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A Family of Optimal Nonlinear Feedback Controllers That Globally Stabilize Angular Velocity

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Abstract In this paper we use Hamilton-Jacobi-Bellman theory to derive optimal nonlinear feedback control laws for a special class of nonlinear systems. The results are applied to a spacecraft angular velocity stabilization problem with two torque inputs. A family of optimal nonlinear feedback controllers that globally asymptotically stabilize angular velocity is established. Special cases of this family of controllers include generalizations of the locally stabilizing controllers of Brockett and Aeyels to global stabilization as well as the globally stabilizing controller of Byrnes and Isidori.

1 Introduction

In a recent paper [1], optimal nonlinear feedback controllers were derived for the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0. \tag{1}$$

Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, the closed-loop system has the form

$$\dot{x}(t) = f(x(t), \phi(x(t))), \quad x(0) = x_0, \quad t \geq 0. \tag{2}$$

For a given nonquadratic performance functional, sufficient conditions for optimality have been given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. These conditions are restated later in Theorem 2.1, while numerous references to prior work in this area can be found in [1]. For the linear time-invariant system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

where $A \in \mathcal{R}^{n\times n}$ and $B \in \mathcal{R}^{n\times m}$ are constant matrices, optimal nonlinear feedback controllers have been derived by assuming nonquadratic state weighting and quadratic control weighting in the performance functional [1]. These results were motivated by the early work of Bass and Webber [2].

In this paper (see section 2), we deal with a class of nonlinear systems of the form

$$\dot{x}(t) = f_1(x(t)) + Bu(t), \tag{4}$$

where $f_1 \colon \mathcal{D} \to \mathcal{R}^n$ satisfies $f_1(0) = 0$. The integrand of the performance functional associated with (4) is chosen to be a polynomial function of x plus linear and quadratic terms in u. The optimal nonlinear feedback control law $u(t) = \phi(x(t))$ is chosen such that the optimality conditions are satisfied (see Corollary 2.2).

To illustrate this result we begin in section 3 by considering an illustrative example from [5]. We then apply this result in section 4 to a controlled version of the Lorenz equations which have been widely studied for their chaotic behavior. Our treatment of this problem was motivated by [6].

This formulation is then specialized in section 5 to the angular velocity stabilization of a rigid spacecraft. If the spacecraft has only two actuators along two principal axes and the uncontrolled principal axis is not an axis of symmetry, then equation (4) has the form

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \\ x_1x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \tag{5}$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$ and $u = [u_1, u_2]^{\mathrm{T}}$. Stabilization of this problem by smooth feedback control has been studied in [3-5]. In [3] a locally asymptotically stabilizing controller was given. Later, Aeyels [4] applied center manifold theory to reduce the problem to one of lower dimension and thereby obtained another locally stabilizing controller. More recently, Byrnes and Isidori [5] used a general methodology of nonlinear zero dynamics to derive a globally stabilizing feedback control law for the system. In the present work, we apply Hamilton-Jacobi-Bellman theory [1] to generate a *family* of optimal feedback controllers that globally asymptotically stabilize (5). It is shown that this family of controllers includes generalizations of the locally stabilizing controllers of Brockett and Aeyels to global stabilization as well as generalizations of the globally stabilizing controller of Byrnes and Isidori.

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2 Optimal Nonlinear Feedback Control

In this section we restate a theorem given in [1] and then specialize this result to a specific class of problems. We begin by considering the problem of characterizing feedback controllers that minimize a given performance functional. For this problem we consider the controlled system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{6}$$

where $x(t) \in \mathcal{D} \subset \mathcal{R}^n$ is the state variable, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{U} \subset \mathcal{R}^m$ is the control variable, $\mathcal{U}$ is an arbitrary set with $0 \in \mathcal{U}$, and $f\colon \mathcal{D} \times \mathcal{U} \to \mathcal{R}^n$ satisfies $f(0,0) = 0$. The control $u(\cdot)$ in (6) is restricted to the class of *admissible controls* consisting of measurable functions $u(\cdot)$ such that $u(t) \in \Omega, t \geq 0$, where the control constraint set $\Omega \subset \mathcal{U}$ is given. We assume $0 \in \Omega$ and Ω is compact.

A measurable mapping $\phi\colon \mathcal{D} \to \Omega$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, where ϕ is a control law and $x(t)$ satisfies (6), then $u(\cdot)$ will be called a *feedback control*. A feedback control is admissible since the control law $\phi(\cdot)$ takes values in Ω, and $x(\cdot)$ is absolutely continuous.

