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Adaptive Control of Plants That Are Practically Impossible to Control by Fixed-Gain Control Laws

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***Abstract*— Some LTI plants are practically impossible to control due to extremely small gain and phase margins. These plants tend to be either unstable or nonminimum phase or both. Since practical control of these plants using fixed-gain controllers is not feasible, it is of interest to determine whether adaptive control can overcome these difficulties. To investigate this question, we apply retrospective cost adaptive control (RCAC) to a collection of plants that are practically impossible to control from an LTI perspective. For each plant, we introduce a destabilizing perturbation in order to determine whether or not RCAC can re-adapt in such a way as to compensate for the loss of margin and restabilize the closed-loop system without manual retuning. Since these plants are inherently difficult to control, it is of interest to determine whether or not restabilization is possible and, if so, assess the severity of the transient response.**

I. Introduction

Feedback control presents numerous challenges due to dimensionality, uncertainty, nonlinearity, state and control constraints, MIMO coupling, delays, disturbances, and noise. Even in the SISO linear time-invariant (LTI) case, some plants are inherently difficult to control due to unstable open-loop poles, nonminimum-phase (NMP) zeros, high relative degree, and time delays. These properties limit the achievable gain and phase margins, thus undermining robust stability and performance [1]. For example, the analysis in [2] shows that the arrangement of the plant poles and zeros constrains the controller bandwidth and the achievable delay margin. These limitations severely limit the feasibility of implementing a feedback control law with the given sensors and actuators. Although robust and adaptive control can account for plant uncertainty, the above limitations apply to all LTI plants under LTI control.

For adaptive control, plants that are inherently difficult to control pose an especially troublesome challenge as explained in [3]:

> Control engineers grounded in classical control know it is possible to formulate control design problems which in practical terms are not possible to solve. An inverted pendulum with more than two rods is a well-known example; again, a plant with nonminimum phase zeros well inside the passband and unstable poles may be near impossible to control, unless additional inputs or outputs are used; another famous example was provided in [4] and so on. When the plant is initially known, as well as the control objective, it will generally become clear at some point in the design process, if not ab initio, that the control objective is impractical.
>
> Now what happens in adaptive control? The catch is that a full description of the plant is lacking. There may be no way to decide on the basis of the a priori information that the projected design task is or is not practical. So what will happen if an adaptive control algorithm is run in such a case? At the least, the algorithm will not converge. At worst, an unstable closed loop will be established.

The present paper is motivated by these concerns. In particular, we consider a collection of plants with severely limited achievable gain and phase margin. We apply retrospective cost adaptive control (RCAC) to each plant, and then we allow the adaptive controller to converge. Once convergence is reached, we determine the gain and phase margin of the closed-loop system. We then introduce a destabilizing perturbation that exceeds either the gain margin or the phase margin. The objective is to determine whether or not RCAC can re-adapt in such a way as to compensate for the loss of margin and restabilize the closed-loop system without manual retuning. Since these plants are inherently difficult to control, it is of interest to determine whether or not restabilization is possible and, if so, assess the severity of the transient response.

We consider several examples. The first example entails an unmodeled change in the static gain, and the second example considers an unmodeled time delay. Finally, we consider discrete-time versions of the well-known examples from [2] and [4].

II. Adaptive Standard Problem

Consider the standard problem consisting of the discrete-time, linear time-invariant system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + D_0 u(k) + D_2 w(k), \tag{2}$$

$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k), \tag{3}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^{l_y}$ is the measurement, $u(k) \in \mathbb{R}^{l_u}$ is the control input, $w(k) \in \mathbb{R}^{l_w}$ is the exogenous input, and $z(k) \in \mathbb{R}^{l_z}$ is the performance variable. The system (1)–(3) may represent a continuous-time, LTI system sampled at a fixed rate. The goal is to develop a feedback or feedforward controller that operates on y to minimize z in the presence of the exogenous signal w. The components of w can represent either a command signal r to be followed, an external disturbance d to be rejected, or sensor noise v that corrupts the measurement, depending on the choice of D_1, D_2, and E_0. Depending on the application,

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components of w may or may not be measured, and, for feedforward control, the measured components of w can be included in y by suitable choice of C and D_2. For fixed-gain control, z need not be measured, whereas, for adaptive control, z is assumed to be measured.

Using the forward shift operator $\mathbf{q}$, we can rewrite (1)–(3) as

$$y(k) = G_{yw}(\mathbf{q})w(k) + G_{yu}(\mathbf{q})u(k), \tag{4}$$

$$z(k) = G_{zw}(\mathbf{q})w(k) + G_{zu}(\mathbf{q})u(k), \tag{5}$$

The controller has the form $u(k) = G_{\rm c}(\mathbf{q})y(k)$. Figure 1 illustrates the adaptive standard problem, which consists of (4)–(5) with the adaptive controller $G_{{\rm c},k}$.

