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Optimal Reduced-Order Observer-Estimators

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Abstract

This paper presents a unified approach to designing reduced-order observer-estimators. Specifically, we seek to design a reduced-order estimator satisfying an observation constraint which involves a pre-specified, possibly unstable, subspace of the system dynamics and which also yields reduced-order estimates of the remaining subspace. The results are obtained by merging the optimal projection approach to reduced-order estimation of Bernstein and Hyland with the subspace-observer results of Bernstein and Haddad. A salient feature of this theory is the treatment of unstable dynamics within reduced-order state-estimation theory. In contrast to the standard full-order estimation problem involving a single algebraic Riccati equation, the solution to the reduced-order observer-estimator problem involves an algebraic system of *four* equations consisting of one modified Riccati equation and three modified Lyapunov equations coupled by *two distinct* oblique projections.

I. Introduction

As is well known, Kalman filter theory addresses the state-estimation problem in guidance and navigation applications by minimizing a least-squares state-estimation error criterion. However, implementation of the standard Kalman filter is often impractical since it is generally of the same order as the system model. Consequently, designers must often implement reduced-order filters to satisfy real-time processing constraints as well as constraints on filter complexity. A further motivation is the fact that although a system model may have many degrees of freedom (such as coloring filter states and vibrational modes), it is often the case that estimates of only a small number of state variables (e.g., rigid body position and rotational modes) are actually required. The literature on reduced-order estimator design is vast and we note a representative collection of papers[1−22] as an indication of longstanding interest in this problem.

Another important issue in estimation theory is the problem of asymptotic observation. As is well-known[23], the steady-state Kalman filter is also an asymptotic observer. However, in reduced-order estimation theory the operations of estimation and observation are distinct, i.e., a reduced-order estimator is not necessarily also an observer. In many practical applications, however, it is necessary to design a reduced-order estimator that also observes a specified portion of the system states. Thus, we seek to design reduced-order *subspace observers* which can asymptotically observe a specified subset of system states.

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The contribution of the present paper is a unified approach to reduced-order observer-estimator design. Specifically, we consider a reduced-order estimation problem which also includes a subspace observation constraint. By merging the optimal projection approach to reduced-order state estimation developed by Bernstein and Hyland[9] with the subspace-observer result of Bernstein and Haddad[17], a reduced-order observer-estimator design theory is developed that includes *optimal observation* of a pre-specified subspace (e.g., rigid body modes and selected vibrational modes) as well as *optimal reduced-order estimation* of the remaining stable subspace (e.g., coloring filter states and remaining vibrational modes).

An additional feature of our approach is that the observed subspace need not be stable, i.e., it may include unstable (for example, neutrally stable) modes. In contrast with the full-order Kalman filter, reduced-order filters for unstable systems may diverge since they may fail to adequately track the unstable modes. The observer-estimator derived in this paper circumvents this problem by including all of the unstable modes within the observed subspace. We note that standard navigational models[26] possess neutrally stable modes, while tracking systems typically model targets as having rigid body dynamics. Additional examples include large flexible space structures undergoing open-loop rotational and/or translational motion.

It is important to stress that our results are not intended to provide a basis for feedback control. As is well known, feedback controllers based upon reduced-order filters may exhibit poor performance including instability. The preferred approach is thus to design reduced-order controllers directly[24,25].

The starting point for the present paper is the Riccati equation approach developed in Ref. 9. There it was shown that optimal reduced-order, steady-state estimators can be characterized by means of an algebraic system of equations consisting of one modified Riccati equation and two modified Lyapunov equations coupled by a projection matrix τ. Specifically, the *order* projection τ is given by

$$\tau \triangleq \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}, \tag{1}$$

where $(\ )^{\#}$ denotes group (Drazin) generalized inverse and $\hat{Q}$ and $\hat{P}$ are rank-deficient nonnegative-definite matrices analogous to the controllability and observability Gramians of the estimator. As discussed in Ref. 10, the order projection τ arises as a direct consequence of optimality and is *not* the result of an *a priori* assumption on the internal structure of the reduced-order estimator.

