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Adaptive Static-Output-Feedback Stabilization Using Retrospective Cost Optimization

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Abstract— **We present a discrete-time, adaptive, static-output-feedback control law that is effective for systems that are unstable, MIMO, and/or nonminimum phase. In particular, we present numerical examples to provide guidelines concerning the modeling information required for controller implementation. This information includes a sufficient number of Markov parameters to capture the relative degree, the sign of the high-frequency gain, and the nonminimum-phase zeros (if any). No further information about the poles or zeros need be known. In addition, we present a stability proof for a full-state-feedback specialization.**

I. INTRODUCTION

Unlike robust control, which fixes the control gains based on a prior, fixed level of modeling uncertainty, adaptive control algorithms tune the feedback gains in response to the behavior of the true plant. Generally speaking, adaptive controllers require less prior modeling information than robust controllers, and thus can be viewed as highly parameter-robust control laws. The price paid for the ability of adaptive control laws to operate with limited prior modeling information is the complexity of analyzing and quantifying the stability and performance of the closed-loop system, especially in light of the fact that adaptive control laws, even for linear plants, are nonlinear.

Stability and performance analysis of adaptive control laws often entails restrictive assumptions on the dynamics of the plant. For example, a widely invoked assumption in adaptive control is passivity [1], which is restrictive and difficult to verify in practice. A weaker but still restrictive assumption is that the plant is minimum phase [2, 3], which may entail the same difficulties, especially since sampled-data control can give rise to nonminimum-phase sampling zeros whether or not the continuous-time system is minimum phase [4]. Beyond these assumptions, adaptive control laws are known to be sensitive to unmodeled dynamics and sensor noise [5, 6], which motivates robust adaptive control laws [7].

In addition to these basic issues, adaptive control laws may entail unacceptable transients during adaptation, which may be exacerbated by actuator limitations [8–10]. In fact, adaptive control under extremely limited modeling information such as uncertainty in the high-frequency gain [11, 12] may yield a transient response that exceeds the practical limits of the plant. Therefore, the type and quality of the available modeling information as well as the speed of adaptation must be considered in the analysis and implementation of adaptive control laws. These issues are discussed in [13].

The goal of this paper is to present a discrete-time, adaptive, MIMO, static-output-feedback controller that is effective for systems that are unstable, nonsquare, and/or nonminimum-phase. Furthermore, the control law specializes to full-state-feedback systems when the plant outputs are direct measurements of the state variables. The algorithm is developed in discrete time based on a discrete-time plant model obtained by either plant discretization or discrete-time system identification so that the controller can be implemented directly as embedded code without an intermediate controller discretization step. Although the discrete-time adaptive control literature is more limited than the continuous-time literature, there are discrete-time versions of many continuous-time algorithms [14–17], as well as adaptive control algorithms unique to discrete time [2, 18].

Fixed-gain static-output-feedback is a challenging problem, especially in the MIMO case and in the presence of transmission zeros [19]. These difficulties are evident in the adaptive control case as well, where the presence of zeros impacts the required prior modeling information.

The adaptive controller presented in this paper is based on retrospective cost optimization. This method is used to adapt dynamic compensators for disturbance rejection, adaptive stabilization, adaptive command following, and model reference adaptive control in [20, 21]. Retrospective cost optimization is a measure of performance at the current time based on a past window of data and without assumptions about the command or disturbance signals. In particular, retrospective cost optimization acts as an inner loop to the adaptive control algorithm by modifying the performance variables based on the difference between the actual past control inputs and the recomputed past control inputs based on the current control law.

We provide a stability proof for a full-state-feedback specialization. In addition, we present numerical examples to illustrate the algorithm's effectiveness in handling systems that are unstable and/or nonminimum phase and to provide insight into the modeling information required for controller implementation. This information includes a sufficient number of Markov parameters to capture the relative degree, the sign of the high-frequency gain, as well as to approximate the nonminimum-phase zeros (if any). In the case of a minimum-phase plant, these numerical results suggest that only the first

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nonzero Markov parameter is required, which is consistent with the discrete-time adaptive control algorithm given in [3]. In particular, for full-state feedback, the retrospective-cost adaptive controller has downward and upward gain margins of 6 dB and ∞ dB, respectively, which is reminiscent of fixed-gain LQR control. These properties are intended to provide motivation for more general proofs of stability and convergence.