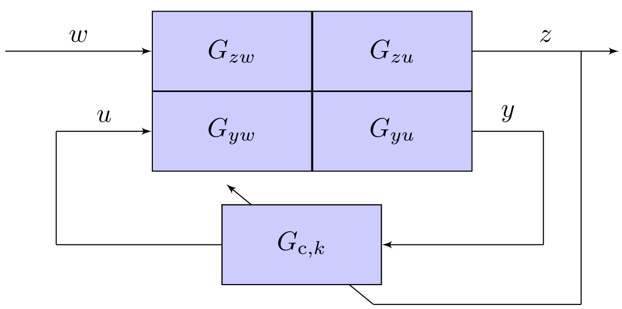

Fig. 1: Adaptive standard problem.

The closed-loop transfer function from the exogenous signal w to the performance variable z is given by

$$\widetilde{G}_{zw} \triangleq G_{zw} + G_{zu}G_{\rm c}(I - G_{yu}G_{\rm c})^{-1}G_{yw}. \tag{6}$$

We refer to the poles of $\widetilde{G}_{zw}$ as the closed-loop poles, and the transmission zeros of $\widetilde{G}_{zw}$ as the closed-loop zeros. In the case where u, w, y, and z are scalar signals, $\widetilde{G}_{zw}$ can be written as

$$\widetilde{G}_{zw} = \frac{N_{zu}N_{yw}N_{\rm c} + N_{zw}(DD_{\rm c} - N_{yu}N_{\rm c})}{D(DD_{\rm c} - N_{yu}N_{\rm c})}, \tag{7}$$

where $G_{\rm c} = D_{\rm c}^{-1}N_{\rm c}$.

A. The Adaptive Servo Problem

The adaptive servo problem is a special case of the adaptive standard problem with

$$y_0(k) = G_u(\mathbf{q})u(k) + G_d(\mathbf{q})d(k), \tag{8}$$

$$e_0(k) = r(k) - y_0(k), \tag{9}$$

$$G_{zw} = [I_{l_y} \quad -G_d \quad 0], \quad G_{zu} = -G_u, \tag{10}$$

$$G_{yw} = [I_{l_y} \quad -G_d \quad -I_{l_y}], \quad G_{yu} = -G_u. \tag{11}$$

We consider the case where $G_d = G_u$, denoted by G.

III. RCAC ALGORITHM

A. Controller Structure

Define the dynamic compensator

$$u(k) = \sum_{i=1}^{n_{\rm c}} P_i(k)u(k-i) + \sum_{i=k_{\rm c}}^{n_{\rm c}} Q_i(k)y(k-i), \tag{12}$$

where $P_i(k) \in \mathbb{R}^{l_u \times l_u}$ and $Q_i(k) \in \mathbb{R}^{l_u \times l_y}$ are the controller coefficient matrices, and $k_{\rm c} \geq 0$. We rewrite (12) as

$$u(k) = \Phi(k)\theta(k), \tag{13}$$

where the regressor matrix $\Phi(k)$ is defined by

$$\Phi(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_{\rm c}) \\ y(k-k_{\rm c}) \\ \vdots \\ y(k-n_{\rm c}) \end{bmatrix}^{\rm T} \otimes I_{l_u} \in \mathbb{R}^{l_u \times l_\theta}, \tag{14}$$

and the controller coefficient vector $\theta(k)$ is defined by

$$\theta(k) \triangleq {\rm vec}\big[P_1(k) \cdots P_{n_{\rm c}}(k)\ Q_{k_{\rm c}}(k) \cdots Q_{n_{\rm c}}(k)\big]^{\rm T} \in \mathbb{R}^{l_\theta}, \tag{15}$$

$l_\theta \triangleq l_u^2 n_{\rm c} + l_u l_y(n_{\rm c}+1-k_{\rm c})$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator. Note that $k_{\rm c} = 0$ allows an exactly proper controller, whereas $k_{\rm c} \geq 1$ yields a strictly proper controller of relative degree at least $k_{\rm c}$. In all examples in this paper, we use $k_{\rm c} = 1$. In terms of $\mathbf{q}$, the transfer function of the controller from y to u is given by

$$G_{{\rm c},k}(\mathbf{q}) = \big(\mathbf{q}^{n_{\rm c}} I_{l_u} - \mathbf{q}^{n_{\rm c}-1}P_1(k) - \cdots - P_{n_{\rm c}}(k)\big)^{-1} \cdot \big(\mathbf{q}^{n_{\rm c}-k_{\rm c}}Q_{k_{\rm c}}(k) + \cdots + Q_{n_{\rm c}}(k)\big). \tag{16}$$

