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Optimal Lyapunov-Based Backward Horizon Adaptive Stabilization

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1 Introduction

Although there is no precise definition of adaptive control, one can say intuitively that an adaptive controller operates by adjusting parameters in response to the behavior of the plant. Adaptive controllers can be used to achieve stabilization, disturbance rejection, and/or tracking objectives. The ultimate objective of adaptive control is to achieve desired performance with minimal prior knowledge of the plant dynamics and disturbance spectrum. Although this objective is shared by robust control, the underlying approaches are distinct. Specifically, robust control is based entirely on prior modeling information and thus does not incorporate learning, while adaptive control is self adjusting in response to measured plant behavior.

In direct adaptive control the gains are adjusted without explicit parameter identification. Direct adaptive control algorithms have been developed for both continuous-time and discrete-time systems. Global stability for several discrete-time systems have been established [1, 2, 3, 4, 5, 6]. However, unlike the continuous-time case, the available discrete-time results are based on RLS or LMS algorithms rather than Lyapunov methods. The approach developed in [1] is based on a convergence result called the Key Technical Lemma (Lemma 6.2.1, pp. 181-182, [7]). This approach is extended to certain classes of nonminimum phase plants in [8] and to plants with disturbances in [9]. Extensions of this approach to smooth stabilization with unknown high frequency gain, were addressed in [10, 11].

In this paper, we use a new method of analysis based on a modified Lyapunov technique and an adaptive step size. We begin by considering a one-step backward-horizon cost function, whose gradient provides an update direction for modifying the feedback gain matrix. The step size in the gradient direction is chosen to minimize the cost function along that direction. An analogous step size is used in [12]. Finally we use a modified Lyapunov technique to prove convergence of the plant states to the origin.

We present the main results in Section 2. Implementability issues are discussed in Section 3, some results from simulation studies are presented in Section 4, and Section 5 presents some conclusions.

2 Adaptive Stabilization Algorithm

Consider the discrete-time system

$$x_{k+1} = Ax_k + Bu_k, \tag{2.1}$$

where $x_k \in \Re^{n_x}$, $u_k \in \Re^{n_u}$ and $k = 0, 1, \ldots$ denotes the time step. We assume that the pair (A, B) is stabilizable and $\text{rank}(B) = n_u$. Furthermore, we assume there exists $K_s \in \Re^{n_u \times n_x}$ such that $A + BK_s$ is asymptotically stable and known. However, we assume that we do not have sufficient knowledge of A and B to determine K_s. Therefore our objective is to determine a full state feedback control law of the form

$$u_k = K_k x_k \tag{2.2}$$

such that the origin of the closed-loop system (2.1), (2.2) is attractive with respect to x_k. The adaptive gain matrix K_k is updated at each time step k to yield the next gain matrix, K_{k+1}.

In certain cases these assumptions can be satisfied with minimal knowledge of the system parameters. For instance, for a single input system in companion form, we do not require knowledge of either the last row of A or the magnitude of the last entry in B. Additional details as well as a multiple input example are given in Section 3.

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To derive an adaptively stabilizing control law, we consider the one-step cost function

$$J_k(K) \stackrel{\triangle}{=} \frac{1}{2}\varepsilon_k^{\mathrm{T}} P \varepsilon_k, \tag{2.3}$$

where $\varepsilon_k \stackrel{\triangle}{=} x_{k+1}(K) - x_{k+1}(K_{\mathrm{s}})$ and $x_{k+1}(K) = (A+BK)x_k$, the state at time $k+1$ when the gain matrix K is used at time k. We also define

$$\hat{J}_k(K) \stackrel{\triangle}{=} \|K - K_{\mathrm{s}}\|_{\mathrm{F}}^2. \tag{2.4}$$

Let $\sigma_{\mathrm{m}}(A)$ denote the maximum singular value of A, let I_n denote the $n \times n$ identity matrix, and let Z^+ denote the set of nonnegative integers.