II. Problem Formulation

Consider the MIMO discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

$$y(k) = Cx(k), \tag{2}$$

$$z(k) = E_1 x(k), \tag{3}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^{l_y}$, $z(k) \in \mathbb{R}^{l_z}$, $u(k) \in \mathbb{R}^{l_u}$, and $k \geq 0$. Our goal is to develop an adaptive output-feedback controller for stabilization, that is, convergence of the *performance variable* z to zero. We assume that measurements of y and z are available for feedback.

For a positive integer r, we define the *extended performance vector* $Z(k) \in \mathbb{R}^{l_z r}$ and the *extended input vector* $U(k) \in \mathbb{R}^{l_u r}$ by

$$Z(k) \triangleq \begin{bmatrix} z(k-r+1) \\ z(k-r+2) \\ \vdots \\ z(k) \end{bmatrix}, \quad U(k) \triangleq \begin{bmatrix} u(k-r) \\ u(k-r+1) \\ \vdots \\ u(k-1) \end{bmatrix}.$$

Note that $Z(k)$, $U(k)$, and $x(k)$ are related by

$$Z(k) = \Gamma x(k-r) + \mathcal{H} U(k), \tag{4}$$

where $\Gamma \in \mathbb{R}^{l_z r}$ and $\mathcal{H} \in \mathbb{R}^{l_z r \times l_u r}$ are given by

$$\Gamma \triangleq \begin{bmatrix} E_1 A \\ E_1 A^2 \\ \vdots \\ E_1 A^r \end{bmatrix}, \quad \mathcal{H} \triangleq \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ H_2 & H_1 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ H_r & H_{r-1} & \cdots & H_1 \end{bmatrix},$$

and, for $i = 1, 2, \ldots$, the Markov parameters H_i of the system (1)–(3) are

$$H_i \triangleq E_1 A^{i-1} B. \tag{5}$$

Let d denote the *relative degree* of (A, B, E_1), that is, the smallest positive integer i such that the ith Markov parameter H_i is nonzero. Note that, if $r < d$, then $\mathcal{H} = 0$. Therefore, we assume that $r \geq d$.

III. Properties of the Markov Parameter Polynomial

From (4), the expression for $z(k)$ is

$$z(k) = E_1 A^r x(k-r) + H_1 u(k-1) + H_2 u(k-2) + \cdots + H_r u(k-r). \tag{6}$$

In terms of the backward-shift operator $\mathbf{q}^{-1}$, (6) can be rewritten as

$$z(k) = E_1 A^r \mathbf{q}^{-r} x(k) + \left[H_1 \mathbf{q}^{-1} + H_2 \mathbf{q}^{-2} + \cdots + H_r \mathbf{q}^{-r}\right] u(k). \tag{7}$$

Shifting (7) forward by r steps gives

$$z(k+r) = E_1 A^r x(k) + p_r(\mathbf{q}) u(k), \tag{8}$$

where

$$p_r(\mathbf{q}) \triangleq H_1 \mathbf{q}^{r-1} + H_2 \mathbf{q}^{r-2} + \cdots + H_r \tag{9}$$

is the *Markov parameter polynomial* and $\mathbf{q}$ is the forward-shift operator. For $r < d$, note that $p_r(\mathbf{q}) = 0$, whereas, if $r \geq d$, then

$$p_r(\mathbf{q}) = H_d \mathbf{q}^{r-d} + H_{d+1} \mathbf{q}^{r-d-1} + \cdots + H_r. \tag{10}$$

Definition III.1. Let $\xi \in \mathbb{C}$ be a transmission zero of G_{zu}. Then, ξ is an *outer zero* of G_{zu} if $|\xi| \geq \rho(A)$. Otherwise, ξ is an *inner zero* of G_{zu}.

The Markov parameter polynomial $p_r(\mathbf{q})$ contains information about the relative degree d, the sign of the high-frequency gain H_d (in the case $l_u = l_z = 1$), and an approximation of each outer nonminimum-phase zero. The following result shows that, for SISO transfer functions, the roots of the Markov parameter polynomial asymptotically approximate each outer nonminimum-phase zero of the transfer function from u to z. As r increases, this approximation improves. For each value of r, the remaining roots play no role in the stability and convergence of the adaptive control algorithm, but what is important is the need to choose r sufficiently large to adequately approximate the nonminimum-phase zeros.