An important point discussed in Ref. 9 is that reduced-order estimators designed by means of either model reduction followed by "full-order" state estimation or full-order state estimation followed by estimator reduction will generally not be optimal for a given order. This point is illustrated by the fact that three matrix equations characterize the optimal reduced-order state estimator with intrinsic coupling between the "operations" of optimal estimator design and optimal estimator reduction.

The solution presented in Ref. 9, however, did not address the issue of observation of a pre-specified subspace. Consequently, the solution given in Ref. 9 was confined to problems in which the plant is asymptotically stable, while in practice it is often necessary to obtain estimators for plants with unstable modes. Intuitively, it is clear that finite, steady-state state-estimation error for unstable plants is only achievable when the estimator retains, or duplicates in some sense, the unstable modes. The solution given in Ref. 9 is inapplicable to unstable systems for the simple reason that the range of the order projection τ may not fully encompass all of the unstable modes. A partial solution to this problem, given in Ref. 17, involves a new and completely distinct reduced-order solution in which the observation subspace of the estimator is *constrained a priori* to include *all* of the unstable modes as well as selected stable modes. Hence the estimator in Ref. 17 effectively serves as an optimal *observer* for a designated plant subspace.

The subspace observation constraint addressed in Ref. 17 was embedded within the optimization process by fixing the internal structure of the reduced-order estimator. This structure gave rise to a new *subspace* projection μ defined by

$$\mu \triangleq \begin{bmatrix} I_{n_u} & P_u^{-1}P_{us} \\ 0_{n_s \times n_u} & 0_{n_s} \end{bmatrix}, \tag{2}$$

where $P_u \in \mathbb{R}^{n_u \times n_u}$ and $P_{us} \in \mathbb{R}^{n_u \times n_s}$ are subblocks of an $n \times n$ nonnegative-definite matrix P satisfying a modified algebraic Lyapunov equation, n_u is the dimension of the observation subspace of the estimator containing all of the unstable modes and selected stable modes, and n_s is the dimension of the remaining subspace containing only stable modes. It turns out that the subspace projection μ, which is completely distinct from the order projection τ defined by (1), plays a crucial role in characterizing the optimal observer gains. Furthermore, it was shown in Ref. 17 that the constrained subspace observer is characterized by one modified Riccati equation and one modified Lyapunov equation coupled by the subspace projection μ. This subspace observer however, was confined to an n_u-dimensional subspace with no estimation of the remaining n_s-dimensional subspace.

The purpose of the present paper is to combine the results of Refs. 9 and 17 in order to obtain a general solution to the Reduced-Order Observer-Estimator Problem. Specifically, we seek a reduced-order observer-estimator of order n_e satisfying $n_u \le n_e \le n$, where n is the dimension of the plant, which includes observation of *all* of a pre-specified n_u-dimensional subspace of the system as well as optimal n_{es} reduced-order estimation of the $n_s = n - n_u$ states in the residual subspace where, $n_{es} = n_e - n_u \le n_s$. As shown in Theorem 1, this general solution to the Reduced-Order Observer-Estimator Problem is characterized by four matrix equations including one modified Riccati equation and three modified Lyapunov equations coupled by both the order projection τ and the subspace projection μ.

Finally, the results of this paper can readily be extended in several directions. These include the treatment of parameter uncertainties[12,16], extensions to nonstrictly proper estimators and singular noise intensity[13,21], worst-case frequency-domain design aspects, i.e., an H_∞ constraint on the estimation error[19,22], and extensions to the discrete-time setting[10,17].

The contents of the paper are as follows. In Section II, the statement of the Reduced-Order Observer-Estimator Problem is given. In Section III, Theorem 1 presents necessary conditions for optimality which characterize solutions to the Reduced-Order Observer-Estimator Problem. To draw connections with the existing literature we specialize Theorem 1 in Section IV to obtain the results of Refs. 9 and 17. We also specialize the results of Theorem 1 to obtain the full-order Kalman filter theory and show that the four matrix equations collapse to the standard observer Riccati equation. To illustrate these results we describe a numerical algorithm in Section V for solving the design equations and apply the algorithm to illustrative numerical examples.

