the problem relies on geometric considerations, the synthesis procedure is based on structure algorithms that make an extensive use of algebraic tools like Toeplitz matrices and Moore-Penrose inverses.

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Appendix A. AN ALGEBRAIC APPROACH FOR COMPUTING THE KEY SUBSPACES OF SYSTEMS WITH DIRECT FEEDTHROUGH TERMS

This appendix presents a direct algebraic approach for computing the maximal output-nulling controlled invariant subspace and the minimal input-containing conditioned invariant subspace, i.e., the basic subspaces of a non-strictly proper linear multivariable system. The computational procedures discussed herein are inspired by the structure algorithm introduced by Silverman (1976) and represent an alternative to those suggested in Basile and Marro (1992), where the basic subspaces of a non-strictly proper system are computed by respectively applying the controlled invariant algorithm and the conditioned invariant algorithm to suitably-defined strictly proper systems.

Consider system (1), (2), with a generic initial state x_0. For any $N\in\mathbb{Z}^+$, let the the input sequence vector U_N and the output sequence vector Y_N be defined by

$$U_N=\begin{bmatrix} u_0\\ u_1\\ \vdots\\ u_{N-1}\end{bmatrix},\quad Y_N=\begin{bmatrix} y_0\\ y_1\\ \vdots\\ y_{N-1}\end{bmatrix}. \tag{A.1}$$

Let the matrices L_N, A_N, and B_N be defined by

$$L_N=\left[\,A^{N-1}B\ A^{N-2}B\ \ldots\ AB\ B\,\right], \tag{A.2}$$

$$A_N=\left[\,C^\top\ A^\top C^\top\ (A^\top)^2C^\top\ \vdots\ (A^\top)^{N-1}C^\top\,\right]^\top, \tag{A.3}$$

$$B_N=\begin{cases} D, & \text{with } N=1,\\ \begin{bmatrix} D & O & \ldots\ \ldots & O\\ CB & D & \ddots & \vdots\\ CAB & CB & D\ \ddots & \vdots\\ \vdots & \vdots & \ddots\ \ddots & O\\ CA^{N-2}B & CA^{N-3}B & \ldots\ CB & D\end{bmatrix}, & \text{with } N\geq 2.\end{cases} \tag{A.4}$$

Then, x_0, x_N, U_N, and Y_N are related by

$$x_N=A^Nx_0+L_NU_N, \tag{A.5}$$

$$Y_N=A_Nx_0+B_NU_N. \tag{A.6}$$

Since $\mathcal{V}^*=\max\mathcal{V}(A,B,C,D)$ is the maximum set of the initial state of controlled trajectories giving rise to zero output, in light of (A.6), a basis matrix V^* of $\mathcal{V}^*$ can be obtained with the following algorithm.

Algorithm 19. Consider system (1), (2). A basis matrix V^* of $\mathcal{V}^*=\max\mathcal{V}(A,B,C,D)$ and the number of steps ν for the algorithm to converge are computed as follows.

1. Set $V_0=I_n$ and $N=1$.
2. Compute $K_N=\left[\begin{smallmatrix}X_N\\ W_N\end{smallmatrix}\right]$ as a basis matrix of $\ker\,[A_N\ B_N]$, partitioned accordingly.
3. Compute V_N as a basis matrix of $\operatorname{im}X_N$.
4. If $\operatorname{im}V_N=\operatorname{im}V_{N-1}$, then set $V^*=V_N$ and $\nu=N-1$, else set $N=N+1$ and return to step 2.

Since $\mathcal{S}^*=\min\mathcal{S}(A,B,C,D)$ is the maximum set of states reachable from the origin along trajectories that give rise to zero output until the last but one step, in light of (A.5) and (A.6), a basis matrix S^* of $\mathcal{S}^*$ can be obtained with the following algorithm.

Algorithm 20. Consider system (1), (2). A basis matrix S^* of $\mathcal{S}^*=\min\mathcal{S}(A,B,C,D)$ and the number of steps ρ for the algorithm to converge are computed as follows.

1. Set $S_0=O_{n\times 1}$ and $N=1$.
2. Compute K_N as a basis matrix of $\ker B_N$.
3. Compute S_N as a basis matrix of $\operatorname{im}(L_NK_N)$.
4. If $\operatorname{im}S_N=\operatorname{im}S_{N-1}$, then set $S^*=S_N$ and $\rho=N-1$, else set $N=N+1$ and return to step 2.