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Retrospective Cost Adaptive Control Using Composite FIR/IIR Controllers

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Abstract— **For stably stabilizable plants, that is, plants that can be stabilized by asymptotically stable controllers, LQG controllers are often unstable. This typically occurs in the case where the plant is unstable or nonminimum phase (NMP). In this paper, we apply retrospective cost adaptive control (RCAC) to stably stabilize plants whose LQG controllers are unstable. RCAC is implemented by using a composite FIR/IIR controller consisting of a high-order FIR component and a low-order IIR component. Asymptotic stability of the IIR component is enforced at each step by reflecting its unstable poles into the open unit disk. These controllers are used for command following and disturbance rejection for Lyapunov-stable and NMP plants.**

I. Introduction

Unstable controllers are undesirable for multiple reasons: they are difficult to start up; they are more susceptible to the adverse effects of saturation; and momentary disconnection from the plant due to delays or data loss can lead to divergence [1]. Stable stabilization, that is, the ability to stabilize a plant using an asymptotically stable controller, is thus a crucial issue in feedback control. As discussed in [2, 3], some unstable plants can be stabilized only by unstable controllers; such plants are pathologically difficult to control but, fortunately, are rare in practice. These plants are outside the scope of the present paper.

For plants that are stably stabilizable, it is obviously desirable to use an asymptotically stable controller, whether the objective is stabilization, command following, or disturbance rejection. For plants that are stably stabilizable, however, it is a well-known but unfortunate fact of feedback control that H_2-optimal and H_∞-optimal dynamic control laws are often unstable; this typically occurs if the plant is unstable or nonminimum phase (NMP). If the optimal controller is unstable, then all asymptotically stable controllers are necessarily suboptimal; it is thus of interest to determine the performance tradeoff due to the restriction to asymptotically stable controllers.

In some cases, it may be possible to obtain asymptotically stable suboptimal control laws by adjusting the weights of the cost function, but such trial-and-error techniques lack guarantees of success. In addition, by modifying the weights, the resulting controller does not address the performance objective associated with the original weights. Although more systematic techniques have been developed [4–7], these techniques add computational complexity and are not guaranteed to be successful.

The present paper focuses on the problem of obtaining asymptotically stable controllers within the context of adaptive control. In particular, we consider retrospective cost adaptive control (RCAC), which is a direct adaptive discrete-time control law that can be used for stabilization, command following (including model reference adaptive control), and disturbance rejection [8–11]. As shown in [11], RCAC controllers are similar to LQG controllers. Consequently, in cases where the LQG controller is unstable, the RCAC controller may converge to an unstable controller. The goal of the present paper is to modify RCAC in order to avoid convergence to unstable controllers.

There are various ad hoc techniques that can be used to enforce asymptotic stability of the RCAC controller. For example, if the updated controller $G_{\mathrm{c},k}$ is unstable, then $G_{\mathrm{c},k}$ can be modified by replacing each unstable pole by its reflection inside the unit circle. Unfortunately, this requires computation at each step of all of the controller poles as well as the construction of the modified controller. In addition, this approach can destabilize the closed-loop system. A more rigorous approach would be to update the controller subject to a stability constraint; however, this constraint is not convex and thus is computationally expensive.

The approach taken in the present paper is to adapt FIR or composite FIR/IIR (CFI) control laws, that is, control laws all or most of whose poles are fixed at the origin. A related approach is developed in [12], where the motivation for sparse controllers is based on computational complexity and accuracy rather than controller stability. For a CFI controller comprised of the product of high-order FIR component and a low-order IIR component, the low-order IIR component provides the ability to adaptively develop an internal model or to facilitate pole placement. Asymptotic stability of the IIR component of the controller is enforced at each step by reflecting its unstable poles into the open unit disk. Since the IIR component is low order, the computational task is alleviated relative to the use of a fully IIR controller.

II. Adaptive Standard Problem

Consider the standard problem consisting of the discrete-time, linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \quad (1)$$
$$y(k) = Cx(k) + D_0 u(k) + D_2 w(k), \quad (2)$$
$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k), \quad (3)$$

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where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^{l_y}$ is the measurement, $u(k) \in \mathbb{R}^{l_u}$ is the control input, $w(k) \in \mathbb{R}^{l_w}$ is the exogenous input, and $z(k) \in \mathbb{R}^{l_z}$ is the performance variable. The goal is to develop a feedback or feedforward controller that operates on y to minimize z in the presence of the exogenous signal w. Depending on the choice of D_1, D_2, and E_0, the components of w can represent either a command signal r to be followed, an external disturbance d to be rejected, or sensor noise v that corrupts the measurement. Depending on the application, components of w may or may not be measured. For fixed-gain control, z need not be measured, whereas, for adaptive control, z is assumed to be measured.