Nomenclature

$\mathbb{R}, \mathbb{R}^{r\times s}, \mathbb{R}^r, \mathbb{E}$	real numbers, $r \times s$ real matrices, $\mathbb{R}^{r\times 1}$, expected value
$I_r, (\ )^{\mathrm{T}}, 0_{r\times s}, 0_r$	$r \times r$ identity matrix, transpose, $r \times s$ zero matrix, $0_{r\times r}$
tr	trace
$\mathcal{N}(Z), \mathcal{R}(Z)$	null space, range of matrix Z
n, n_u, n_s, n_e, n_{es} ℓ, q	positive intergers; $n_u \le n_e \le n$, $n = n_u + n_s, n_e = n_u + n_{es}$
x, x_u, x_s, x_e, x_{eu} y, y_e	$n, n_u, n_s, n_e, n_u, n_{es}, \ell, q$- dimensional vectors
A, C, L	$n \times n, \ell \times n, q \times n$ matrices
A_u, A_{us}, A_s	$n_u \times n_u, n_u \times n_s, n_s \times n_s$ matrices
C_u, C_s	$\ell \times n_u, \ell \times n_s$ matrices
L_u, L_s	$q \times n_u, q \times n_s$ matrices
R	$q \times q$ positive-definite matrix
asymptotically stable matrix	matrix with eigenvalues in open left half plane
A_e, B_e, C_e	$n_e \times n_e, n_e \times \ell, q \times n_e$ matrices
$A_{eu}, A_{eus}, A_{esu}, A_{es}$	$n_u \times n_u, n_u \times n_{es}, n_{es} \times n_u, n_{es} \times n_{es}$ matrices
B_{eu}, B_{es}	$n_u \times \ell, n_{es} \times \ell$ matrices
C_{eu}, C_{es}	$q \times n_u, q \times n_{es}$ matrices
$w_1(t), t \ge 0$	n-dimensional white noise process with nonnegative-definite intensity V_1
$w_2(t), t \ge 0$	ℓ-dimensional white noise process with positive-definite intensity V_2
V_{12}	$n \times \ell$ cross intensity of $w_1(t), w_2(t)$
F, F_e, H	$[I_{n_u} \quad 0_{n_u\times n_s}]$, $[I_{n_u} \quad 0_{n_u\times n_{es}}]$, $[0_{n_s\times n_u} \quad I_{n_s}]$
$\tilde{A}$	$\begin{bmatrix} A - F^{\mathrm{T}}B_{eu}C & -F^{\mathrm{T}}A_{eus} \\ B_{es}C & A_{es} \end{bmatrix}$
$\tilde{L}$	$[L \quad -C_{es}]$
$\tilde{R}$	$\tilde{L}^{\mathrm{T}}R\tilde{L}$
$\tilde{w}(t)$	$\begin{bmatrix} w_1(t) - F^{\mathrm{T}}B_{eu}w_2(t) \\ B_{es}w_2(t) \end{bmatrix}$
$\tilde{V}$	$\begin{bmatrix} V_1 - V_{12}B_{eu}^{\mathrm{T}}F - F^{\mathrm{T}}B_{eu}V_{12}^{\mathrm{T}} + F^{\mathrm{T}}B_{eu}V_2B_{eu}^{\mathrm{T}}F & V_{12}B_{es}^{\mathrm{T}} - F^{\mathrm{T}}B_{eu}V_2B_{es}^{\mathrm{T}} \\ B_{es}V_{12}^{\mathrm{T}} - B_{es}V_2B_{eu}^{\mathrm{T}}F & B_{es}V_2B_{es}^{\mathrm{T}} \end{bmatrix}$

II. The Reduced-Order Observer-Estimator Problem

The following problem is addressed.