B. Retrospective Performance Variable

We define the retrospective performance variable as

$$\hat{z}(k,\hat{\theta}) \triangleq z(k) + G_{\rm f}(\mathbf{q})[\Phi(k)\hat{\theta} - u(k)], \tag{17}$$

where $\hat{\theta} \in \mathbb{R}^{l_\theta}$ and $G_{\rm f}$ is an $n_z \times n_u$ filter specified below. The rationale underlying (17) is to replace the control $u(k)$ with $\Phi(k)\hat{\theta}^*$, where $\hat{\theta} = \hat{\theta}^*$ is the retrospectively optimized controller coefficient vector obtained by optimization below. The updated controller thus has coefficients $\theta(k+1) = \hat{\theta}^*$. Consequently, the implemented control at step $k+1$ is given by

$$u(k+1) = \Phi(k+1)\theta(k+1). \tag{18}$$

The filter $G_{\rm f}$ is constructed below based on the required modeling information. This filter has the form

$$G_{\rm f} \triangleq D_{\rm f}^{-1}N_{\rm f}, \tag{19}$$

where $D_{\rm f}$ is an $l_z \times l_z$ polynomial matrix with leading coefficient I_{l_u}, and $N_{\rm f}$ is an $l_z \times l_u$ polynomial matrix. For reasons given below, we henceforth refer to $G_{\rm f}$ as the *target model*. By defining the filtered versions $\Phi_{\rm f}(k) \in \mathbb{R}^{l_z \times l_\theta}$ and $u_{\rm f}(k) \in \mathbb{R}^{l_z}$ of $\Phi(k)$ and $u(k)$, respectively, (17) can be written as

$$\hat{z}(k,\hat{\theta}) = z(k) + \Phi_{\rm f}(k)\hat{\theta} - u_{\rm f}(k), \tag{20}$$

where

$$\Phi_{\rm f}(k) \triangleq G_{\rm f}(\mathbf{q})\Phi(k), \quad u_{\rm f}(k) \triangleq G_{\rm f}(\mathbf{q})u(k). \tag{21}$$

C. Retrospective Cost

Using the retrospective performance variable $\hat{z}(k,\hat{\theta})$ defined by (17), we define the cumulative retrospective cost function

$$J(k,\hat{\theta}) \triangleq \sum_{i=1}^{k} \lambda^{k-i}\hat{z}^{\mathrm{T}}(i,\hat{\theta})R_z(i)\hat{z}(i,\hat{\theta}) + \sum_{i=1}^{k}\lambda^{k-i}(\Phi_{\mathrm{f}}(i)\hat{\theta})^{\mathrm{T}}R_u(i)\Phi_{\mathrm{f}}(i)\hat{\theta} + \lambda^k(\hat{\theta}-\theta(0))^{\mathrm{T}}R_\theta(\hat{\theta}-\theta(0)), \tag{22}$$

where $\lambda \in (0,1]$ is the forgetting factor, R_θ is positive definite, and, for all $i \geq 1$, $R_z(i)$ is positive definite and $R_u(i)$ is positive semidefinite. The performance-variable and control-input weighting matrices $R_z(i)$ and $R_u(i)$ are time-dependent and thus may depend on present and past values of y, z, and u. Recursive minimization of (22) is used to update the controller coefficient vector $\hat{\theta}$. The following result uses recursive least squares to obtain the minimizer of (22).

Proposition: Let $P(0) = R_\theta^{-1}$. Then, for all $k \geq 1$, the retrospective cost function (22) has the unique global minimizer $\theta(k+1) = \hat{\theta}^*$, which is given by

$$\theta(k+1) = \theta(k) - P(k)\Phi_{\mathrm{f}}^{\mathrm{T}}(k)\Upsilon^{-1}(k) \cdot \left[\Phi_{\mathrm{f}}(k)\theta(k)+\bar{R}(k)R_z(k)(z(k)-u_{\mathrm{f}}(k))\right], \tag{23}$$

$$P(k+1) = \frac{1}{\lambda}P(k) - \frac{1}{\lambda}P(k)\Phi_{\mathrm{f}}^{\mathrm{T}}(k)\Upsilon^{-1}(k)\Phi_{\mathrm{f}}(k)P(k), \tag{24}$$

where

$$\bar{R}(k) \triangleq \left(R_z(k)+R_u(k)\right)^{-1}, \tag{25}$$

$$\Upsilon(k) \triangleq \lambda\bar{R}(k) + \Phi_{\mathrm{f}}(k)P(k)\Phi_{\mathrm{f}}^{\mathrm{T}}(k). \tag{26}$$

For all examples in this paper, we initialize $\theta(0) = 0$ in order to reflect the absence of additional prior modeling information. Furthermore, for all $i \geq 1$, we use $R_z(i) = I_{l_z}$.