Lemma 1 *Consider the gain update law*

$$K_{k+1} = K_k - \eta_k \frac{\partial J_k}{\partial K_k} = K_k - \eta_k B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}, \tag{2.5}$$

where $\eta_k \in \Re$ and $k \in Z^+$. Then the following statements hold:
i) If $\varepsilon_k = 0$, then $K_{k+1} = K_k$ for all $\eta_k \in \Re$.
ii) If $\varepsilon_k \neq 0$, then $\hat{\eta}_k$ given by

$$\hat{\eta}_k = \frac{\|N\varepsilon_k\|_2^2}{\|B^T P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2} \tag{2.6}$$

is positive and minimizes $\hat{J}_k(K_{k+1})$ with

$$\hat{J}_k(K_{k+1}) = \hat{J}_k(K_k) - \frac{\|N\varepsilon_k\|_2^4}{\|B^T P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2}. \tag{2.7}$$

iii) Suppose $\varepsilon_k \neq 0$. Then $\hat{J}_k(K_{k+1}) < \hat{J}_k(K_k)$ if and only if $\eta_k \in (0, 2\hat{\eta}_k)$. Furthermore, $\hat{J}_k(K_{k+1}) = \hat{J}_k(K_k)$ if and only if either $\eta_k = 0$ or $\eta_k = 2\hat{\eta}_k$.
Let $\{\eta_k\}_{k\in Z^+}$ be a sequence of positive real numbers, let $K_0 \in \Re^{n_u \times n_x}$, let $\{K_k\}_{k=0}^{\infty}$ be the sequence generated by (2.5), and let $\mathcal{S} \stackrel{\triangle}{=} \{k \in Z^+ : \varepsilon_k \neq 0\}$. Then the following statements hold
iv) If $\mathcal{S}$ is empty, then $x_k \to 0$ as $k \to \infty$.
v) If $\mathcal{S}$ is not empty and

$$\sup_{k\in\mathcal{S}} \left| \frac{\eta_k}{\hat{\eta}_k} - 1 \right| < 1, \tag{2.8}$$

then $\frac{\|N\varepsilon_k\|}{\|Nx_k\|} \to 0$ as $k \to \infty$.

Proof: To prove *i)* let $\varepsilon_k = 0$. Then (2.5) implies $K_{k+1} = K_k$ for all η_k.

To prove *ii)*, define

$$\hat{K}_k \stackrel{\triangle}{=} K_k - K_{\mathrm{s}}, \tag{2.9}$$

and rewrite (2.5) as

$$\hat{K}_{k+1} = \hat{K}_k - \eta_k B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}. \tag{2.10}$$

Now using (2.1), (2.2) we can write

$$x_{k+1}(K_k) = (A_{\mathrm{s}} + B\hat{K}_k)x_k, \tag{2.11}$$

which implies

$$\varepsilon_k = B\hat{K}_k x_k. \tag{2.12}$$

From (2.10) and (2.12) it follows that

$$\begin{aligned}
\hat{J}_k(K_{k+1}) &= \|\hat{K}_{k+1}\|_{\mathrm{F}}^2 \\
&= \|\hat{K}_k + \eta_k B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\
&= \hat{J}_k(K_k) + \|B^{\mathrm{T}} P B \hat{K}_k x_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \eta_k^2 \\
&\quad -2\mathrm{tr}\left(B^{\mathrm{T}} P B \hat{K}_k x_k x_k^{\mathrm{T}} \hat{K}_k^{\mathrm{T}}\right)\eta_k \\
&= \hat{J}_k(K_k) + \|B^{\mathrm{T}} P B \hat{K}_k x_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \eta_k^2 \\
&\quad -2\|NB\hat{K}_k x_k\|_2^2 \eta_k \\
&= \hat{J}_k(K_k) + \|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \eta_k^2 \\
&\quad -2\|N\varepsilon_k\|_2^2 \eta_k \\
&= \hat{J}_k(K_k) + (\eta_k - 2\hat{\eta}_k)\|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\
&= \hat{J}_k(K_k) + ((\eta_k - \hat{\eta}_k)^2)\|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\
&\quad -\hat{\eta}_k^2 \|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2.
\end{aligned} \tag{2.13}$$