Letting $L(x, u)$ be the performance integrand, where $L\colon \mathcal{D} \times \mathcal{U} \to \mathcal{R}$, the corresponding Hamiltonian is defined as

$$H(x, u, p) \triangleq L(x, u) + p^T f(x, u),$$

where $p \in \mathcal{R}^n$. Furthermore, we define the set of asymptotically stabilizing admissible controllers $\mathcal{S}(x_0)$ for each initial condition $x_0 \in \mathcal{D}$, that is,

$$\mathcal{S}(x_0) \triangleq \{u(\cdot)\colon u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (6) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

Although this set plays a role in the following theorem, note that no explicit characterization of this set is required.

Theorem 2.1.[1] Consider the controlled system (6) with performance functional

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))dt. \tag{7}$$

Assume that there exists a C^1 function $V\colon \mathcal{D} \to \mathcal{R}$ and a control law $\phi\colon \mathcal{D} \to \Omega$ such that

$$V(0) = 0, \tag{8}$$

$$V(x) > 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{9}$$

$$\phi(0) = 0, \tag{10}$$

$$V'(x) f(x, \phi(x)) < 0, \quad x \in \mathcal{D}, \quad x \neq 0, \tag{11}$$

$$H(x, \phi(x), V'^{\mathrm{T}}(x)) = 0, \quad x \in \mathcal{D}, \tag{12}$$

$$H(x, u, V'^{\mathrm{T}}(x)) \geq 0, \quad x \in \mathcal{D}, \quad u \in \Omega. \tag{13}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system (2) is locally asymptotically stable, and

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad \text{for all } x_0 \in \mathcal{D}. \tag{14}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad \text{for all } x_0 \in \mathcal{D}. \tag{15}$$

Finally, if $\mathcal{D} = \mathcal{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then the asymptotic stability is global.

The proof of this theorem is given in [1].

Remark 2.1. The classical Hamilton-Jacobi-Bellman (HJB) equation is of the form

$$\frac{\partial}{\partial t} V(t, x(t)) + \min_{u \in \Omega} H(t, x(t), u, \frac{\partial}{\partial x} V(t, x(t))) = 0. \tag{16}$$

If V is independent of t, then the HJB equation reduces to the algebraic (time-invariant) relations (12), (13).

Remark 2.2. Theorem 2.1 gives sufficient conditions for optimality and asymptotic stability of the feedback control law $\phi(x(\cdot))$ and the controlled system (6). Necessary conditions for the existence of a continuously differentiable control law that asymptotically stabilizes (6) are given in [3].

Next, we consider a special case of Theorem 2.1. Let $\mathcal{D} = \mathcal{R}^n, \Omega = \mathcal{U} = \mathcal{R}^m$, and consider the nonlinear system

$$\dot{x}(t) = f_1(x(t)) + Bu(t) \tag{17}$$

as in (4), with performance integrand $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}} R u, \tag{18}$$

where $L_1\colon \mathcal{R}^n \to \mathcal{R}$, $R \in \mathcal{R}^{m \times m}$ is a positive definite matrix, and $L_2\colon \mathcal{R}^n \to \mathcal{R}^{1 \times m}$ with $L_2(0) = 0$. With the specialization (17), (18), we have the following corollary of Theorem 2.1.

Corollary 2.2. Consider the controlled system (17), assume that there exists a C^1 function $V\colon \mathcal{R}^n \to \mathcal{R}$ and a function $L_2\colon \mathcal{R}^n \to \mathcal{R}^{1 \times m}$, such that

$$V(0) = 0, \tag{19}$$

$$V(x) > 0, \quad x \in \mathcal{R}^n, \quad x \neq 0, \tag{20}$$

$$V'(x)[f_1(x) - \frac{1}{2} B R^{-1} L_2^{\mathrm{T}}(x) - \frac{1}{2} B R^{-1} B^{\mathrm{T}} V'^{\mathrm{T}}(x)] < 0, \quad x \in \mathcal{R}^n, \quad x \neq 0, \tag{21}$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Furthermore, define the feedback control $u(\cdot) = \phi(x(\cdot))$, where

$$\phi(x) \triangleq -\frac{1}{2} R^{-1} [L_2^{\mathrm{T}}(x) + B^{\mathrm{T}} V'^{\mathrm{T}}(x)]. \tag{22}$$