IV. Virtual External Control Perturbation and Target Model G_{f}

The target model G_{f} is a key feature of RCAC. In [5], G_{f} is viewed as a model of G_{zu} that captures the sign of the leading coefficient of N_{zu} along with the NMP zeros of G_{zu}. In [6], the analysis of RCAC involves an ideal filter $\bar{G}_{\mathrm{f}}$, which is a closed-loop transfer function involving an ideal feedback controller $\bar{G}_{\mathrm{c}}$. These insights lead to an alternative interpretation of G_{f} as a target model for a specific closed-loop transfer function. These properties are demonstrated below. Using (13), the retrospective performance variable (17) can be written as

$$\hat{z}(k,\hat{\theta}) = z(k) - G_{\mathrm{f}}(\mathbf{q})[u(k) - \Phi(k)\hat{\theta}]. \tag{27}$$

It can be seen from (27) that minimizing (22) determines the controller coefficient vector $\hat{\theta}$ that best fits $G_{\mathrm{f}}(\mathbf{q})[\Phi(k)\theta(k) - \Phi(k)\hat{\theta}]$ to the performance data $z(k)$. In terms of the optimal controller coefficient vector $\hat{\theta}^*$, (27) can be written as

$$\hat{z}(k,\hat{\theta}^*) = z(k) - G_{\mathrm{f}}(\mathbf{q})[\Phi(k)\theta(k) - \Phi(k)\hat{\theta}^*]. \tag{28}$$

For convenience, we define

$$u^*(k) \triangleq \Phi(k)\hat{\theta}^*, \tag{29}$$

$$\tilde{u}(k) \triangleq u(k) - u^*(k), \tag{30}$$

so that $u(k) = u^*(k) + \tilde{u}(k)$. Using this notation, (28) can be written as

$$\hat{z}(k,\hat{\theta}^*) = z(k) - G_{\mathrm{f}}(\mathbf{q})\tilde{u}(k). \tag{31}$$

Replacing u in Φ by $u^* + \tilde{u}$, it follows from (12)–(14) and (29) that u^* satisfies

$$u^*(k) = \sum_{i=1}^{n_{\mathrm{c}}} P_i^* u^*(k-i) + \sum_{i=1}^{n_{\mathrm{c}}} P_i^*\tilde{u}(k-i) + \sum_{i=k_{\mathrm{c}}}^{n_{\mathrm{c}}} Q_i^* y(k-i). \tag{32}$$

Note that the actual input to the plant at step k is $u(k)$. However, we may write $u(k)$ as the sum of the *pseudo control input* $u^*(k)$ and the *virtual external control perturbation* $\tilde{u}(k)$. Note that the signals u^* and $\tilde{u}$ are not explicitly used by RCAC. From (32) it follows that

$$u^*(k) = D_{\mathrm{c}}^{*-1}(\mathbf{q})[(\mathbf{q}^{n_{\mathrm{c}}}I_{l_u} - D_{\mathrm{c}}^*(\mathbf{q}))\tilde{u}(k) + N_{\mathrm{c}}^*(\mathbf{q})y(k)], \tag{33}$$

where

$$D_{\mathrm{c}}^*(\mathbf{q}) \triangleq \mathbf{q}^{n_{\mathrm{c}}}I_{l_u} - \mathbf{q}^{n_{\mathrm{c}}-1}P_1^* - \cdots - P_{n_{\mathrm{c}}}^*, \tag{34}$$

$$N_{\mathrm{c}}^*(\mathbf{q}) \triangleq \mathbf{q}^{n_{\mathrm{c}}-k_{\mathrm{c}}}Q_{k_{\mathrm{c}}}^* + \cdots + Q_{n_{\mathrm{c}}}^*, \tag{35}$$

$$G_{\mathrm{c}}^* \triangleq D_{\mathrm{c}}^{*-1}N_{\mathrm{c}}^*. \tag{36}$$

It follows from (4), (5), and (33) that

$$z(k) = G_{zw}(\mathbf{q})w(k) + G_{zu}(\mathbf{q})\left[\left(\frac{\mathbf{q}^{n_{\mathrm{c}}}}{D_{\mathrm{c}}^*(\mathbf{q})} - 1\right)\tilde{u}(k) + G_{\mathrm{c}}^*(\mathbf{q})y(k) + \tilde{u}(k)\right], \tag{37}$$

$$y(k) = G_{yw}(\mathbf{q})w(k) + G_{yu}(\mathbf{q})\left[\left(\frac{\mathbf{q}^{n_{\mathrm{c}}}}{D_{\mathrm{c}}^*(\mathbf{q})} - 1\right)\tilde{u}(k) + G_{\mathrm{c}}^*(\mathbf{q})y(k) + \tilde{u}(k)\right]. \tag{38}$$