To minimize $\hat{J}_k(K_{k+1})$, we proceed as follows. By (2.12), $\varepsilon_k \neq 0$ implies $\hat{K}_k x_k \neq 0$ and $x_k \neq 0$. Hence $\hat{K}_k x_k x_k^{\mathrm{T}} \neq 0$. Since $B^{\mathrm{T}}PB$ and N are nonsingular and B has full column rank, it follows that $\|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 = \|B^{\mathrm{T}} P B \hat{K}_k x_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \neq 0$. Therefore $\hat{\eta}_k$ can be defined by 2.6 and $\eta_k = \hat{\eta}_k$ minimizes $\hat{J}_k(K_{k+1})$ with(2.7).

To prove *iii)* assume $\hat{J}_k(K_{k+1}) - \hat{J}_k(K_k) < 0$. Then by (2.13)

$$\eta_k(\eta_k - 2\hat{\eta}_k)\|B^{\mathrm{T}} P \varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 < 0, \tag{2.14}$$

which implies $0 < \eta_k < 2\hat{\eta}_k$. Conversely, $0 < \eta_k < 2\hat{\eta}_k$ implies (2.14), which implies $\hat{J}_k(K_{k+1}) - \hat{J}_k(K_k) < 0$ by (2.13). Setting $\hat{J}_k(K_{k+1}) = \hat{J}_k(K_k)$ in (2.13) yields $\eta_k = 0$ or $\eta_k = 2\hat{\eta}_k$.

To prove *iv)* let $\varepsilon_k = 0$ for all $k \in Z^+$. This implies $x_{k+1} = A_{\mathrm{s}} x_k$ for all $k \in Z^+$. Since A_{s} is asymptotically stable, it follows that $x_k \to 0$.

To prove *v)*, define

$$\gamma \stackrel{\triangle}{=} \sup_{k\in\mathcal{S}} \left| \frac{\eta_k}{\hat{\eta}_k} - 1 \right|. \tag{2.15}$$

By (2.8), $\gamma < 1$, hence $\eta_k \in [(1-\gamma)\hat{\eta}_k, (1+\gamma)\hat{\eta}_k] \subset (0, 2\hat{\eta}_k)$ for all $k \in \mathcal{S}$. Hence $\eta_k \neq 0$ and $\eta_k \neq 2\hat{\eta}_k$.

Furthermore, as in the proof of *ii)*, $\|B^{\mathrm{T}}P\varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \neq 0$ and $\eta_k < 2\hat{\eta}_k$. Now let $k \in \mathcal{S}$. Using (2.6) and (2.13) we have

$$\begin{aligned}\|\hat{K}_{k+1}\|_{\mathrm{F}}^2 - \|\hat{K}_k\|_{\mathrm{F}}^2 &= \eta_k(\eta_k\|B^{\mathrm{T}}P\varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\ &\quad -2\|N\varepsilon_k\|_2^2) \\ &= \eta_k(\eta_k - 2\hat{\eta}_k)\|B^{\mathrm{T}}P\varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2 \\ &< 0. \end{aligned} \tag{2.16}$$

Since $\mathcal{S}$ is non-empty, there exists $n > 0$ such that $\varepsilon_n(K_n) \neq 0$. Let $r_0 > n$ and, for all $r > r_0$, define the non-empty set $\mathcal{S}_r \triangleq \{k : 0 \leq k \leq r \text{ and } \varepsilon_k \neq 0\}$. For $r > r_0$, it follows from (2.16) that

$$\begin{aligned}\|\hat{K}_0\|_{\mathrm{F}}^2 &\geq \|\hat{K}_0\|_{\mathrm{F}}^2 - \|\hat{K}_{r+1}\|_{\mathrm{F}}^2 \\ &= \sum_{k=0}^{r}\left(\|\hat{K}_k\|_{\mathrm{F}}^2 - \|\hat{K}_{k+1}\|_{\mathrm{F}}^2\right) \\ &= \sum_{k\in\mathcal{S}_r}\eta_k(2\hat{\eta}_k - \eta_k)\|B^{\mathrm{T}}P\varepsilon_k\|_{\mathrm{F}}^2 \\ &> 0. \end{aligned} \tag{2.17}$$