Fact III.1. Consider $l_u = l_z = 1$ and let $\xi \in \mathbb{C}$ be an outer zero of the transfer function from u to z. For each r, let $\mathcal{R}_r \triangleq \{\xi_{r,1}, \ldots, \xi_{r,r-d}\}$ be the set of roots of $p_r(\mathbf{q})$. Then, there exists a sequence $\{\xi_{r,i_r}\}_{r=1}^{\infty}$ that converges to ξ as $r \to \infty$.

IV. Retrospective Cost Optimization

Let

$$u(k) = K(k) y(k), \tag{11}$$

where $K(k) \in \mathbb{R}^{l_u \times l_y}$ is the *gain matrix*. From (11), it follows that $U(k)$ can be rewritten as

$$U(k) \triangleq \sum_{i=1}^{r} L_i K(k-i) y(k-i), \tag{12}$$

where

$$L_i \triangleq \begin{bmatrix} 0_{(r-i)l_u \times l_u} \\ I_{l_u} \\ 0_{(i-1)l_u \times l_u} \end{bmatrix} \in \mathbb{R}^{l_u r \times l_u}. \tag{13}$$

Next, for $\mathcal{K} \in \mathbb{R}^{l_u \times l_y}$, define the *retrospective performance vector* $\hat{Z}(\mathcal{K}, k) \in \mathbb{R}^{l_z r}$ by

$$\hat{Z}(\mathcal{K}, k) \triangleq \Gamma x(k-r) + \mathcal{H} \hat{U}(\mathcal{K}, k), \tag{14}$$

where $\hat{U}(\mathcal{K},k) \in \mathbb{R}^{l_u r}$ is the *recomputed input vector*, given by

$$\hat{U}(\mathcal{K},k) \triangleq \sum_{i=1}^{r} L_i \mathcal{K} y(k-i). \tag{15}$$

Subtracting (4) from (14) yields

$$\hat{Z}(\mathcal{K},k) = Z(k) - \mathcal{H}\left[U(k) - \hat{U}(\mathcal{K},k)\right], \tag{16}$$

and hence,

$$\hat{Z}(\mathcal{K},k) = f(k) + D(k)\text{vec}\ \mathcal{K}, \tag{17}$$

where

$$f(k) \triangleq Z(k) - \mathcal{H}U(k) \in \mathbb{R}^{l_z r}, \tag{18}$$

$$D(k) \triangleq \sum_{i=1}^{r} y^{\mathrm{T}}(k-i) \otimes (\mathcal{H}L_i) \in \mathbb{R}^{l_z r \times l_u l_y}, \tag{19}$$

vec is the column-stacking operator, and $\otimes$ represents the Kronecker product.

Now consider the *retrospective cost function*

$$J(\mathcal{K},k) \triangleq \hat{Z}^{\mathrm{T}}(\mathcal{K},k)\hat{Z}(\mathcal{K},k) \tag{20}$$
$$+ \alpha(k)\text{tr}\left[(\mathcal{K} - K(k))^{\mathrm{T}}(\mathcal{K} - K(k))\right],$$

where $\alpha(k) > 0$ is the *learning rate*. Substituting (17) into (20) yields

$$J(\mathcal{K},k) = c(k) + b^{\mathrm{T}}(k)\text{vec}\ \mathcal{K} + (\text{vec}\ \mathcal{K})^{\mathrm{T}} M(k)\text{vec}\mathcal{K},$$

where

$$M(k) \triangleq D^{\mathrm{T}}(k)D(k) + \alpha(k)I_{l_u l_y}, \tag{21}$$

$$b(k) \triangleq 2D^{\mathrm{T}}(k)f(k) - 2\alpha(k)\text{vec}\ K(k), \tag{22}$$

$$c(k) \triangleq f^{\mathrm{T}}(k)f(k) + \alpha(k)\text{tr}\left[K^{\mathrm{T}}(k)K(k)\right]. \tag{23}$$

Since $M(k)$ is positive definite, $J(\mathcal{K},k)$ has the strict global minimizer $K(k+1)$ given by

$$K(k+1) = -\frac{1}{2}\text{vec}^{-1}\left[M^{-1}(k)b(k)\right], \tag{24}$$

which requires the inverse of a positive-definite matrix of size $l_z r \times l_z r$. Equation (24) is the adaptive control update law. Note that $\mathcal{H}$ (which appears in $f(k)$ and $D(k)$) must be specified in order to implement (24).