Using the forward shift operator $\mathbf{q}$, we can rewrite (1)–(3) as

$$y(k) = G_{yw}(\mathbf{q})w(k) + G_{yu}(\mathbf{q})u(k), \tag{4}$$

$$z(k) = G_{zw}(\mathbf{q})w(k) + G_{zu}(\mathbf{q})u(k), \tag{5}$$

where

$$G_{yw}(\mathbf{q}) \triangleq D^{-1}(\mathbf{q})N_{yw}(\mathbf{q}) = C(\mathbf{q}I - A)^{-1}D_1 + D_2, \tag{6}$$

$$G_{yu}(\mathbf{q}) \triangleq D^{-1}(\mathbf{q})N_{yu}(\mathbf{q}) = C(\mathbf{q}I - A)^{-1}B + D_0, \tag{7}$$

$$G_{zw}(\mathbf{q}) \triangleq D^{-1}(\mathbf{q})N_{zw}(\mathbf{q}) = E_1(\mathbf{q}I - A)^{-1}D_1 + E_0, \tag{8}$$

$$G_{zu}(\mathbf{q}) \triangleq D^{-1}(\mathbf{q})N_{zu}(\mathbf{q}) = E_1(\mathbf{q}I - A)^{-1}B + E_2. \tag{9}$$

The controller has the form $u(k) = G_{\mathrm{c},k}(\mathbf{q})y(k)$, where the adaptive controller $G_{\mathrm{c},k}$ is updated at each step. Figure 1 illustrates the adaptive standard problem, which consists of (4)–(9) with the adaptive controller $G_{\mathrm{c},k}$.

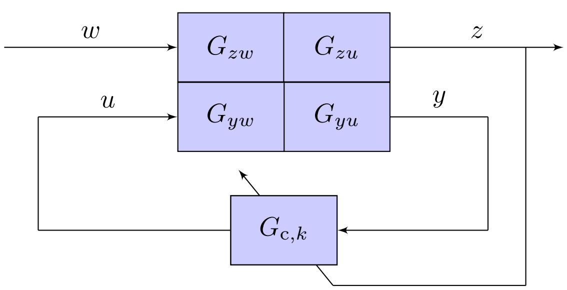

Fig. 1: Adaptive standard problem. The controller $G_{\mathrm{c},k}$ is updated at each step and thus is linear time varying.

A. Adaptive Servo Problem

The adaptive servo problem is a special case of the adaptive standard problem with

$$x(k+1) = Ax(k) + Bu(k) + \bar{D}_1 d(k), \tag{10}$$

$$y_0(k) = \bar{C}x(k) + \bar{D}_0 u(k), \tag{11}$$

$$e_0(k) = r(k) - y_0(k), \tag{12}$$

$$y_{\mathrm{n}}(k) = y_0(k) + v(k), \tag{13}$$

$$e_{\mathrm{n}}(k) = r(k) - y_{\mathrm{n}}(k). \tag{14}$$

We can rewrite (11) in terms of $\mathbf{q}$ as

$$y_0(k) = G_u(\mathbf{q})u(k) + G_d(\mathbf{q})d(k), \tag{15}$$

where

$$G_u(\mathbf{q}) \triangleq \bar{C}(\mathbf{q}I - A)^{-1}B + \bar{D}_0, \tag{16}$$

$$G_d(\mathbf{q}) \triangleq \bar{C}(\mathbf{q}I - A)^{-1}\bar{D}_1. \tag{17}$$

In the notation of the standard problem,

$$w = \begin{bmatrix} r \\ d \\ v \end{bmatrix}, \quad y = e_{\mathrm{n}}, \quad z = e_{\mathrm{n}}. \tag{18}$$

The measured error signal e_{n} is the difference between the command r and the measurement y_{n}, which may be corrupted by noise. Since only the measured error signal available for feedback, it serves as the performance variable within RCAC. However, the true error signal e_0, which is the difference between the command r and the plant output y_0, provides a true measure of the command-following performance. Since this signal is not available for feedback, it is used only as a diagnostic. If, however, sensor noise is absent, then e_{n} and e_0 are identical. In all examples in this paper, we consider the case where $G_d = G_u$, which is denoted by G.

III. RCAC Algorithm

A. Controller Structure

Define the IIR dynamic controller

$$u(k) = \sum_{i=1}^{n_{\mathrm{c}}} P_i(k)u(k-i) + \sum_{i=1}^{n_{\mathrm{c}}} Q_i(k)y(k-i), \tag{19}$$

where $P_i(k) \in \mathbb{R}^{l_u \times l_u}$ and $Q_i(k) \in \mathbb{R}^{l_u \times l_y}$ are the controller coefficient matrices. We rewrite (19) as

$$u(k) = \Phi(k)\theta(k), \tag{20}$$

where the regressor matrix $\Phi(k)$ is defined by

$$\Phi(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_{\mathrm{c}}) \\ y(k-1) \\ \vdots \\ y(k-n_{\mathrm{c}}) \end{bmatrix}^{\mathrm{T}} \otimes I_{l_u} \in \mathbb{R}^{l_u \times l_\theta}, \tag{21}$$

and the controller coefficient vector $\theta(k)$ is defined by

$$\theta(k) \triangleq \mathrm{vec}\big[P_1(k) \cdots P_{n_{\mathrm{c}}}(k)\ Q_1(k) \cdots Q_{n_{\mathrm{c}}}(k)\big]^{\mathrm{T}} \in \mathbb{R}^{l_\theta}, \tag{22}$$