Solving (38) for $y(k)$ and substituting $y(k)$ into (37) yields

$$z(k) = \widetilde{G}_{zw}^*(\mathbf{q})w(k) + \widetilde{G}_{z\tilde{u}}^*(\mathbf{q})\tilde{u}(k), \tag{39}$$

where $\widetilde{G}_{zw}^*$ is given by (6) with G_{c} replaced by G_{c}^*, that is,

$$\widetilde{G}_{zw}^* \triangleq G_{zw} + G_{zu}G_{\mathrm{c}}^*(I - G_{yu}G_{\mathrm{c}}^*)^{-1}G_{yw}, \tag{40}$$

and where

$$\widetilde{G}_{z\tilde{u}}^*(\mathbf{q}) \triangleq \frac{N_{zu}(\mathbf{q})\mathbf{q}^{n_{\mathrm{c}}}}{D(\mathbf{q})D_{\mathrm{c}}^*(\mathbf{q})}\left[1 + \frac{N_{\mathrm{c}}^*(\mathbf{q})N_{yu}(\mathbf{q})}{[D(\mathbf{q})D_{\mathrm{c}}^*(\mathbf{q}) - N_{yu}(\mathbf{q})N_{\mathrm{c}}^*(\mathbf{q})]}\right]. \tag{41}$$

$$= \frac{N_{zu}(\mathbf{q})\mathbf{q}^{n_{\mathrm{c}}}}{D(\mathbf{q})D_{\mathrm{c}}^*(\mathbf{q}) - N_{yu}(\mathbf{q})N_{\mathrm{c}}^*(\mathbf{q})}. \tag{42}$$

It can be seen from (31) that $\hat{z}(k,\hat{\theta}^*) = z(k) - G_{\rm f}(\mathbf{q})\tilde{u}(k)$ is the residual of the fit between z and the output of the target model $G_{\rm f}$ with input $\tilde{u}$. However, it follows from (39) that $\widetilde{G}^*_{z\tilde{u}}$, whose coefficients are given by $\hat{\theta}^*$, is the actual transfer function from $\tilde{u}(k)$ to $z(k)$. Therefore, minimizing the retrospective cost function (22) yields the value $\theta(k+1) = \hat{\theta}^*$ of $\hat{\theta}$ and thus the controller $G_{{\rm c},k+1}$ that provides the best fit of $G_{\rm f}(\mathbf{q})$ by the transfer function $\widetilde{G}_{z\tilde{u},k+1}$ from $\tilde{u}(k,\theta(k+1))$ to $z(k)$. In other words, RCAC determines $G_{{\rm c},k+1}$ so as to optimally fit $\widetilde{G}_{z\tilde{u},k+1}$ to $G_{\rm f}(\mathbf{q})$.

V. Modeling Information Required for $G_{\rm f}$

In this section we specify the modeling information required by RCAC. For the standard problem, this information includes the relative degree, first nonzero Markov parameter, and NMP zeros of G_{zu}.

A. Relative Degree

Since $\widetilde{G}_{z\tilde{u},k+1}$ approximates $G_{\rm f}$, it is advantageous to choose the relative degree of $G_{\rm f}$ to be equal to the relative degree of $\widetilde{G}_{z\tilde{u},k+1}$. It follows from (42) that the relative degree of $\widetilde{G}_{z\tilde{u},k+1}$ is equal to the relative degree of G_{zu}. We thus choose the relative degree of $G_{\rm f}$ to be equal to the relative degree of G_{zu}. This choice requires knowledge of the relative degree ${\rm d}_{zu}$ of G_{zu} [6].

B. NMP Zeros

A key feature of $\widetilde{G}_{z\tilde{u},k+1}$ is the factor N_{zu} in its numerator. This means that, since RCAC adapts $G_{{\rm c},k}$ so as to match $\widetilde{G}_{z\tilde{u},k+1}$ to $G_{\rm f}$, RCAC may cancel NMP zeros of G_{zu} that are not included in the roots of $N_{\rm f}$ in order to remove them from $\widetilde{G}_{z\tilde{u},k+1}$. This observation motivates the need to include all of the NMP zeros of G_{zu} in $N_{\rm f}$ [6], [7].

C. FIR Target Model

In the case where G_{zu} is minimum phase, we define the FIR target model

$$G_{\rm f}(\mathbf{q}) \triangleq \frac{H_{{\rm d}_{zu}}}{\mathbf{q}^{{\rm d}_{zu}}}. \tag{43}$$

In the case where G_{zu} is NMP, we define the FIR target model

$$G_{\rm f}(\mathbf{q}) \triangleq \frac{H_{{\rm d}_{zu}} N_{zu,{\rm u}}(\mathbf{q})}{\mathbf{q}^{{\rm d}_{zu}+\deg(N_{zu,{\rm u}})}}, \tag{44}$$

where $N_{zu,{\rm u}}$ denotes the NMP portion of N_{zu}.

D. IIR Target Model for Pole Placement

Since $\widetilde{G}_{z\tilde{u},k+1}$ approximates $G_{\rm f}$, RCAC attempts to place the poles of $\widetilde{G}_{z\tilde{u},k+1}$ at the locations of the poles of $G_{\rm f}$. Note that the poles of $\widetilde{G}_{z\tilde{u},k+1}$ are equal to the closed-loop poles, that is, the poles of $\widetilde{G}_{zw,k+1}$. Consequently, RCAC attempts to place the closed-loop poles at the locations of the poles of $G_{\rm f}$. In order to use $G_{\rm f}$ for pole placement, let $D_{\rm p}$ be a monic polynomial of degree $n_{\rm p}$ whose roots are the desired closed-loop pole locations. Then, in the case where G_{zu} is minimum phase, we define the IIR target model