The learning rate $\alpha(k)$ affects convergence speed of the adaptive control algorithm. As $\alpha(k)$ is increased, a higher weight is placed on the difference between the previous controller coefficients and the updated controller coefficients, and, as a result, convergence speed is lowered. Likewise, as $\alpha(k)$ is decreased, convergence speed is raised.

V. Full-State-Feedback Stability Analysis

Let $z(k) = y(k) = x(k)$, and thus $C = E_1 = I_n$. Therefore, we have a full-state-feedback system. Furthermore, let $l_u = r = 1$. Then, for all $k \geq 0$, the closed-loop system with gain matrix $K(k)$ is given by

$$x(k+1) = [A + BK(k)]\,x(k), \tag{25}$$

$$K(k+1) = K(k) - \frac{x^{\mathrm{T}}(k+1)\hat{B}}{\alpha(k) + \hat{B}^{\mathrm{T}}\hat{B}x^{\mathrm{T}}(k)x(k)}x^{\mathrm{T}}(k), \tag{26}$$

where $\hat{B} \triangleq \delta B$ is an estimate of the input matrix B, and $\delta \in \mathbb{R}$ is a scale factor.

Let $K^* \in \mathbb{R}^{l_u \times n}$ be a gain matrix that renders the ideal closed-loop system nilpotent, that is,

$$x^*(k+1) = \mathcal{N}x^*(k), \tag{27}$$

where $x^*(k) \in \mathbb{R}^n$, and the matrix $\mathcal{N} \triangleq A + BK^* \in \mathbb{R}^{n\times n}$ is nilpotent. Consequently, for all $k \geq n$, $x^*(k) = 0$. Define the error states $\tilde{x}(k) \in \mathbb{R}^n$ and $\tilde{K}(k) \in \mathbb{R}^{l_u \times n}$ by

$$\tilde{x}(k) \triangleq x(k) - x^*(k), \tag{28}$$

$$\tilde{K}(k) \triangleq K(k) - K^*. \tag{29}$$

Thus, for all $k \geq n$, $\tilde{x}(k) = x(k)$. Therefore, for all $k \geq n$, substituting $K(k) = \tilde{K}(k) + K^*$ into (25) and (26) yields the closed-loop error system

$$x(k+1) = \left[\mathcal{N} + B\tilde{K}(k)\right]x(k), \tag{30}$$

$$\tilde{K}(k+1) = \tilde{K}(k) - \frac{x^{\mathrm{T}}(k+1)\hat{B}}{\alpha(k) + \hat{B}^{\mathrm{T}}\hat{B}x^{\mathrm{T}}(k)x(k)}x^{\mathrm{T}}(k). \tag{31}$$

Now, let $n = l_u = l_y = l_z = r = 1$ and define $K^* \triangleq -A/B$, which yields $x^*(k) \equiv 0$ for all $k \geq 1$. Consequently, for all $k \geq 1$, $\tilde{x}(k) = x(k)$. Therefore, for all $k \geq 1$, it follows from (30), (31) that the closed-loop error system is

$$x(k+1) = B\tilde{K}(k)x(k), \tag{32}$$

$$\tilde{K}(k+1) = \Gamma(\gamma(k)x^2(k))\tilde{K}(k), \tag{33}$$

where, for $\lambda \geq 0$,

$$\Gamma(\lambda) \triangleq \frac{1+\eta\lambda}{1+\lambda}, \tag{34}$$

$\eta \triangleq 1 - 1/\delta$, $\delta = \hat{B}/B$, and $\gamma(k) \triangleq \hat{B}^2/\alpha(k)$. Note that $\Gamma(0) = 1$, $\Gamma(\lambda) \to \eta$ as $\lambda \to \infty$, and $\Gamma(\lambda)$ is a decreasing function of λ on $[0,\infty)$. Also, note that $\eta \in (-1,1)$ if and only if $\delta > \frac{1}{2}$.

Lemma V.1. Assume that $\delta > \frac{1}{2}$ and consider (32), (33). Then, for all $k \geq 1$, $\eta < \Gamma(\gamma(k)x^2(k)) \leq 1$. Furthermore, for all $k \geq 1$ such that $x(k) \neq 0$, $\eta < \Gamma(\gamma(k)x^2(k)) < 1$, and thus $|\Gamma(\gamma(k)x^2(k))| < 1$.