$l_\theta \triangleq l_u^2 n_{\mathrm{c}} + l_u l_y(n_{\mathrm{c}})$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator. In terms of $\mathbf{q}$, the transfer function of the controller from y to u is given by

$$G_{\mathrm{c},k}(\mathbf{q}) = \left(\mathbf{q}^{n_{\mathrm{c}}} I_{l_u} - \mathbf{q}^{n_{\mathrm{c}}-1}P_1(k) - \cdots - P_{n_{\mathrm{c}}}(k)\right)^{-1} \cdot \left(\mathbf{q}^{n_{\mathrm{c}}-1}Q_1(k) + \cdots + Q_{n_{\mathrm{c}}}(k)\right). \tag{23}$$

B. Retrospective Performance Variable

We define the retrospective performance variable as

$$\hat{z}(k,\hat{\theta}) \triangleq z(k) + G_{\rm f}(\mathbf{q})[\Phi(k)\hat{\theta} - u(k)], \tag{24}$$

where $\hat{\theta} \in \mathbb{R}^{l_\theta}$ and $G_{\rm f}$ is an $n_z \times n_u$ filter specified below. The rationale underlying (24) is to replace the control $u(k)$ with $\Phi(k)\hat{\theta}^*(k)$, where $\hat{\theta}^*$ is the retrospectively optimized controller coefficient vector obtained by optimization below. The updated controller thus has coefficients $\theta(k+1) = \hat{\theta}^*$. Consequently, the implemented control at step $k+1$ is given by

$$u(k+1) = \Phi(k+1)\theta(k+1). \tag{25}$$

The filter $G_{\rm f}$ is constructed below based on the required modeling information. This filter has the form

$$G_{\rm f} \triangleq D_{\rm f}^{-1} N_{\rm f}, \tag{26}$$

where $D_{\rm f}$ is an $l_z \times l_z$ polynomial matrix with leading coefficient I_{l_u}, and $N_{\rm f}$ is an $l_z \times l_u$ polynomial matrix. For reasons given below, we henceforth refer to $G_{\rm f}$ as the *target model*. By defining the filtered versions $\Phi_{\rm f}(k) \in \mathbb{R}^{l_z \times l_\theta}$ and $u_{\rm f}(k) \in \mathbb{R}^{l_z}$ of $\Phi(k)$ and $u(k)$, respectively, (24) can be written as

$$\hat{z}(k,\hat{\theta}) = z(k) + \Phi_{\rm f}(k)\hat{\theta} - u_{\rm f}(k), \tag{27}$$

where

$$\Phi_{\rm f}(k) \triangleq G_{\rm f}(\mathbf{q})\Phi(k), \quad u_{\rm f}(k) \triangleq G_{\rm f}(\mathbf{q})u(k). \tag{28}$$

C. Retrospective Cost

Using the retrospective performance variable $\hat{z}(k,\hat{\theta})$ defined by (24), we define the cumulative retrospective cost function

$$\begin{aligned} J(k,\hat{\theta}) \triangleq & \sum_{i=1}^{k} \lambda^{k-i} \hat{z}^{\rm T}(i,\hat{\theta}) R_z(i) \hat{z}(i,\hat{\theta}) \\ & + \sum_{i=1}^{k} \lambda^{k-i} (\Phi_{\rm f}(i)\hat{\theta})^{\rm T} R_u(i) \Phi_{\rm f}(i)\hat{\theta} \\ & + \lambda^k (\hat{\theta} - \theta(0))^{\rm T} R_\theta (\hat{\theta} - \theta(0)), \end{aligned} \tag{29}$$

where $\lambda \in (0,1]$ is the forgetting factor, R_θ is positive definite, and, for all $i \geq 1$, $R_z(i)$ is positive definite and $R_u(i)$ is positive semidefinite. The performance-variable and control-input weighting matrices $R_z(i)$ and $R_u(i)$ are time-dependent and thus may depend on present and past values of y, z, and u. Recursive minimization of (29) is used to update the controller coefficient vector $\hat{\theta}$. The following result uses recursive least squares to obtain the minimizer of (29).