Reduced-Order Observer-Estimator Problem

For the nth-order system

$$\dot{x}(t) = Ax(t) + w_1(t), \quad t \in [0, \infty), \tag{3}$$

with noisy measurements

$$y(t) = Cx(t) + w_2(t), \tag{4}$$

design an n_eth-order reduced-order strictly proper observer-estimator

$$\dot{x}_e(t) = A_ex_e(t) + B_ey(t), \tag{5}$$

$$y_e(t) = C_e x_e(t), \tag{6}$$

that satisfies the following design criteria:

(i) the observer-estimator (5), (6) is a steady-state asymptotic observer for a specified n_u-dimensional subspace of the plant (3) where $n_u \le n_e \le n$; and

(ii) the observer-estimator is an optimal estimator which minimizes the least-squares state-estimation error criterion

$$J(A_e, B_e, C_e) \triangleq \lim_{t\to\infty} \mathbb{E}[Lx(t) - y_e(t)]^{\mathrm{T}} R[Lx(t) - y_e(t)]. \tag{7}$$

To make condition (i) more precise, partition (3), (4) according to

$$x(t) = \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} \in \mathbb{R}^n, \quad x_u(t) \in \mathbb{R}^{n_u}, \quad x_s(t) \in \mathbb{R}^{n_s}, \quad n = n_u + n_s, \tag{8}$$

$$\begin{bmatrix} \dot{x}_u(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A_u & A_{us} \\ 0_{n_s\times n_u} & A_s \end{bmatrix} \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} w_{1u}(t) \\ w_{1s}(t) \end{bmatrix}, \tag{9}$$

$$y(t) = [C_u \ C_s] \begin{bmatrix} x_u(t) \\ x_s(t) \end{bmatrix} + w_2(t), \tag{10}$$

and (5), (6) as

$$x_e(t) = \begin{bmatrix} x_{eu}(t) \\ x_{es}(t) \end{bmatrix} \in \mathbb{R}^{n_e}, \quad x_{eu}(t) \in \mathbb{R}^{n_u}, \quad x_{es}(t) \in \mathbb{R}^{n_{es}}, \quad n_e = n_u + n_{es}, \tag{11}$$

$$\begin{bmatrix} \dot{x}_{eu}(t) \\ \dot{x}_{es}(t) \end{bmatrix} = \begin{bmatrix} A_{eu} & A_{eus} \\ A_{esu} & A_{es} \end{bmatrix} \begin{bmatrix} x_{eu}(t) \\ x_{es}(t) \end{bmatrix} + \begin{bmatrix} B_{eu} \\ B_{es} \end{bmatrix} y(t), \tag{12}$$

$$y_e(t) = [C_{eu} \ C_{es}] \begin{bmatrix} x_{eu}(t) \\ x_{es}(t) \end{bmatrix}. \tag{13}$$

We note that the partitioned form of the matrix A appearing in (9) allows us to characterize the two subspaces corresponding to $x_u(t)$ and $x_s(t)$. The $n_s \times n_u$ zero matrix in the (2,1)-block of A is needed in order to achieve asymptotic observation of $x_u(t)$ independently of $x_s(t)$. If necessary, the matrix A can be recast in the form (9) by utilizing a similarity transformation to a modal basis. Of course, the coupling matrix A_{us} may be either zero or nonzero.

Furthermore, in (8)–(13) we implicitly assume that $0 < n_u < n_e$. The special cases $n_u = 0$ and $n_u = n_e$ will be discussed later in this section and in Section IV. The observation condition (i) is captured by imposing the additional constraint

$$\lim_{t\to\infty} [x_u(t) - x_{eu}(t)] = 0, \tag{14}$$

for all $x(0)$ and $x_e(0)$ when $w_1(t) \equiv 0$ and $w_2(t) \equiv 0$. The requirement (14) implies that zero asymptotic observation error for a specified n_u-dimensional subspace is achieved under zero external disturbances and arbitrary initial conditions.

To require that the observer-estimator is also an optimal reduced-order estimator, the matrix L identifies the states or linear combinations of states whose estimates are desired. In accordance with the partitioning given in (8), L is partitioned as

$$L \triangleq [L_u \quad L_s]. \tag{15}$$

Thus, the goal of the Reduced-Order Observer-Estimator Problem is to design a reduced-order observer-estimator of order n_e which observes a specified plant subspace and provides optimal estimates of specified linear combinations of plant states. Since the observer-estimator (5), (6) serves as a reduced-order observer for an n_u-dimensional subspace of the plant (3), its order n_e must satisfy $n_u \le n_e \le n$.