Then the solution $x(t) = 0$, $t \geq 0$, of the closed-loop system

$$\dot{x}(t) = f_1(x(t)) + B\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{23}$$

is globally asymptotically stable, and the performance functional

$$J(x_0, u(\cdot)) = \int_0^\infty (L_1(x) + L_2(x)u + u^{\mathrm{T}}Ru)\mathrm{d}t, \tag{24}$$

where

$$L_1(x) \triangleq \phi^{\mathrm{T}}(x)R\phi(x) - V'(x)f_1(x), \quad x \in \mathcal{R}^n, \tag{25}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \quad \text{for all } x_0 \in \mathcal{R}^n. \tag{26}$$

Furthermore,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \quad \text{for all } x_0 \in \mathcal{R}^n. \tag{27}$$

Proof. For (17), (18), the Hamiltonian is

$$H(x, u, V'^{\mathrm{T}}(x)) = L_1(x) + L_2(x)u + u^{\mathrm{T}}Ru + V'(x)(f_1(x) + Bu). \tag{28}$$

The optimal feedback control law (22) is obtained by setting $\frac{\partial H}{\partial u} = 0$. With (22), it can be seen that (19)-(21) imply (8), (9), (11). Next, since V is C^1 and $x = 0$ is a local minimum of V, it follows that $V'(0) = 0$. In addition, since by assumption $L_2(0) = 0$, it follows that $\phi(0) = 0$. This proves (10). Next, (12) holds because of the choice of $L_1(x)$ given by (25). Finally, since

$$H(x, u, V'(x)) = (u - \phi(x))^{\mathrm{T}}R(u - \phi(x)),$$

and R is positive definite, equation (13) holds. The results of the corollary now follow directly from Theorem 2.1. □

Remark 2.3. With $L_1(x)$ in equation (25) and $\phi(x)$ in equation (22), $L(x, u)$ can be expressed as

$$\begin{aligned} L(x,u) &= u^{\mathrm{T}}Ru - \phi^{\mathrm{T}}(x)R\phi(x) + L_2(x)[u - \phi(x)] \\ &\quad - V'(x)[f_1(x) + B\phi(x)], \end{aligned}$$

which can be rearranged as

$$\begin{aligned} L(x,u) &= [u + \frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)]^{\mathrm{T}}R[u + \frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)] \\ &\quad - [V'(x)f_1(x) - \frac{1}{2}V'(x)BR^{-1}L_2^{\mathrm{T}}(x) \\ &\quad - \frac{1}{2}V'(x)BR^{-1}B^{\mathrm{T}}V'^{\mathrm{T}}(x)] \\ &\quad - \frac{1}{4}V'(x)BR^{-1}B^{\mathrm{T}}V'^{\mathrm{T}}(x). \end{aligned} \tag{29}$$

In the above expression, the first term is nonnegative, while the second term is $-\dot{V}(x)$, which, according to (21), is also nonnegative, so we have

$$L(x, u) \geq -\frac{1}{4}V'(x)BR^{-1}B^{\mathrm{T}}V'^{\mathrm{T}}(x).$$

Hence, the performance integrand $L(x, u)$ may be negative, which allows the possibility that the performance functional $J(x_0, u(\cdot))$ be negative for some control law. However, if we confine $u(\cdot) \in \mathcal{S}(x_0)$ so that $u(\cdot)$ is a stabilizing controller, then, according to (26), (27), we have

$$J(x_0, u(\cdot)) \geq V(x_0) \geq 0, \quad \text{for all } x_0 \in \mathcal{R}^n \text{ and } u(\cdot) \in \mathcal{S}(x_0).$$

In addition, as will be seen in section 5, in certain special cases $L(x, u)$ is actually nonnegative.

Note that the function $L_2(x)$, which appears in the linear term $L_2(x)u$ in the performance functional, provides greater flexibility in adjusting the control law (22). This term is crucial in satisfying (21), which implies that the Lyapunov derivative $\dot{V}(x)$ is negative. Once $L_2(x)$ is determined, $L_1(x)$ can be obtained by direct calculation. If, however, there does not exist an $L_2(x)$ satisfying (21), then another $V(x)$ should be considered. It should be noted that although the present design scheme does not provide a systematic technique for generating suitable Lyapunov functions, it does provide a quick and easy method for checking whether the chosen function $V(x)$ qualifies as a Lyapunov function. Furthermore, by varying the parameters characterizing $V(x)$ and R, one can generate a family of optimal controllers which provide different response rates for the closed-loop system.