Proof. Let $k \geq 1$. Since $\eta \in (-1,1)$, it follows that

$$\eta < 1 \leq 1 + (1-\eta)\,\gamma(k)x^2(k).$$

Therefore,

$$\eta\left[1+\gamma(k)x^2(k)\right] < 1+\eta\gamma(k)x^2(k) \le 1+\gamma(k)x^2(k),$$

and thus,

$$\eta < \Gamma(\gamma(k)x^2(k)) \le 1.$$

Furthermore, for all $k \ge 1$ such that $x(k) \ne 0$, it follows that $-1 < \eta < \Gamma(\gamma(k)x^2(k)) < 1$. □

Theorem V.1. Assume that $n = l_u = l_y = l_z = r = 1$, assume that $\delta > \frac{1}{2}$, and consider the open-loop system (1)–(3) and the adaptive feedback controller (11), (24). Then, for all initial conditions $x(0)$ and $K(0)$, the following statements hold:

(i) $K(k)$ is bounded.
(ii) $\lim_{k\to\infty} x(k) = 0$.
(iii) $\{|\tilde{K}(k)|\}_{k=1}^{\infty}$ is nonincreasing.
(iv) $\lim_{k\to\infty} |\tilde{K}(k)| < 1/|B|$.
(v) There exists $k_0 \ge 1$ such that $\{|x(k)|\}_{k=k_0}^{\infty}$ is decreasing.
(vi) The zero solution of the closed-loop error system (32), (33) is Lyapunov stable.

Proof. Let $k \ge 1$ so that $\tilde{x}(k) = x(k)$. Consider the positive-definite, radially unbounded Lyapunov candidate

$$V(x,\tilde{K}) \triangleq \ln\left(1+\gamma_0 x^2\right) + a\tilde{K}^2, \tag{35}$$

where $\gamma_0 \triangleq \hat{B}^2/\alpha_{\mathrm{u}} > 0$ and $a > 0$ is specified below. The Lyapunov difference is thus given by

$$\Delta V(k) \triangleq V(x(k+1),\tilde{K}(k+1)) - V(x(k),\tilde{K}(k)). \tag{36}$$

Evaluating $\Delta V(k)$ along the trajectories of the closed-loop error system (32), (33) yields

$$\begin{aligned}\Delta V(k) &= \ln\left(1+\gamma_0 x^2(k+1)\right) - \ln\left(1+\gamma_0 x^2(k)\right)\\ &\quad + a\left(\tilde{K}^2(k+1) - \tilde{K}^2(k)\right)\\ &= \ln\left[1+\frac{\left(B^2\tilde{K}^2(k)-1\right)\gamma_0 x^2(k)}{b_1(k)}\right]\\ &\quad + a\left[\frac{2(\eta-1)\gamma(k)x^2(k)+(\eta^2-1)\gamma^2(k)x^4(k)}{b_2^2(k)}\right]\tilde{K}^2(k),\end{aligned}$$

where $b_1(k) \triangleq 1+\gamma_0 x^2(k)$ and $b_2(k) \triangleq 1+\gamma(k)x^2(k)$.

Since, for all $z > 0$, $\ln z \le z-1$, we have

$$\begin{aligned}\Delta V(k) &\le \frac{\left(B^2\tilde{K}^2(k)-1\right)\gamma_0 x^2(k)}{b_1(k)}\\ &\quad + a\left[\frac{2(\eta-1)\gamma(k)x^2(k)+(\eta^2-1)\gamma^2(k)x^4(k)}{b_2^2(k)}\right]\tilde{K}^2(k).\end{aligned}$$

Letting $a \triangleq \frac{\hat{B}^2}{2\delta-1} > 0$ and noting that, for all $k \ge 0$, $\gamma_0 \le \gamma(k)$, it follows that

$$\Delta V(k) \le \frac{-b_3\gamma_0\left[1+\gamma(k)x^2(k)\right]x^2(k)\tilde{K}^2(k) - \gamma_0 b_2^2(k)x^2(k)}{b_1(k)b_2^2(k)}, \tag{37}$$

where $b_3 \triangleq \frac{B^2}{2\delta-1}$. Thus,

$$\Delta V(k) \le -W(x(k),\tilde{K}(k)), \tag{38}$$

where

$$W(x(k),\tilde{K}(k)) \triangleq \frac{b_3\gamma_0\left[1+\gamma(k)x^2(k)\right]\tilde{K}^2(k)+\gamma_0 b_2^2(k)}{b_1(k)b_2^2(k)}x^2(k). \tag{39}$$