$$G_{\rm f}(\mathbf{q}) \triangleq \frac{H_{{\rm d}_{zu}}\mathbf{q}^{n_{\rm p}-{\rm d}_{zu}}}{D_{\rm p}(\mathbf{q})}, \tag{45}$$

and, in the case where G_{zu} is NMP, we define the IIR target model

$$G_{\rm f}(\mathbf{q}) \triangleq \frac{H_{{\rm d}_{zu}}\mathbf{q}^{n_{\rm p}-{\rm d}_{zu}-\deg(N_{zu,{\rm u}})}N_{zu,{\rm u}}(\mathbf{q})}{D_{\rm p}(\mathbf{q})}. \tag{46}$$

The target models (43) and (45) for minimum-phase G_{zu} along with the target models (44) and (46) for NMP G_{zu} represent the modeling information required by RCAC.

VI. Examples

Example 1. Unmodeled time-varying time delay for the adaptive servo problem. Consider the asymptotically stable, minimum-phase plant

$$G(\mathbf{q}) = \frac{\mathbf{q}-0.9}{(\mathbf{q}-0.95)(\mathbf{q}^2-1.6\mathbf{q}+0.89)}. \tag{47}$$

Let r be the harmonic command $r(k) = \cos\omega k$, where $\omega = 0.5$ rad/sample, and let $d = v = 0$. We use the FIR target model (43), and set $n_{\rm c} = 15$, $R_\theta = 0.1$, and $R_u = 0.1z^2$. At step $k = 15000$, a 1-step delay is introduced into the closed-loop system. At step $k = 30000$, an additional 2-step delay is introduced, and at step $k = 60000$, an additional 6-step delay is introduced. Table I shows the magnitude crossover frequency $\omega_{\rm mco}$, the phase margin PM, and the delay margin DM prior to the insertion of additional delays. Note that each delay exceeds the delay margin. In each case, RCAC re-adapts and restabilizes the closed-loop system, as shown in Figure 2. After the third delay, RCAC restabilizes the system at step $k = 100000$ (not shown). ■

k	$\omega_{\rm mco}$ (rad/sample)	PM (deg)	DM (steps)
15000	1.8872	19.0799	0.1765
30000	1.4003	86.1123	1.0733
60000	0.5742	181.7065	5.5233

TABLE I: Example 1: Unmodeled time-varying time delay for the adaptive servo problem. Magnitude crossover frequency, phase margin, and delay margin prior to inserting additional delays.

Example 2. Unmodeled change in NMP zeros for the adaptive standard problem. Consider the asymptotically stable, minimum-phase plant

$$G(\mathbf{q}) = \frac{(\mathbf{q}-0.5)(\mathbf{q}^2-1.92\mathbf{q}+1.44)}{(\mathbf{q}-0.35)(\mathbf{q}-0.6)(\mathbf{q}^2-0.8\mathbf{q}+0.32)}, \tag{48}$$

and let w be zero-mean Gaussian white noise with standard deviation $\sigma = 0.01$. We set $n_{\rm c} = 4$, $R_\theta = 10^{-5}$, $R_u = 0.1z^2$, and we use the IIR target model (46) with $D_{\rm p}(\mathbf{q})$ chosen to contain the closed-loop poles of high-authority LQG. RCAC approximates the closed-loop frequency response of high-authority LQG. At step $k = 5000$, the NMP zeros move

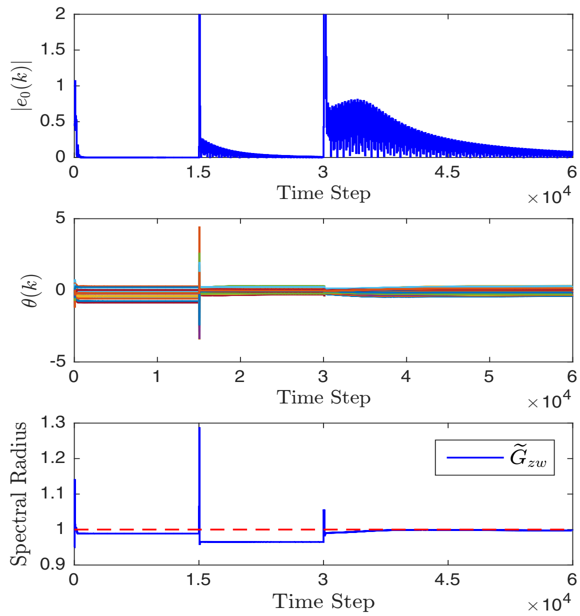

Fig. 2: Example 1: Unmodeled time-varying time delay for the adaptive servo problem. At step $k = 15000$, a 1-step delay is introduced into the closed-loop system. At step $k = 30000$, an additional 2-step delay is introduced, and at step $k = 60000$, an additional 6-step delay is introduced. With G_c fixed, each delay is destabilizing. In each case, RCAC re-adapts and restabilizes the closed-loop system