Let $r > r_0$, let $k \in \mathcal{S}_r$, and consider the quadratic function $g(\eta) = \eta(2\hat{\eta}_k - \eta)$ defined on the interval $L = [(1-\gamma)\hat{\eta}_k, (1+\gamma)\hat{\eta}_k]$. We have

$$(1-\gamma^2)\hat{\eta}_k^2 = \min_{\eta\in L} g(\eta) = g((1-\gamma)\hat{\eta}_k) = g((1+\gamma)\hat{\eta}_k).$$

Since $g(\cdot)$ is quadratic, it follows that

$$\eta(2\hat{\eta}_k - \eta) \geq (1-\gamma^2)\hat{\eta}_k^2 \text{ for all } \eta \in L. \tag{2.18}$$

Using (2.18), we can rewrite (2.17) as

$$\begin{aligned}\frac{\|\hat{K}_0\|_{\mathrm{F}}^2}{1-\gamma^2} &\geq \sum_{k\in\mathcal{S}_r}\frac{\|N\varepsilon_k\|_2^4}{\|B^{\mathrm{T}}P\varepsilon_k x_k^{\mathrm{T}}\|_{\mathrm{F}}^2} \\ &= \sum_{k\in\mathcal{S}_r}\frac{\|N\varepsilon_k\|_2^4}{\|B^{\mathrm{T}}P\varepsilon_k x_k^{\mathrm{T}}N^{\mathrm{T}}N^{-\mathrm{T}}\|_{\mathrm{F}}^2} \\ &\geq \sum_{k\in\mathcal{S}_r}\frac{\|N\varepsilon_k\|_2^4}{\|NB\|_{\mathrm{F}}^2\|N\varepsilon_k\|_2^2\|Nx_k\|_2^2\|N^{-1}\|_{\mathrm{F}}^2} \\ &\geq \sum_{k\in\mathcal{S}_r}\frac{\|N\varepsilon_k\|_2^2}{\|NB\|_{\mathrm{F}}^2\|Nx_k\|_2^2\|N^{-1}\|_{\mathrm{F}}^2}\end{aligned}$$

or

$$\sum_{k\in\mathcal{S}_r}\frac{\|N\varepsilon_k\|_2^2}{\|Nx_k\|_2^2} \leq \beta,$$

where $\beta \triangleq \|\hat{K}_0\|_{\mathrm{F}}^2\|NB\|_{\mathrm{F}}^2\|N^{-1}\|_{\mathrm{F}}^2/(1-\gamma^2)$. Letting $r \to \infty$ yields

$$\sum_{k\in\mathcal{S}}\frac{\|N\varepsilon_k\|_2^2}{\|Nx_k\|_2^2} \leq \beta. \tag{2.19}$$

Next, define the set $\mathcal{S}' \triangleq Z^+\backslash\mathcal{S}$ and note that $\varepsilon_k = 0$ for all $k \in \mathcal{S}'$. If $k \in \mathcal{S}'$ and $x_k = 0$ then $x_l = 0$ for all $l \geq k$. Hence assume that $x_k \neq 0$ for all $k \in \mathcal{S}'$. For $k \in \mathcal{S}'$, we have $\frac{\|N\varepsilon_k\|_2^2}{\|Nx_k\|_2^2} = 0$. Therefore it follows from (2.19) that

$$\lim_{k\to\infty}\frac{\|N\varepsilon_k\|_2}{\|Nx_k\|_2} = 0. \tag{2.20}$$

■

Theorem 1 *Assume there exists $K_{\mathrm{s}} \in \Re^{n_u\times n_x}$ such that $A_{\mathrm{s}} \triangleq A + BK_{\mathrm{s}}$ is asymptotically stable, let $R \in \Re^{n_x\times n_x}$ be positive definite, let $P \in \Re^{n_x\times n_x}$ be the positive-definite solution to*