To show *(i)*, summing (38) from 1 to $k-1$ and noting that, for all $k \ge 0$, $W(x(k),\tilde{K}(k)) \ge 0$, yields

$$\begin{aligned}V(x(k),\tilde{K}(k)) &= V(x(1),\tilde{K}(1)) + \sum_{j=1}^{k-1}\Delta V(j)\\ &\le V(x(1),\tilde{K}(1)) - \sum_{j=1}^{k-1}W(x(j),\tilde{K}(j))\\ &\le V(x(1),\tilde{K}(1)).\end{aligned} \tag{40}$$

Thus, $V(x(k),\tilde{K}(k))$ is bounded. Since $V(x(k),\tilde{K}(k))$ is positive definite and radially unbounded, it follows that $x(k)$ and $\tilde{K}(k)$ are bounded. Thus, $K(k) = \tilde{K}(k) + K^*$ is bounded.

Now, we show *(ii)*. Since V is positive definite, it follows from (38) that

$$\begin{aligned}0 &\le \lim_{k\to\infty}\sum_{j=1}^{k}W(x(j),\tilde{K}(j))\\ &\le -\lim_{k\to\infty}\sum_{j=1}^{k}\Delta V(j)\\ &= V(x(1),\tilde{K}(1)) - \lim_{k\to\infty}V(x(k),\tilde{K}(k))\\ &\le V(x(1),\tilde{K}(1)),\end{aligned} \tag{41}$$

where all three limits exist. Thus $\lim_{k\to\infty} W(x(k),\tilde{K}(k)) = 0$. It now follows from (39) that $\lim_{k\to\infty} x(k) = 0$.

We now show *(iii)*. Since, by Lemma V.1, $-1 < \Gamma(\gamma(k)x^2(k)) \le 1$ for all $k \ge 1$, it follows from (33) that $\{|\tilde{K}(k)|\}_{k=1}^{\infty}$ is nonincreasing. Let $\kappa \triangleq \lim_{k\to\infty}|\tilde{K}(k)|$, and note that $\kappa \ge 0$ and, for all $k \ge 1$, $|\tilde{K}(k)| \ge \kappa$.

To show *(iv)*, suppose that $\kappa \ge 1/|B|$. Then, for all $k \ge 1$, it follows that $|x(k+1)| \ge \kappa|B||x(k)| \ge |x(k)|$. Consequently, $\{|x(k)|\}_{k=1}^{\infty}$ is nondecreasing. Therefore, if $x(1) \ne 0$, then $\{|x(k)|\}_{k=1}^{\infty}$ does not converge to zero. Hence $\kappa < 1/|B|$.

We now show *(v)*. Since $\{|\tilde{K}(k)|\}_{k=1}^{\infty}$ is nonincreasing and $\kappa < 1/|B|$, it follows that there exists $k_0 \ge 1$ such that, for all $k \ge k_0$, $|\tilde{K}(k)| < 1/|B|$, and thus $|B\tilde{K}(k)| < 1$. Consequently, it follows from (32) that $\{|x(k)|\}_{k=k_0}^{\infty}$ is decreasing.

Finally, to show *(vi)*, let $\mathcal{X}(k) \triangleq \begin{bmatrix} x(k)\\ \tilde{K}(k)\end{bmatrix}$ be the state of the closed-loop error system (32), (33). Since V is positive definite and, by (38), ΔV is negative semidefinite, it follows from [22, Lemma A.3.12] that the zero solution of the closed-loop error system is Lyapunov stable. □

r	roots$(p_r(\mathbf{q}))$
3	$\{$**2.1,-2.74**$\}$
4	$\{$0.73,**1.6,-2.97**$\}$
5	$\{$0.274$\pm$0.72$\jmath$,**1.91,-3.03**$\}$
6	$\{$-0.49,0.43$\pm$0.94$\jmath$,**2.0,-3.01**$\}$

TABLE I

ROOTS OF $p_r(\mathbf{Q})$ AS A FUNCTION OF r FOR THE UNSTABLE, NONMINIMUM-PHASE PLANT IN EXAMPLE VI.1. AS r INCREASES, THE u TO z NONMINIMUM-PHASE ZEROS ARE MORE CLOSELY APPROXIMATED.

VI. STATIC-OUTPUT-FEEDBACK EXAMPLE

We now present a numerical example to investigate the performance of the adaptive control algorithm in the presence of nonminimum-phase zeros. The adaptive controller gains are initialized to zero, that is $K(0) = 0$.