Proposition: Let $P(0) = R_\theta^{-1}$. Then, for all $k \geq 1$, the retrospective cost function (29) has the unique global minimizer $\theta(k+1) = \hat{\theta}^*$, which is given by

$$\begin{aligned} \theta(k+1) = & \theta(k) - P(k)\Phi_{\rm f}^{\rm T}(k)\Upsilon^{-1}(k) \\ & \cdot \left[\Phi_{\rm f}(k)\theta(k) + \bar{R}(k)R_z(k)(z(k) - u_{\rm f}(k))\right], \end{aligned} \tag{30}$$

and where $P(k)$ satisfies

$$P(k+1) = \frac{1}{\lambda}P(k) - \frac{1}{\lambda}P(k)\Phi_{\rm f}^{\rm T}(k)\Upsilon^{-1}(k)\Phi_{\rm f}(k)P(k), \tag{31}$$

where

$$\bar{R}(k) \triangleq (R_z(k) + R_u(k))^{-1}, \tag{32}$$

$$\Upsilon(k) \triangleq \lambda\bar{R}(k) + \Phi_{\rm f}(k)P(k)\Phi_{\rm f}^{\rm T}(k). \tag{33}$$

For all examples in this paper, we initialize $\theta(0) = 0$ in order to reflect the absence of additional prior modeling information. Furthermore, for all $i \geq 1$, we use $R_z(i) = I_{l_z}$.

IV. Target Model $G_{\rm f}$

Using (20), the retrospective performance variable (24) can be written as

$$\hat{z}(k,\hat{\theta}) = z(k) - G_{\rm f}(\mathbf{q})[u(k) - \Phi(k)\hat{\theta}]. \tag{34}$$

It can be seen from (34) that minimizing (29) determines the controller coefficient vector $\hat{\theta}$ that best fits $G_{\rm f}(\mathbf{q})[\Phi(k)\theta(k) - \Phi(k)\hat{\theta}]$ to the performance data $z(k)$. In terms of the optimal controller coefficient vector $\hat{\theta}^*$, (34) can be written as

$$\hat{z}(k,\hat{\theta}^*) = z(k) - G_{\rm f}(\mathbf{q})[\Phi(k)\theta(k) - \Phi(k)\hat{\theta}^*]. \tag{35}$$

For convenience, we define

$$u^*(k) \triangleq \Phi(k)\hat{\theta}^*, \tag{36}$$

$$\tilde{u}(k) \triangleq u(k) - u^*(k), \tag{37}$$

so that

$$u(k) = u^*(k) + \tilde{u}(k). \tag{38}$$

Using this notation, (35) can be written as

$$\hat{z}(k,\hat{\theta}^*) = z(k) - G_{\rm f}(\mathbf{q})\tilde{u}(k). \tag{39}$$

Using (38) to replace u in Φ by $u^* + \tilde{u}$, it follows from (19)–(21) and (36) that $u^*(k)$ satisfies

$$u^*(k) = \sum_{i=1}^{n_{\rm c}} P_i^*[u^*(k-i) + \tilde{u}(k-i)] + \sum_{i=1}^{n_{\rm c}} Q_i^* y(k-i). \tag{40}$$

Note that, in (38), the actual input $u(k)$ to the plant at step k is written as the sum of the *pseudo control input* $u^*(k)$ and the *virtual external control perturbation* $\tilde{u}(k)$. From (40) it follows that

$$u^*(k) = D_{\rm c}^{*-1}(\mathbf{q})[(\mathbf{q}^{n_{\rm c}} I_{l_u} - D_{\rm c}^*(\mathbf{q}))\tilde{u}(k) + N_{\rm c}^*(\mathbf{q})y(k)], \tag{41}$$

where

$$D_{\rm c}^*(\mathbf{q}) \triangleq \mathbf{q}^{n_{\rm c}} I_{l_u} - \mathbf{q}^{n_{\rm c}-1}P_1^* - \cdots - P_{n_{\rm c}}^*, \tag{42}$$

$$N_{\rm c}^*(\mathbf{q}) \triangleq \mathbf{q}^{n_{\rm c}-1}Q_1^* + \cdots + Q_{n_{\rm c}}^*, \tag{43}$$

$$G_{\rm c}^* \triangleq D_{\rm c}^{*-1} N_{\rm c}^*. \tag{44}$$

It follows from (38) and (41) that

$$z(k) = G_{zw}(\mathbf{q})w(k) + G_{zu}(\mathbf{q})\left[\left(\frac{\mathbf{q}^{n_c}}{D_c^*(\mathbf{q})}-1\right)\tilde{u}(k)+G_c^*(\mathbf{q})y(k)+\tilde{u}(k)\right], \quad (45)$$

$$y(k) = G_{yw}(\mathbf{q})w(k) + G_{yu}(\mathbf{q})\left[\left(\frac{\mathbf{q}^{n_c}}{D_c^*(\mathbf{q})}-1\right)\tilde{u}(k)+G_c^*(\mathbf{q})y(k)+\tilde{u}(k)\right]. \quad (46)$$