3 An Illustrative Example

To illustrate Corollary 2.2, we consider the controlled system

$$\begin{aligned} \dot{x}_1 &= x_2^2 - x_1^5, \\ \dot{x}_2 &= x_1^2 + u \end{aligned}$$

considered by Byrnes and Isidori [5]. For this system, let $V(x)$ be defined by

$$V(x) = p_1x_1^2 + p_2x_2^2 > 0,$$

where $p_1 > 0$ and $p_2 > 0$, and consider the performance functional

$$L(x, u) = L_1(x) + L_2(x)u + Ru^2,$$

where $R > 0$. To satisfy (21), it can be shown that, although $L_2(x)$ is not unique, the simplest form for $L_2(x)$ is

$$L_2(x) = 2R\frac{p_1}{p_2}x_1x_2 + 2Rx_1^2.$$

The optimal nonlinear feedback control law given by (22) is thus

$$\phi(x) = -\frac{p_1}{p_2}x_1x_2 - x_1^2 - \frac{p_2}{R}x_2.$$

Then, from (25), $L_1(x)$ is of the form

$$L_1(x) = \frac{1}{R}[R\frac{p_1}{p_2}x_1x_2+Rx_1^2+p_2x_2]^2-[2p_1x_1(x_2^2-x_1^5)+2p_2x_1^2x_2].$$

Hence, $V(x)$ is a Lyapunov function since

$$\dot{V}(x) = -2p_1x_1^6 - 2\frac{p_2^2}{R}x_2^2 < 0, \quad \text{for all } x \in \mathcal{R}^2, \quad x \neq 0,$$

which confirms (21). Note that for larger R, the decay rate $\dot{V}(x)$ of Lyapunov function is smaller and thus the closed-loop response is slower. Finally, taking $p_1 = p_2 = R = \frac{1}{2}$ yields the controller developed by Byrnes and Isidori [5].

4 Stabilization of the Controlled Lorenz Equations

Consider the controlled Lorenz equations

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2, \\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3 + u, \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned}$$

studied by Vincent and Yu [6], where σ, r, b are positive constants. In the notation of (17), we have

$$f_1(x) = \begin{bmatrix} -\sigma(x_1 - x_2) \\ rx_1 - x_2 - x_1x_3 \\ x_1x_2 - bx_3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $x = [x_1, x_2, x_3]^{\mathrm{T}}$. If $r < 1$, the uncontrolled Lorenz equation has only one equilibrium state, namely, $[0,0,0]^{\mathrm{T}}$, which is locally asymptotically stable. If $r > 1$, the uncontrolled Lorenz equation has three equilibrium states, namely, $[\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1]^{\mathrm{T}}$, $[-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1]^{\mathrm{T}}$, and $[0,0,0]^{\mathrm{T}}$. The stability of the fisrt two equilibrium states depends on the values of σ, b, and r, while the last equilibrium state is unstable. For the parameter values $\sigma = 10$, $b = 8/3, r = 28$, chosen by Vincent and Yu [6], all these three equilibrium states are unstable. In [6], Vincent and Yu established a linear feedback control law and a bounded bang-bang control law both of which locally stabilized the controlled Lorenz equation around the unstable equilibrium state $[6\sqrt{2}, 6\sqrt{2}, 27]^{\mathrm{T}}$, which represents steady convection motion. Here we apply Corollary 2.2 to obtain a globally stabilizing control law for the no-convection equilibrium state $[0,0,0]^{\mathrm{T}}$. Our controller is valid for both $r < 1$ and $r > 1$. We choose $V(x)$ to be

$$V(x) = p_1x_1^2 + p_2x_2^2 + p_3x_3^2 > 0,$$

where $p_1 > 0, p_2 > 0$, and $p_3 > 0$. Following the same procedures as in the previous example, we take $p_2 = p_3$, and $L_2(x)$ to be of the form

$$L_2(x) = \frac{R}{p_2}(2p_1\sigma + 2p_2r)x_1 + ax_2.$$

The optimal feedback control law is thus

$$\phi(x) = -(\frac{p_1}{p_2}\sigma + r)x_1 - (\frac{a}{2R} + \frac{p_2}{R})x_2$$

with Lyapunov derivative

$$\dot{V}(x) = -2p_1\sigma x_1^2 - 2p_2(1 + \frac{p_2}{R} + \frac{a}{2R})x_2^2 - 2p_2x_3^2.$$