As will be seen, the observation constraint (14) can be satisfied even if the subspace corresponding to $x_u(t)$ is unstable. Thus we allow A_u to possess unstable as well as stable modes. Of course, our results remain valid even if A_u is asymptotically stable. The subscript "u," however, reminds us that A_u is permitted to be unstable. Furthermore, we require that A_s be an asymptotically stable matrix. In applications, the matrix A_s may include the dynamics of all coloring filter states as well as damped vibrational modes.

Before continuing it is useful to point out that several simpler problems are included as special cases within the above formulation. For example, consider the full-order case $n_e = n$ or, equivalently, $n_{es} = n_s$. In this case the observer-estimator can observe all of $x(t)$ and the matrix A_e is given by[23] $A_e = A - B_e C$. Note that the subblocks of A_e are thus given by

$$\begin{bmatrix} A_{eu} & A_{eus} \\ A_{esu} & A_{es} \end{bmatrix} = \begin{bmatrix} A_u - B_{eu}C_u & A_{us} - B_{eu}C_s \\ -B_{es}C_u & A_s - B_{es}C_s \end{bmatrix}. \tag{16}$$

The optimal value of B_e for the least-squares estimator in this case is, of course, the steady-state Kalman filter gain characterized by the algebraic observer Riccati equation.

Next, consider the case $n_e < n$ without the observation constraint (14), i.e., $n_u = 0$. Thus, with $x_u(t)$ and $x_{eu}(t)$ absent, we can identify $n_s = n$, $n_{es} = n_e$, and $A_s = A$. This problem is precisely the reduced-order estimation problem considered in Ref. 9.

Finally, suppose that $n_e = n_u < n$ so that the estimator states $x_{eu}(t) = x_e(t)$ are required to satisfy the observation constraint (14) but that no additional degrees of freedom are permitted in the estimator, i.e., $x_{es}(t)$ is absent. In this case the estimator acts solely as an optimal reduced-order *subspace observer* whose gains are dictated by the optimality criterion (7). This problem was considered in Ref. 17.

To analyze the observation constraint (14), define the error states

$$z_u(t) \triangleq x_u(t) - x_{eu}(t) \tag{17}$$

so that the observation constraint (14) can be written as

$$\lim_{t\to\infty} z_u(t) = 0. \tag{18}$$

Note that the error states $z_u(t)$ satisfy

$$\begin{aligned} \dot{z}_u(t) = \dot{x}_u(t) - \dot{x}_{eu}(t) = {} & (A_u - B_{eu}C_u)x_u(t) - A_{eu}x_{eu}(t) \\ & + (A_{us} - B_{eu}C_s)x_s(t) - A_{eus}x_{es}(t) \\ & + w_{1u}(t) - B_{eu}w_2(t). \end{aligned} \tag{19}$$

Using (9), (12), and (19) the overall augmented system (3)–(6) become

$$\begin{bmatrix} \dot{z}_u(t) \\ \dot{x}_s(t) \\ \dot{x}_{eu}(t) \\ \dot{x}_{es}(t) \end{bmatrix} = \begin{bmatrix} A_u - B_{eu}C_u & A_{us} - B_{eu}C_s & A_u - B_{eu}C_u - A_{eu} & -A_{eus} \\ 0_{n_s\times n_u} & A_s & 0_{n_s\times n_u} & 0_{n_s\times n_{es}} \\ B_{eu}C_u & B_{eu}C_s & A_{eu} + B_{eu}C_u & A_{eus} \\ {}_{es}C_u & B_{es}C_s & A_{esu} + B_{es}C_u & A_{es} \end{bmatrix} \begin{bmatrix} z_u(t) \\ x_s(t) \\ x_{eu}(t) \\ x_{es}(t) \end{bmatrix} + \begin{bmatrix} w_{1u}(t) - B_{eu}w_2(t) \\ w_{1s}(t) \\ B_{eu}w_2(t) \\ B_{es}w_2(t) \end{bmatrix}. \tag{20}$$