Solving (46) for $y(k)$ and substituting $y(k)$ into (45) yields

$$z(k) = \widetilde{G}_{zw}^*(\mathbf{q})w(k) + \widetilde{G}_{z\tilde{u}}^*(\mathbf{q})\tilde{u}(k), \quad (47)$$

where

$$\widetilde{G}_{zw}^* \triangleq \frac{N_{zw}}{D} + \frac{N_{zu}N_{yw}N_c^*}{D(DD_c^* - N_{yu}N_c^*)} \quad (48)$$

and

$$\widetilde{G}_{z\tilde{u}}^*(\mathbf{q}) \triangleq \frac{N_{zu}(\mathbf{q})\mathbf{q}^{n_c}}{D(\mathbf{q})D_c^*(\mathbf{q})}\left[1 + \frac{N_c^*(\mathbf{q})N_{yu}(\mathbf{q})}{[D(\mathbf{q})D_c^*(\mathbf{q}) - N_{yu}(\mathbf{q})N_c^*(\mathbf{q})]}\right] \quad (49)$$

$$= \frac{N_{zu}(\mathbf{q})\mathbf{q}^{n_c}}{D(\mathbf{q})D_c^*(\mathbf{q}) - N_{yu}(\mathbf{q})N_c^*(\mathbf{q})}. \quad (50)$$

It can be seen from (39) that $\hat{z}(k, \hat{\theta}^*) = z(k) - G_f(\mathbf{q})\tilde{u}(k)$ is the residual of the fit between $z(k)$ and the output of the target model G_f with input $\tilde{u}(k)$. However, it follows from (47) that $\widetilde{G}_{z\tilde{u}}^*$, whose coefficients are given by $\hat{\theta}^*$, is the actual transfer function from $\tilde{u}$ to z. Therefore, minimizing the retrospective cost function (29) yields the value $\theta(k+1) = \hat{\theta}^*$ of $\hat{\theta}$ and thus the controller $G_{c,k+1}$ that provides the best fit of G_f by the transfer function $\widetilde{G}_{z\tilde{u},k+1}$ from $\tilde{u}$ to z. In other words, RCAC determines $G_{c,k+1}$ so as to optimally fit $\widetilde{G}_{z\tilde{u},k+1}$ to G_f.

V. Modeling Information Required for G_f

In this section we specify the modeling information required by RCAC. This information includes the relative degree, leading numerator coefficient, and NMP zeros of G_{zu}.

A. NMP Zeros

A key feature of $\widetilde{G}_{z\tilde{u}}$ is the factor N_{zu} in its numerator. This means that, since RCAC adapts $G_{c,k}$ so as to match $\widetilde{G}_{z\tilde{u}}$ to G_f. In particular, by placing controller poles at the locations of the NMP zeros, RCAC may cancel NMP zeros of G_{zu} that are not included in the roots of N_f in order to remove them from $\widetilde{G}_{z\tilde{u}}$. This phenomenon motivates the need to include all of the NMP zeros of G_{zu} in N_f [10, 13].

B. FIR Target Model

If there is no possibility of unstable pole-zero cancellation, then we use the FIR target model

$$G_f(\mathbf{q}) \triangleq \frac{H_{d_{zu}}}{\mathbf{q}^{d_{zu}}}. \quad (51)$$

The target model (51) can be used in the case where either G_{zu} is minimum phase or the poles of G_c are fixed. If, however, G_{zu} is NMP and the poles of G_c are optimized, we use the FIR target model

$$G_f(\mathbf{q}) \triangleq \frac{H_{d_{zu}}N_{zu,u}(\mathbf{q})}{\mathbf{q}^{d_{zu}+\deg(N_{zu,u})}}, \quad (52)$$

where $N_{zu,u}$ denotes the NMP portion of N_{zu}. We use (52) in the case where G_{zu} is NMP and unstable pole-zero cancellation is possible.

VI. Pole Reflection

In order to enforce asymptotic stability of the controller, we apply a reflection technique. In particular, if the updated controller $G_{c,k}$ is unstable, then $G_{c,k}$ is modified by replacing each unstable pole by its reciprocal inside the unit disk. For an IIR controller of order n_c, all of the poles may need to be reflected. However, in the next section we consider controllers all or most of whose poles are fixed inside the open unit disk. In this case, only a small number of controller poles may require reflection in order to enforce controller stability.

VII. CFI Controller Structure

As an alternative to the IIR controller (19), we consider the n_cth-order CFI controller

$$u(k) = \sum_{i=1}^{l} P_i(k)u(k-i) + \sum_{i=1}^{n_c} Q_i(k)y(k-i), \quad (53)$$

where $l \le n_c$ is the number of possibly nonzero poles of the controller. The controller (53) has at most l nonzero poles as well as $n_c - l$ poles fixed at zero. At each step k, if one or more of the l free poles is unstable, then we reflect each unstable pole to its reciprocal inside the unit disk. This technique relocates at most l poles, whereas, for the IIR controller, as many as n_c poles may need to be reflected within the unit disk. We use (53) to obtain asymptotically stable controllers in the case where the high-authority LQG controller is unstable. This approach implicitly assumes that the plant can be stabilized by a controller with the CFI structure (53).