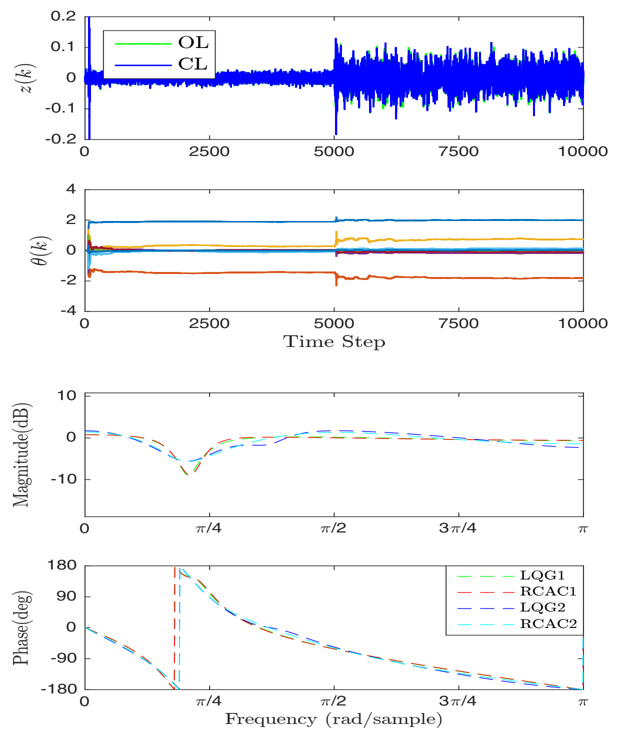

Fig. 3: Example 2: Unmodeled change in NMP zeros for the adaptive standard problem. At step $k = 5000$, the NMP zeros move to $0.99 \pm 1.38\jmath$. If G_c is fixed to be $G_{c,5000}$, then the closed-loop system becomes unstable (not shown). However, under adaptation, the plant is restabilized.

from $0.96 \pm 0.72\jmath$ to $0.99 \pm 1.38\jmath$. If G_c is fixed to be $G_{c,5000}$, then Figure 3 shows that the closed-loop system becomes unstable. However, under adaptation, the plant is restabilized, and RCAC approximates the closed-loop frequency response of LQG for the modified plant, as shown in Figure 3. ■

Example 3. Unstable plant with severely limited gain margin. Consider the unstable, minimum-phase, continuous-time plant from [4] given by

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{49}$$

$$C = E_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_2 = 1. \tag{50}$$

For the standard problem, we discretize (49) and (50) with a sampling period of 0.01 sec. Figure 4 shows that RCAC approximates the closed-loop frequency response of high-authority LQG. The LQG controller yields a gain margin of 0.04, and the RCAC controller yields a gain margin of 0.0034. At step $k = 10000$, the nominal plant G is replaced by $1.1G$. If G_c is fixed to be $G_{c,10000}$, then the closed-loop system becomes unstable. However, under adaptation, the plant is restabilized, as shown by Figure 4.■

Example 4. Unstable plant with limited delay margin. Consider the unstable, minimum-phase, continuous-time plant from [2] given by

$$A = \begin{bmatrix} -0.08 & -0.03 & 0.2 \\ 0.2 & -0.04 & -0.005 \\ -0.06 & 0.2 & -0.07 \end{bmatrix}, \tag{51}$$

$$B = \begin{bmatrix} -0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad C = E_1 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}. \tag{52}$$

This plant has an unstable pole at 0.1081. It is shown in [2] that the maximum achievable delay margin for the plant is 18.51 sec. For the standard problem, we discretize (51) and (52) with a sampling period of 0.1 sec. Using the controller given by (23) in [2], and discretizing with a sampling period of 0.1 sec, the delay margin of the discrete-time closed-loop system is 6.07 steps.

Next, we apply RCAC to the adaptive standard problem in order to stabilize (51) and (52). We use the FIR target model (43), and set $n_c = 3$, $R_\theta = 100$, and $R_u = 0.1z^2$. The delay margin of the closed-loop system using RCAC is 0.31 steps, as shown by Table II. Figure 5 shows the closed-loop responses for the initial condition $x(0) = [0.1\ 0.1\ 0.1]^{\mathrm{T}}$ for both RCAC and the controller given in [2] discretized with a sampling period of 0.1 sec. At step $k = 3000$, a 7-step delay is introduced into the system, which destabilizes both closed-loop systems. However, RCAC re-adapts and restabilizes the closed-loop system. ■