It can be seen that in order to make $\dot{V}(x) < 0$, a must be chosen such that $a > -2R - 2p_2$. Some simplification of $\phi(x)$ and possible reduction in control effort is obtained by choosing $a = -2p_2$. In this case, the optimal feedback control law and Lyapunov derivative are given by

$$\phi(x) = -(\frac{p_1}{p_2}\sigma + r)x_1,$$
$$\dot{V}(x) = -2p_1\sigma x_1^2 - 2p_2x_2^2 - 2p_2x_3^2.$$

Note that with the control law $\phi(x)$ given above, the function $f(x,\phi(x)) = f_1(x) + B\phi(x)$ has only one equilibrium state $[0,0,0]^{\mathrm{T}}$. Also note that the globally stabilizing feedback control law $\phi(x)$ only requires knowledge of x_1.

5 Angular Velocity Stabilization

Consider the angular velocity stabilization of a rigid spacecraft with two actuators along principal axes and whose uncontrolled principal axis is not an axis of symmetry. The dynamical equation for this problem is given by (5). The associated linearized system has one uncontrollable eigenvalue on the imaginary axis, which corresponds to the critical case [3]. For this system, let $V(x)$ and $L(x,u)$ be of the form

$$V(x) = p_1(x_1 + \alpha x_3^k)^2 + p_2(x_2 + \beta x_3^{k+1})^2 + p_3x_3^2,$$
$$L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}Ru,$$

where p_1, p_2, p_3 are positive real numbers, k is a positive integer, and α, β are real numbers. Furthermore, let R have the form

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \quad r_1 > 0, r_2 > 0.$$

To satisfy (21), it can be shown that $L_2(x)$ must be one of the forms

$$L_2^{\mathrm{T}}(x) = \begin{bmatrix} 2r_1k\alpha x_1x_2x_3^{k-1} + 2r_1(p_3/p_1)x_2x_3 \\ 2r_2(k+1)\beta x_1x_2x_3^k - 2r_2\alpha(p_3/p_2)x_3^{k+1} \end{bmatrix}, \quad (30)$$

$$L_2^{\mathrm{T}}(x) = \begin{bmatrix} 2r_1k\alpha x_1x_2x_3^{k-1} - 2r_1\beta(p_3/p_1)x_3^{k+2} \\ 2r_2(k+1)\beta x_1x_2x_3^k + 2r_2(p_3/p_2)x_1x_3 \end{bmatrix}. \quad (31)$$

Substituting $L_2(x)$ into (22), we obtain a family of optimal nonlinear feedback control laws

$$\phi(x) = \begin{bmatrix} -k\alpha x_1x_2x_3^{k-1} - (p_3/p_1)x_2x_3 \\ -(p_1/r_1)(x_1 + \alpha x_3^k), \\ -(k+1)\beta x_1x_2x_3^k + \alpha(p_3/p_2)x_3^{k+1} \\ -(p_2/r_2)(x_2 + \beta x_3^{k+1}) \end{bmatrix}, \quad (32)$$

$$\phi(x) = \begin{bmatrix} -k\alpha x_1x_2x_3^{k-1} + \beta(p_3/p_1)x_3^{k+2} \\ -(p_1/r_1)(x_1 + \alpha x_3^k), \\ -(k+1)\beta x_1x_2x_3^k - (p_3/p_2)x_1x_3 \\ -(p_2/r_2)(x_2 + \beta x_3^{k+1}) \end{bmatrix}, \quad (33)$$

corresponding to $L_2(x)$ given by (30) and (31), respectively. The decay rates of the function $V(x)$ are the same for both cases, which is

$$\dot{V}(x) = -2\frac{p_1^2}{r_1}(x_1+\alpha x_3^k)^2 - 2\frac{p_2^2}{r_2}(x_2+\beta x_3^{k+1})^2 + 2\alpha\beta p_3 x_3^{2(k+1)}.$$