We also consider a controller with an FIR controller implemented in parallel with an integrator

$$u(k) = \sum_{i=1}^{n_c} Q_i(k)y(k-i) + K_I(k)\gamma(k), \quad (54)$$

where the integrator state satisfies

$$\gamma(k) = \gamma(k-1) + Fy(k), \quad (55)$$

$\gamma(k) \in \mathbb{R}^{l_\gamma}$ and $F \in \mathbb{R}^{l_\gamma \times l_y}$ selects components of $y(k)$. The motivation for (54) is to fix the poles of the controller in order to remove the possibility of unstable pole-zero cancellation. We use (54) for step-command following for the adaptive servo problem.

VIII. H_2 Cost of Strictly Proper Controllers

For the plant (1)–(3), given a strictly proper controller $G_c \sim \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & 0 \end{array}\right]$, we can compute the H_2 cost as follows. We define

$$\widetilde{D} \triangleq \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \quad \widetilde{V} = \widetilde{D}\widetilde{D}^{\mathrm{T}}. \tag{56}$$

Then the H_2 cost is given by

$$J(A_c, B_c, C_c) = \mathrm{tr}(Q_1 R_1) + \mathrm{tr}(Q_2 C_c^{\mathrm{T}} R_2 C_c), \tag{57}$$

where $Q_1 \in \mathbb{R}^{n \times n}$, $Q_2 \in \mathbb{R}^{n_c \times n_c}$ satisfy

$$\widetilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \tag{58}$$

and $\widetilde{Q} \in \mathbb{R}^{n+n_c \times n+n_c}$ is the solution of the discrete-time Lyapunov equation

$$\widetilde{Q} = \widetilde{A}\widetilde{Q}\widetilde{A}^{\mathrm{T}} + \widetilde{V}. \tag{59}$$

IX. Examples

Example 1. Broadband disturbance rejection for a NMP plant using an IIR controller. We consider the Lyapunov-stable, NMP plant

$$G(\mathbf{q}) = \frac{(\mathbf{q}-0.8)(\mathbf{q}-1.1)}{(\mathbf{q}-0.85)(\mathbf{q}^2-1.9\mathbf{q}+1)}, \tag{60}$$

and let w be zero-mean Gaussian white noise with standard deviation 0.01. For this plant, the high-authority LQG controller is unstable with a pole at 1.0235. The H_2 cost of the LQG controller is 2.4587. Next, we apply RCAC with a high-order IIR controller structure (see [11] for details) and the FIR target model (52) with $n_c = 50$, $R_\theta = 0.02 I_{l_\theta}$, and $R_u = 0$. The RCAC controller has an unstable pole at 1.03 with H_2 cost 2.4606. Figure 2 shows that the closed-loop frequency response of the final RCAC controller, which is unstable, approximates the closed-loop frequency response of the unstable high-authority LQG controller.

Alternatively, we apply RCAC with the CFI controller (53) with $l = 2$ and the FIR target model (52). The RCAC controller is asymptotically stable with H_2 cost 2.5129, as compared to the H_2 cost 2.4587 of the unstable LQG controller. Figure 3 shows the closed-loop response of the RCAC controller and the closed-loop frequency response of the LQG controller and the final RCAC controller. Note that the closed-loop frequency response of the final RCAC controller, which is asymptotically stable, approximates the closed-loop frequency response of the unstable high-authority LQG controller. During the adaptation, RCAC performs a total of 12 pole reflections. Note that the pole reflections have no discernible effect on either the closed-loop spectral radius or the H_2 cost, as shown in Figure 4. ◇