$$P = A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}} + R, \tag{2.21}$$

and let $N \in \Re^{n_x\times n_x}$ satisfy $N^{\mathrm{T}}N = P$. Let the control be given by (2.2) with the gain update (2.5) and with $\{\eta_k\}_{k\in Z^+}$ satisfying (2.8). Then

$$\lim_{k\to\infty} x_k = 0. \tag{2.22}$$

Proof: If $\mathcal{S}$ is empty, the result follows from *iv)* of Lemma 1. Hence assume $\mathcal{S}$ is not empty and consider the Lyapunov candidate $V(x_k) = x_k^{\mathrm{T}}Px_k + \|\hat{K}_k\|_{\mathrm{F}}^2$. Let $\Delta V \triangleq V(x_{k+1}) - V(x_k)$. Then using (2.21) and *iii)* of Lemma 1, we have

$$\begin{aligned}\Delta V &= x_{k+1}^{\mathrm{T}}Px_{k+1} - x_k^{\mathrm{T}}Px_k + \|\hat{K}_{k+1}\|_{\mathrm{F}}^2 - \|\hat{K}_k\|_{\mathrm{F}}^2 \\ &\leq x_{k+1}^{\mathrm{T}}Px_{k+1} - x_k^{\mathrm{T}}Px_k \\ &= (A_{\mathrm{s}}x_k + \varepsilon_k)^{\mathrm{T}}P(A_{\mathrm{s}}x_k + \varepsilon_k) - x_k^{\mathrm{T}}Px_k \\ &= x_k^{\mathrm{T}}(A_{\mathrm{s}}^{\mathrm{T}}PA_{\mathrm{s}} - P)x_k + \varepsilon_k^{\mathrm{T}}P\varepsilon_k + 2\varepsilon_k^{\mathrm{T}}Px_k \\ &= -x_k^{\mathrm{T}}Rx_k + \varepsilon_k^{\mathrm{T}}P\varepsilon_k + 2\varepsilon_k^{\mathrm{T}}Px_k \\ &\leq -x_k^{\mathrm{T}}Rx_k + 2\|N\varepsilon_k\|\|NA_{\mathrm{s}}x_k\| + \|N\varepsilon_k\|^2 \\ &= -x_k^{\mathrm{T}}Rx_k + 2\|N\varepsilon_k\|\|NA_{\mathrm{s}}N^{-1}Nx_k\| \\ &\quad +\|N\varepsilon_k\|^2 \\ &\leq -x_k^{\mathrm{T}}Rx_k + 2\|N\varepsilon_k\|\|NA_{\mathrm{s}}N^{-1}\|\|Nx_k\| \\ &\quad +\|N\varepsilon_k\|^2 \\ &\leq -x_k^{\mathrm{T}}Rx_k + 2\sigma_{\mathrm{m}}(NA_{\mathrm{s}}N^{-1})\|N\varepsilon_k\|\|Nx_k\| \\ &\quad +\|N\varepsilon_k\|^2. \end{aligned} \tag{2.23}$$

Now since $N^{\mathrm{T}}N = P$, (2.21) implies

$$I_{n_x} = \hat{A}_s^{\mathrm{T}}\hat{A}_s + \hat{R}, \quad (2.24)$$

where $\hat{A}_s \triangleq NA_sN^{-1}$ and $\hat{R} \triangleq N^{-\mathrm{T}}RN^{-1}$ is positive definite. Thus, $\sigma_m(\hat{A}_s) < 1$. Therefore,

$$\begin{aligned}\Delta V &\le -x_k^{\mathrm{T}}Rx_k + 2\|N\varepsilon_k\|\|Nx_k\| \\ &\quad +\|N\varepsilon_k\|^2. \end{aligned} \quad (2.25)$$

Let $\delta > 0$. By *v)* of Lemma 1, there exists k_δ such that $\|N\varepsilon_k\|/\|x_k\| < \delta$ for all $k > k_\delta$. Then for $k > k_\delta$ we can write

$$\begin{aligned}\Delta V &< -x_k^{\mathrm{T}}Rx_k + 2\delta\|Nx_k\|^2 \\ &\quad +\delta^2\|Nx_k\|^2 \\ &< -x_k^{\mathrm{T}}(R-(2\delta+\delta^2)P)x_k. \end{aligned} \quad (2.26)$$