It is obvious that if $\alpha\beta < 0$ then $\dot{V}(x) < 0$, for all nonzero $x \in \mathcal{R}^3$. Thus, if α, β are chosen to be nonzero and have opposite sign, then the control law $\phi(x)$ globally asymptotically stabilizes (5), and $V(x)$ is a Lyapunov function for the closed-loop system. Finally, $L_1(x)$ can be calculated directly from (25) for both cases as

$$\begin{aligned}L_1(x)= {} & r_1[k\alpha x_1 x_2 x_3^{k-1} + \frac{p_3}{p_1}x_2x_3 + \frac{p_1}{r_1}(x_1+\alpha x_3^k)]^2 \\ & + r_2[(k+1)\beta x_1 x_2 x_3^k - \alpha\frac{p_3}{p_2}x_3^{k+1} + \frac{p_2}{r_2}(x_2+\beta x_3^{k+1})]^2 \\ & - x_1x_2[2p_1 k\alpha x_3^{k-1}(x_1+\alpha x_3^k) \\ & \quad + 2p_2(k+1)\beta x_3^k(x_2+\beta x_3^{k+1}) + 2p_3x_3],\end{aligned}$$

or

$$\begin{aligned}L_1(x)= {} & r_1[k\alpha x_1 x_2 x_3^{k-1} - \beta\frac{p_3}{p_1}x_3^{k+2} + \frac{p_1}{r_1}(x_1+\alpha x_3^k)]^2 \\ & + r_2[(k+1)\beta x_1 x_2 x_3^k + \frac{p_3}{p_2}x_1x_3 + \frac{p_2}{r_2}(x_2+\beta x_3^{k+1})]^2 \\ & - x_1x_2[2p_1 k\alpha x_3^{k-1}(x_1+\alpha x_3^k) \\ & \quad + 2p_2(k+1)\beta x_3^k(x_2+\beta x_3^{k+1}) + 2p_3x_3].\end{aligned}$$

Returning to Remark 2.3, the performance integrand for this problem is from (29)

$$\begin{aligned}L(x,u)= {} & [u+\frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)]^{\mathrm{T}}R[u+\frac{1}{2}R^{-1}L_2^{\mathrm{T}}(x)] \\ & + \frac{p_1^2}{r_1}(x_1+\alpha x_3^k)^2 + \frac{p_2^2}{r_2}(x_2+\beta x_3^{k+1})^2 - 2\alpha\beta p_3 x_3^{2(k+1)},\end{aligned}$$

where R and $L_2(x)$ are as defined previously. Since $\alpha\beta < 0$, the above expression shows that $L(x,u)$ is nonnegative for all u in this problem.

In the special case of the control law (32) that $k=1, p_1 = p_2 = r_1 = r_2 = \frac{1}{2}, p_3 = 1, \alpha = 1$, and $\beta = -1$, we obtain the globally asymptotically stabilizing control law

$$\phi(x) = \begin{bmatrix} -x_1x_2 - 2x_2x_3 - (x_1+x_3) \\ 2x_1x_2x_3 + 2x_3^2 - (x_2 - x_3^2) \end{bmatrix}, \qquad (34)$$

which is the controller obtained by Byrnes and Isidori [5]. Deleting all but the last terms in ϕ yields the locally stabilizing controller obtained by Brockett [3]. If, however, $k = 2, p_1 = p_2 = r_1 = r_2 = \frac{1}{2}$, and $p_3 = 1$, are chosen in (32), then we obtain the globally asymptotically stabilizing control law

$$\phi(x) = \begin{bmatrix} -2\alpha x_1x_2x_3 - 2x_2x_3 - (x_1+\alpha x_3^2) \\ -3\beta x_1x_2x_3^2 + 2\alpha x_3^3 - (x_2+\beta x_3^3) \end{bmatrix}. \qquad (35)$$

Deleting all but the last terms in ϕ yields the locally stabilizing controller proposed by Aeyels [4]. Thus the control laws (34) and (35) can be viewed as globally asymptotically stabilizing generalizations of the controllers obtained in [3,4], while the family of controllers (32) yields the control law of [5] as a special case.

In studying the rigid body angular velocity stabilization problem, one needs to consider the rates of response, the maximum control effort (torque) available, and the total energy expenditure in control. The family of control laws obtained in this paper allows us to make tradeoffs among these factors. Some simulations were performed by varying the parameters involving in equations (32), (33). It was found that the response of angular velocity depends upon the parameters chosen and the initial conditions in a fairly complicated way. The only obvious observation on these parameters is that k should be kept small, that is, $k = 1$ will be the best choice. If k is taken to be larger, the response tends to approach the equilibrium point slowly. Four sets of parameters are selected to yield the following controllers.