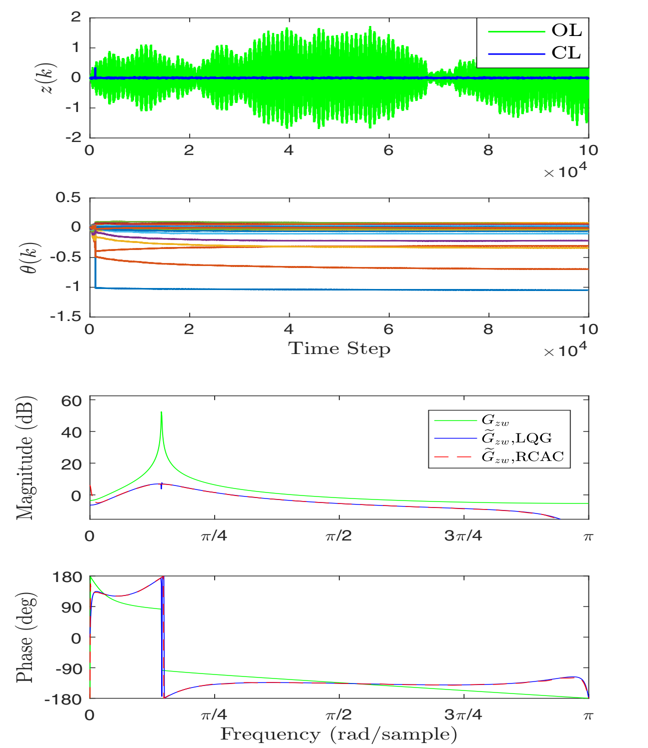

Fig. 2: Example 1: Broadband disturbance rejection for the NMP plant (60) using the IIR controller structure (19). RCAC approximates the closed-loop frequency response of the high-authority LQG controller. The frequency-response plots are shown at step $k = 10^5$. However, RCAC converges to an unstable controller (not shown).

Example 2. Step command following for a MISO NMP plant using a CFI controller. Consider the asymptotically stable, NMP, MISO plant

$$G(\mathbf{q}) = \begin{bmatrix} \frac{(\mathbf{q}-0.99)(\mathbf{q}^2+0.98)}{D(\mathbf{q})} & \frac{(\mathbf{q}-0.925)(\mathbf{q}-0.975)(\mathbf{q}-1.2)}{D(\mathbf{q})} \end{bmatrix}, \tag{61}$$

where $D(\mathbf{q}) = (\mathbf{q}-0.995)(\mathbf{q}-0.975)(\mathbf{q}^2-1.9\mathbf{q}+0.9125)$. Let r be an alternating sequence of step commands with heights ± 1, let $v = 0$, and let d be zero-mean Gaussian white noise with standard deviation 2×10^{-6}. Applying RCAC with the IIR controller (19) yields an unstable controller (not shown). Alternatively, we apply RCAC with the CFI controller (54). Since (54) consists of an FIR portion and an integrator, there is no possibility of unstable pole-zero cancellation, and thus we use the FIR target model (51) for both channels, with $n_c = 10$, $R_\theta = 10^{10} I_{l_\theta}$, and $R_u = 0$. Figure 5 shows the closed-loop response. Note that RCAC follows the sequence of step commands without knowledge of the NMP zeros of G. ◇

X. Conclusions

This paper considered retrospective cost adaptive control (RCAC) with composite FIR/IIR (CFI) controllers along with a reflection technique to enforce asymptotic stability of the controller. In several examples for which the high-authority LQG controller is unstable, this method was used

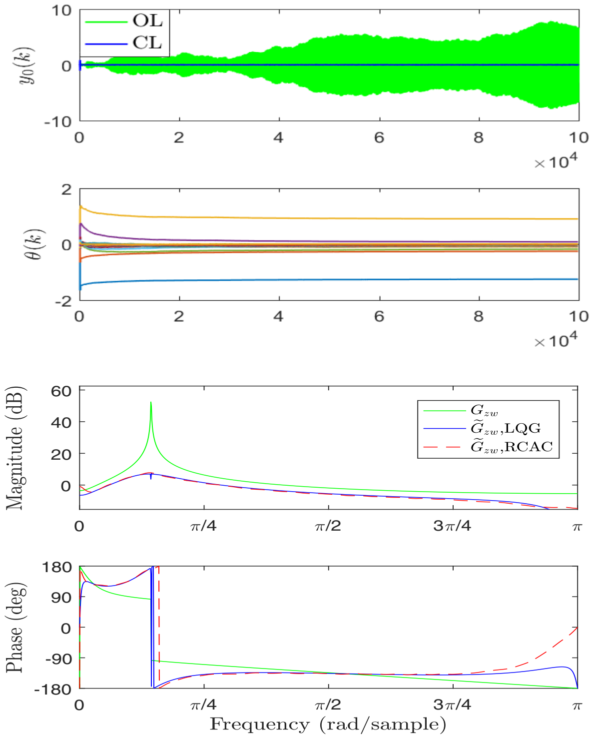

Fig. 3: Example 1: Broadband disturbance rejection for the NMP plant (60) using the CFI controller (53). RCAC approximates the closed-loop frequency response of the high-authority LQG controller. The frequency-response plots are shown at step $k = 10^5$.

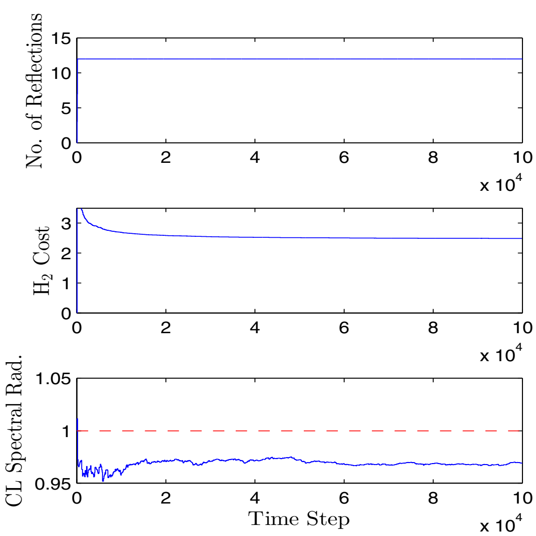

Fig. 4: Example 1: Broadband disturbance rejection for the minimum-phase plant (60) using the CFI controller (53). During the adaptation, RCAC performs 12 pole reflections. Note that the pole reflections have no discernible effect on either the closed-loop spectral radius or the H_2 cost.

to obtain asymptotically stable, stabilizing controllers whose H_2 performance is close to the performance of the high-authority LQG controller. In addition, an FIR controller in parallel with an integrator was used to achieve command following without knowledge of the NMP zeros of the plant.