Example VI.1 (SISO, nonminimum-phase, unstable plant)**.** Consider the unstable, nonminimum-phase plant

$$x(k+1) = \begin{bmatrix} -0.36 & 0.48 & 1.05 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(k),$$

$$y(k) = \begin{bmatrix} 0 & 1 & -4 \end{bmatrix} x(k),$$

$$z(k) = \begin{bmatrix} 1 & 1 & -6 \end{bmatrix} x(k),$$

with poles $-\sqrt{2}/2 \pm \sqrt{2}/2, 1.05$, u to y outer nonminimum-phase zero 4, and u to z outer nonminimum-phase zeros $2, -3$. Table I lists the roots of the Markov-parameter polynomial $p_r(\mathbf{q})$ as a function of r. Note that, as r increases, the nonminimum-phase zeros are more closely approximated, but $p_r(\mathbf{q})$ also contains additional spurious roots. For $r \le 3$, the closed-loop simulation fails. Therefore, taking $r = 4$ with $\alpha(k) \equiv 100$, the open- and closed-loop responses are shown in Figure 1. The adaptive controller stabilizes the plant. ■

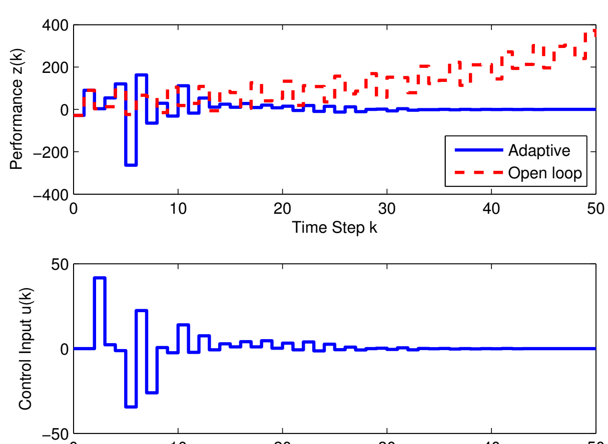

Fig. 1. Open- and closed-loop responses of the unstable, nonminimum-phase, SISO plant in Example VI.1 with $\alpha(k) \equiv 100$ and $r = 5$. The adaptive controller stabilizes the plant.

These results suggest that, for nonminimum-phase plants, the adaptive controller requires a sufficient number of Markov parameters to capture the approximate locations of any nonminimum-phase zeros. In particular, we require $r > n$. Furthermore, as the order of the Markov-parameter polynomial increases, and hence r increases, the estimation accuracy of all nonminimum-phase zeros improves.

VII. FULL-STATE-FEEDBACK EXAMPLES

In the special case $z(k) = y(k) = x(k)$, and thus $C = E_1 = I_n$, we have a full-state-feedback system. The following numerical examples investigate the response of the adaptive control algorithm in this case. In each example, the adaptive controller gains are initialized to zero, that is $K(0) = 0$.

Example VII.1 (Scalar input, unstable plant)**.** Consider the unstable plant

$$x(k+1) = \begin{bmatrix} -0.38 & 0.46 & 1.03 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k),$$

with poles located at $\{-\sqrt{2}/2 \pm \sqrt{2}/2, 1.03\}$. To demonstrate the effect of the learning rate, we take $r = 1$ with either $\alpha(k) \equiv 1$ or $\alpha(k) \equiv 1000$. The open- and closed-loop responses are shown in Figure 2. With $\alpha(k) \equiv 1$, x approaches zero within 10 time steps, while, with $\alpha(k) \equiv 1000$, x approaches zero within 20 time steps.

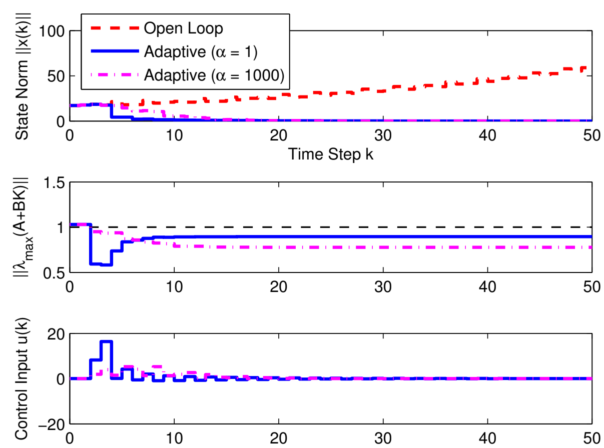

Fig. 2. Open- and closed-loop responses of the unstable, scalar-input plant in Example VII.1 with $r = 1$ and either $\alpha(k) \equiv 1$ or $\alpha(k) \equiv 1000$. With $\alpha(k) \equiv 1$, x approaches zero within 10 time steps, while, with $\alpha(k) \equiv 1000$, x approaches zero within 20 time steps.