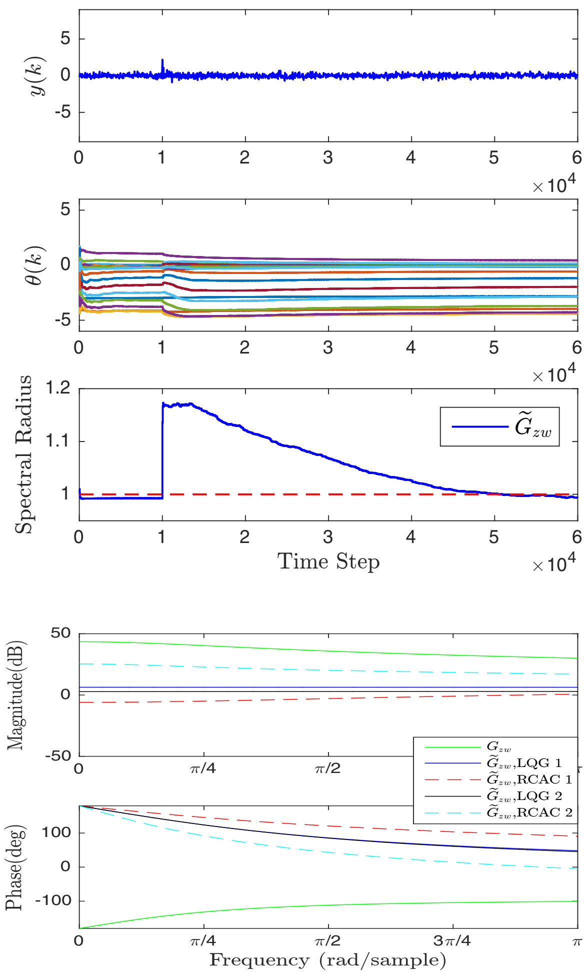

Fig. 4: Example 3: Discretized plant from [4] with severely limited gain margin. At step $k = 10000$, the gain margin is 0.0034, and the nominal plant G is replaced by $1.1G$. If $G_{\rm c}$ is fixed to be $G_{{\rm c},10000}$, then the closed-loop system becomes unstable. However, under adaptation, the plant is restabilized.

Controller	$\omega_{\rm mco}$ (rad/sec)	PM (deg)	DM (steps)
[2]	0.3708	129.12	6.07
RCAC	2.1834	38.83	0.31

TABLE II: Margins for the controller from [2] and RCAC at step $k = 3000$.

VII. Conclusions

This paper demonstrated that retrospective cost adaptive control (RCAC) is based on the matching of a closed-loop transfer function arising from intercalated virtual exogenous signal injection. RCAC matches this closed-loop transfer function to the target model used to define the retrospective cost.

Next, we applied RCAC to a collection of numerical examples involving plants that are practically impossible to control using fixed-gain controllers due to extremely small gain and phase margins. Plants of this type are viewed in [3] as potentially problematic for adaptive control as well. At convergence, the closed-loop systems possessed small

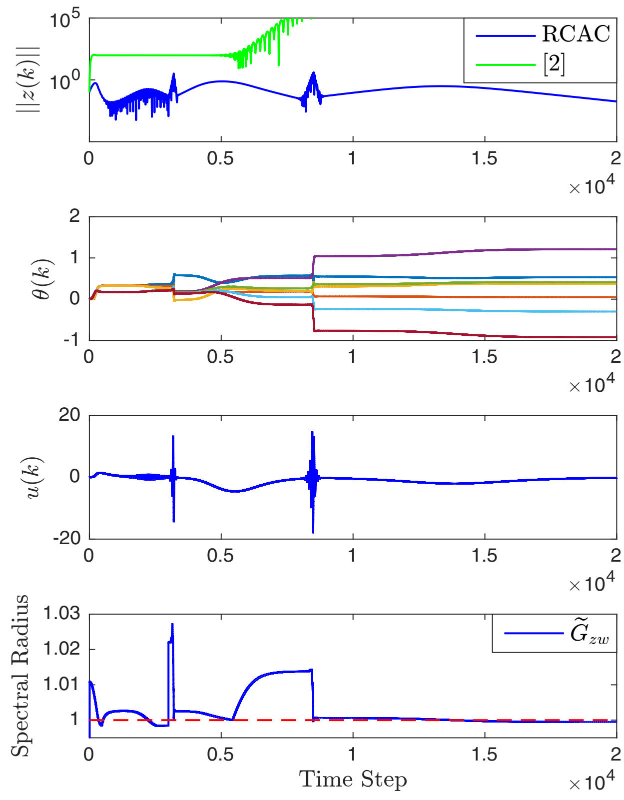

Fig. 5: Example 4: Unstable plant with limited delay margin. Closed-loop responses for the initial condition $x(0) = [0.1\ 0.1\ 0.1]^{\rm T}$ for both the controller given by [2] and RCAC. At step $k = 3000$, a 7-step delay is introduced into the system, which destabilizes both closed-loop systems. However, RCAC re-adapts and restabilizes the closed-loop system.

gain or phase margin, as expected, and thus the insertion of additional gain or time delay caused instability. However, with continued adaptation using RCAC, it was shown that RCAC was able to re-adapt and restabilize the plant. The recoverable range of perturbation was assessed numerically. Future research will focus on deriving analytical bounds for recoverability.

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