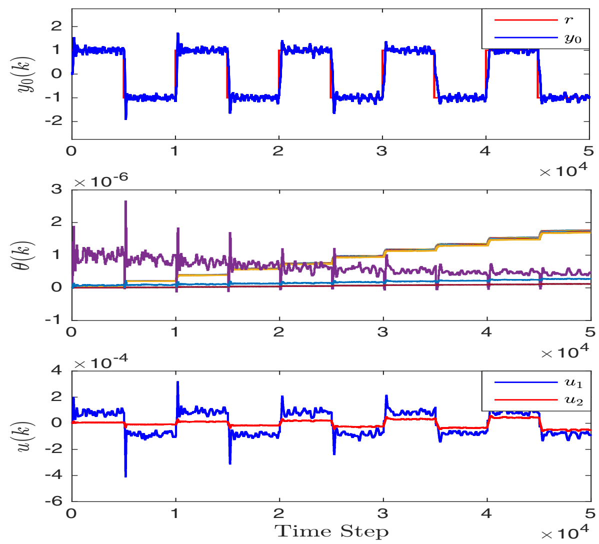

Fig. 5: Example 2: Step command following for the MISO, NMP plant (61) using the CFI controller (54). RCAC follows the sequence of step commands without knowledge of the NMP zeros of G, and does not converge to an unstable controller.

Using CFI controllers, future work will focus on refinements of the reflection technique for the case of NMP plants with unmodeled NMP zeros.

REFERENCES

[1] D. G. MacMartin and J. P. How, "Implementation and Prevention of Unstable Optimal Compensators," in *Proc. Amer. Contr. Conf.*, Baltimore, MD, June 1994, pp. 2190–2195.

[2] M. Vidyasagar, *Control System Synthesis: A Factorization Approach*. MIT Press, 1985.

[3] P. Dorato, *Analytic Feedback System Design: An Interpolation Approach*. Brooks/Cole, 1999.

[4] Y. Halevi, "Stable LQG Controllers," *IEEE Trans. Autom. Contr.*, vol. 39, no. 10, pp. 2104–2106, 1994.

[5] C. Ganesh and J. B. Pearson, "H_2-Optimization with Stable Controllers," *Automatica*, vol. 25, no. 4, pp. 629–634, 1989.

[6] D. U. Campos-Delgado and K. Zhou, "H_∞ Strong Stabilization," *IEEE Trans. Autom. Contr.*, vol. 46, no. 12, pp. 1968–1972, 2001.

[7] A. A. Saif, D. Gu, and I. Postlethwaite, "Strong Stabilization of MIMO Systems via H_∞ Optimization," *Sys. Contr. Lett.*, vol. 32, no. 2, pp. 111–120, 1997.

[8] R. Venugopal and D. S. Bernstein, "Adaptive Disturbance Rejection Using ARMARKOV System Representations," *IEEE Trans. Contr. Sys. Tech.*, vol. 8, pp. 257–269, 2000.

[9] M. A. Santillo and D. S. Bernstein, "Adaptive Control Based on Retrospective Cost Optimization," *J. Guid. Contr. Dyn.*, vol. 33, pp. 289–304, 2010.

[10] J. B. Hoagg and D. S. Bernstein, "Retrospective Cost Model Reference Adaptive Control for Nonminimum-Phase Systems," *J. Guid. Contr. Dyn.*, vol. 35, pp. 1767–1786, 2012.

[11] Y. Rahman, A. Xie, and D. S. Bernstein, "Retrospective Cost Adaptive Control: Pole Placement, Frequency Response, and Connections with LQG Control," *IEEE Contr. Sys. Mag.*, pp. 28–69, October 2017.

[12] S. Formentin and A. Karimi, "Direct Data-Driven Design of Sparse Controllers," in *Proc. Amer. Contr. Conf.*, Washington, DC, June 2013, pp. 3099–3104.

[13] Y. Rahman, K. Aljanaideh, E. D. Sumer, and D. S. Bernstein, "Adaptive Control of Aircraft Lateral Motion with an Unknown Transition to Nonminimum-Phase Dynamics," in *Proc. Amer. Contr. Conf.*, Portland, OR, June 2014, pp. 2359–2364.