- **Controller 1.** This control law, which is due to [5], is given by (32) with $k = 1, \alpha = 1, \beta = -1, p_1 = p_2 = r_1 = r_2 = 1/2$, and $p_3 = 1$.

- **Controller 2.** This control law is given by (32) with $k = 1, \alpha = 1, \beta = -1, p_1 = 2, p_2 = 1/2, p_3 = 4, r_1 = 1/2$, and $r_2 = 1/2$.

- **Controller 3.** This control law is given by (32) with $k = 1, \alpha = -1, \beta = 1, p_1 = p_2 = r_1 = r_2 = 1/2$, and $p_3 = 1$.

- **Controller 4.** This control law is given by (33) with $k = 1, \alpha = 1, \beta = -1, p_1 = p_2 = r_1 = r_2 = 1/2$, and $p_3 = 1$.

These controllers were simulated with the initial condition $(-1,-1,-1)^{\mathrm{T}}$. The simulation results for angular velocities and control efforts are shown in Figure 1 - 3, where solid, dashed, dotted, and dashed-dotted lines represent controllers 1, 2, 3, and 4, respectively. Figure 1 shows the phase plane of x_1 and x_2 from $t = 0$ to $t = 5$ sec, while Figure 2 shows the decay of state x_3 with respect to time. Figure 3 shows the control action from $t = 0$ to $t = 5$ sec. It can be seen that, for this initial condition, Controller 2 has the fastest decay rate, but it requires the largest control effort, while Controller 3 uses the least control effort and results in a satisfactory decay rate. Hence, these simulation results enable us to make tradeoffs in designing the control law for rigid body angular velocity stabilization problems.

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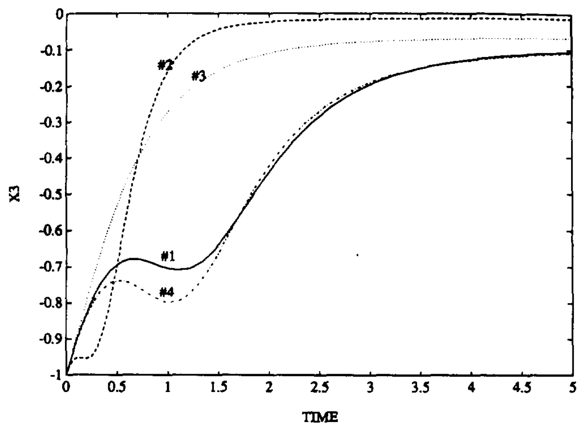

Figure 2.

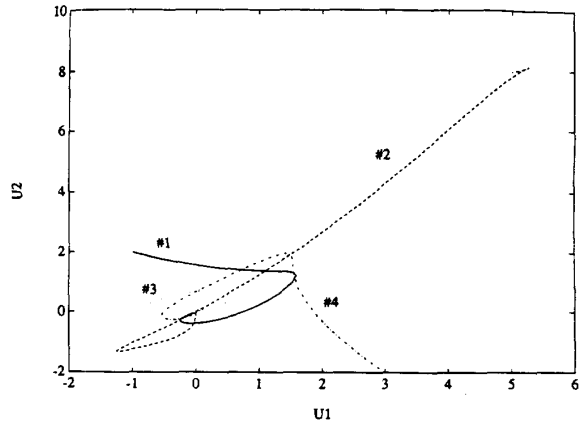

Figure 3.

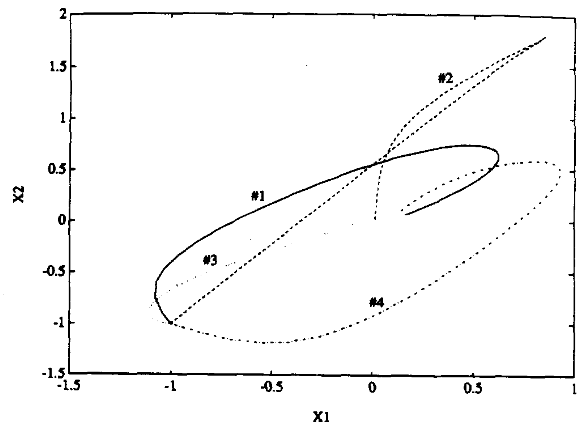

Figure 1.