Now choose δ sufficiently small such that $R-(2\delta+\delta^2)P$ is positive definite. Next, define the translated system with $\hat{k} = k - k_\delta$ and $\hat{x}_{\hat{k}} \triangleq x_{k_\delta+\hat{k}}$

$$\hat{x}_{\hat{k}+1} = (A+BK_k)\hat{x}_{\hat{k}}. \quad (2.27)$$

Using (2.26), it follows from the discrete version of Theorem 4.4 in [13] that for the translated system (2.27), with initial condition $\hat{x}_0 = x_{k_\delta}$ at $\hat{k} = 0$, $\hat{x}_{\hat{k}}^{\mathrm{T}}(R-(2\delta+\delta^2)P)\hat{x}_{\hat{k}} \to 0$ as $\hat{k} \to \infty$. It follows that for the system (2.1), (2.2), (2.5)

$$\lim_{k\to\infty} x_k = 0.$$

■

Remark 1 Lyapunov analysis for adaptive control is difficult in discrete time because the Lyapunov candidate cannot usually be chosen such that the derivative is linear in the error states, which makes it difficult to show negative definiteness. Our analysis uses a modified Lyapunov approach which provides convergence but not Lyapunov stability.

The following result provides an alternative step size that guarantees decrease of the cost function $J_k(K_{k+1})$. This result provides a 1-step backward horizon interpretation for the gradient update scheme.

Proposition 1 *Let* $\varepsilon_k \ne 0$ *and define*

$$\eta_k^* = \frac{\left\|B^TP\varepsilon_k\right\|_2^2}{\left\|NBB^TP\varepsilon_kx_k^{\mathrm{T}}.\right\|_{\mathrm{F}}^2} \quad (2.28)$$

Then
i) η_k^* *is positive and minimizes* $J_k(K_{k+1})$ *with*

$$J_k(K_{k+1}) = J_k(K_k) - \frac{x_k^{\mathrm{T}}x_k\left\|B^TP\varepsilon_k\right\|_2^4}{2\left\|NBB^TP\varepsilon_kx_k^{\mathrm{T}}\right\|_{\mathrm{F}}^2}. \quad (2.29)$$

ii) $J_k(K_{k+1}) < J_k(K_k)$ *if and only if* $\eta_k \in (0, 2\eta_k^*)$. *Furthermore,* $J_k(K_{k+1}) = J_k(K_k)$ *if and only if either* $\eta_k = 0$ *or* $\eta_k = 2\eta_k^*$.
iii) If $\varepsilon_k \ne 0$, *then* $\eta_k^* \le \hat{\eta}_k$.
iv) If $\varepsilon_k \ne 0$ *and* $n_u = 1$, *then* $\eta_k^* = \hat{\eta}_k$.

Proof: The proofs of *i)* and *ii)* are analogous to the proofs of Lemma 1, parts *ii)* and *iii)*. To prove *iii)*, let $C = \begin{bmatrix} \nu & B^{\mathrm{T}}PB\nu \end{bmatrix}^{\mathrm{T}}$, where $\nu \triangleq (B^{\mathrm{T}}PB)^{1/2}\hat{K}_kx_k$. Since $\det(CC^{\mathrm{T}}) \ge 0$ we have

$$(\nu^{\mathrm{T}}\nu)[\nu^{\mathrm{T}}(B^{\mathrm{T}}PB)^2\nu] - (\nu^{\mathrm{T}}B^{\mathrm{T}}PB\nu)^2 \ge 0. \quad (2.30)$$

Since $\varepsilon_k \ne 0$, it follows that $\hat{K}_kx_k \ne 0$, $x_k \ne 0$ and $\nu \ne 0$. Therefore,

$$\eta_k^* = \frac{\nu^{\mathrm{T}}B^{\mathrm{T}}PB\nu}{(x_k^{\mathrm{T}}x_k)\nu^{\mathrm{T}}(B^{\mathrm{T}}PB)^2\nu} \le \frac{\nu^{\mathrm{T}}\nu}{(x_k^{\mathrm{T}}x_k)\nu^{\mathrm{T}}B^{\mathrm{T}}PB\nu} = \hat{\eta}_k. \quad (2.31)$$

To prove *iv)*, let $n_u = 1$. Then $B^{\mathrm{T}}PB$ is a scalar and (2.31) holds with equality.