To demonstrate the effect of a scale-factor error, we take $r = 1$, $\alpha(k) \equiv 1$, and $\hat{\mathcal{H}} = \delta\mathcal{H}$, where $\delta \in \mathbb{R}$ is a scale-factor error. A closed-loop performance comparison for $\delta \in (0.5, 5]$ is shown in Figure 3 where the performance metric is given by

$$k_0 \triangleq \min\{k : \frac{1}{5}\sum_{i=0}^{4} ||x(k-i)|| < 0.1\}, \qquad (42)$$

that is, k_0 is the minimum time step k such that the average of $\{||x(k-i)||\}_{i=0}^{4}$ is less than 0.1. The best performance is obtained for $\delta \approx 1.25$. Furthermore, the controller fails for $\delta < 0.5$. ■

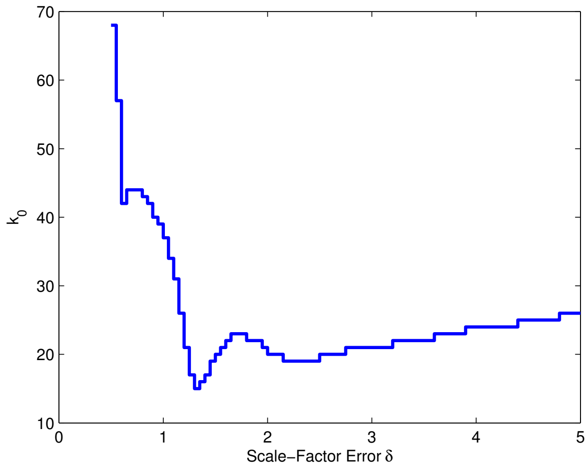

Fig. 3. Closed-loop performance comparison of the unstable, scalar-input plant in Example VII.1. We take $r = 1$, $\alpha(k) \equiv 1$, and $\hat{\mathcal{H}} = \delta\mathcal{H}$ where $\delta \in (0.5, 5]$. The best performance is obtained for $\delta \approx 1.25$. Furthermore, the controller fails for $\delta < 0.5$.

These results suggest that, for (1)–(3) with $z(k) = y(k) = x(k)$ and feedback control law (11), the closed-loop system has downward and upward gain margins of 6 dB and ∞ dB, respectively, which is consistent with the results of Theorem V.1.

Example VII.2 (Scalar input, nilpotent plant)**.** Consider the nilpotent plant

$$x(k+1) = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} u(k),$$

where, for $i = 1, \ldots, n-1$ and $j = 2, \ldots, n$, $a_{i,j}$ is chosen randomly. To determine a rule-of-thumb for choosing r, we set $\alpha(k) \equiv 1$ and vary n and r for various choices of $\{a_{i,j}\}$. In all cases, $r > n$ results in controller convergence and thus, an asymptotically stable closed-loop system. However, the controller may fail for $r \leq n$.

Note that the dynamics matrix given above is singular. In practice, every dynamics matrix obtained through discretization is nonsingular. Numerical tests suggest that, if the dynamics matrix A is nonsingular, then $r = 1$ is sufficient for closed-loop asymptotic stabilization. ■

These results suggest that, for (1)–(3) with $z(k) = y(k) = x(k)$ and feedback control law (11), $r > n$ is a sufficient condition such that the closed-loop system is asymptotically stable. Furthermore, if A is nonsingular, then $r = 1$ is a sufficient condition such that the closed-loop system is asymptotically stable.

VIII. Conclusions

We presented a discrete-time, adaptive, static-output-feedback control algorithm based on retrospective cost optimization. We presented a stability proof for a full-state-feedback specialization. In addition, through numerical examples, we demonstrated the algorithm's effectiveness in handling nonminimum-phase zeros as well as developed rules of thumb for choosing the parameters necessary for controller implementation. These numerical studies serve as motivation for more general Lyapunov-based stability, robustness, and convergence proofs of the adaptive control algorithm.

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