■

Remark 2 Note that K_{k+1} is computed using the knowledge of x_k and x_{k+1} at time $k+1$. The updated gain K_{k+1} is used to propagating the state from x_{k+1} to x_{k+2}.

To compute the updated gain matrix K_{k+1} we need the gradient direction of the cost function J_k, as well as a step size η_k to move along this direction. Convergence of plant states is proved for an open interval around a step size, $\hat{\eta}_k$ that minimizes the norm of the distance between K_{k+1} and K_s. The smaller step size η_k^* minimizes the *one-step backward horizon* cost $J_k(K_{k+1})$, but convergence has not been shown for this case.

3 Implementation

The simplest application of Theorem 1, is a single-input system in companion form. The only quantity that needs to be known is the sign of the high-frequency gain.

In general, we can implement the algorithm without knowledge of K_s for systems with decoupled inputs. We require knowledge of the rows of A which are not assignable by any input. We also require that B be of the form $B = |b|B_0$, where B_0 is

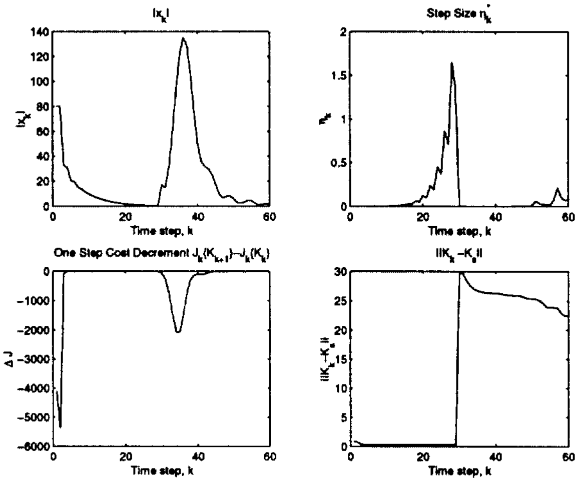

Figure 1: Two Input Example

known. An example of such a system is the double companion form with decoupled inputs and coupled states

$$A = \begin{bmatrix} 0_{n\times 1} & I_n & 0_{n\times n+1} \\ & a_1 \cdots & \\ 0_{n\times 1} & I_n & 0_{n\times n+1} \\ & a_2 \cdots & \end{bmatrix}, \; B = |b| \begin{bmatrix} 0_{n-1\times 2} \\ 1 \quad 0 \\ 0_{n-1\times 2} \\ 0 \quad 1 \end{bmatrix}. \tag{3.1}$$

This system can be stabilized without knowledge of the row vectors a_1, a_2, the positive scalar $|b|$ or the matrix K_s.

4 Numerical Results

In this section we consider simulation results for a plant of the form (3.1) with 4 states and 2 inputs. A is initially stable and suddenly becomes unstable at $k = 30$, and B is scaled by a scalar. The states are also perturbed at $k = 30$. Only the relative magnitudes of the various elements of B are known to the controller. The states, step size, one step decrement in cost and the norm of the controller error are shown in Figure 1. The step size η_k^* was used for the simulation. Though η_k^* has not been shown to satisfy (2.8), the simulation results show that η_k^* and $\hat{\eta}_k$ yield nearly identical trajectories.

5 Conclusions

In this paper we derived a discrete-time adaptive stabilization algorithm and proved closed-loop attractivity with respect to the plant states. An unstable and abruptly varying plant was simulated. Future work will focus on output feedback and disturbance rejection.

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