At this point we make the crucial observation that the explicit dependence of the error states $z_u(t)$ on the states $x_{eu}(t)$ can be eliminated in favor of $z_u(t)$ by constraining the (1,3) and (4,3) blocks of the block 4×4 matrix in (20) to be zero, i.e.,

$$A_{eu} \triangleq A_u - B_{eu}C_u, \tag{21}$$

$$A_{esu} \triangleq -B_{es}C_u. \tag{22}$$

With (21) and (22) A_e becomes

$$A_e = \begin{bmatrix} A_u - B_{eu}C_u & A_{eus} \\ -B_{es}C_u & A_{es} \end{bmatrix}. \tag{23}$$

Now the error states $z_u(t)$ satisfy

$$\dot{z}_u = A_{eu}z_u(t) + (A_{us} - B_{eu}C_s)x_s(t) - A_{eus}x_{es}(t) + w_{1u}(t) - B_{eu}w_2(t), \tag{24}$$

where A_{eu} is given by (21).

Next, note that the least-squares state-estimation error criterion (7) can be written as

$$\begin{aligned} J(A_e, B_e, C_e) = \lim_{t\to\infty} \mathbb{E}[&L_u z_u(t) + L_s x_s(t) \\ &+ (L_u - C_{eu})x_{eu}(t) - C_{es}x_{es}(t)]^{\mathrm{T}} R[L_u z_u(t) \\ &+ L_s x_s(t) + (L_u - C_{eu})x_{eu}(t) - C_{es}x_{es}(t)]. \end{aligned} \tag{25}$$

Now, to eliminate the explicit dependence of the estimation error (25) on $x_{eu}(t)$ in favor of $z_u(t)$, we constrain

$$C_{eu} \triangleq L_u. \tag{26}$$

The constraints (21), (22), and (26) on the reduced-order observer-estimator gains A_{eu}, A_{esu}, and C_{eu} are thus imposed in order for the reduced-order observer-estimator to asymptotically observe the $x_u(t)$ subspace of the plant (9). Note that constraints (21) and (22) are consistent with the full-order Kalman filter result (16) in which A_{eu} and A_{esu} are given by the constraints (21) and (22).

Next, using constraints (21) and (22) to eliminate the explicit dependence on $x_{eu}(t)$, it follows that the augmented system (20) has the form

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{w}(t), \quad t \in [0, \infty), \tag{27}$$

where

$$\tilde{x}(t) \triangleq \begin{bmatrix} z_u(t) \\ x_s(t) \\ x_{es}(t) \end{bmatrix} \in \mathbb{R}^{n+n_{es}}, \quad \tilde{w}(t) \triangleq \begin{bmatrix} w_{1u}(t) - B_{eu}w_2(t) \\ w_{1s}(t) \\ B_{es}w_2(t) \end{bmatrix}, \tag{28}$$

and

$$\begin{aligned} \tilde{A} &\triangleq \begin{bmatrix} A_u - B_{eu}C_u & A_{us} - B_{eu}C_s & -A_{eus} \\ 0_{n_s\times n_u} & A_s & 0_{n_s\times n_{es}} \\ B_{es}C_u & B_{es}C_s & A_{es} \end{bmatrix} \\ &= \begin{bmatrix} A - F^{\mathrm{T}}B_{eu}C & -F^{\mathrm{T}}A_{eus} \\ B_{es}C & A_{es} \end{bmatrix}. \end{aligned} \tag{29}$$

We now show that the stability of $\tilde{A}$ is equivalent to the stability of A_e.

Lemma 1. $\tilde{A}$ is asymptotically stable if and only if A_e is asymptotically stable. In this case, $\lim_{t\to\infty} z_u(t) = 0$ for $w_1(t) \equiv 0, w_2(t) \equiv 0$, and for all initial conditions $x(0), x_e(0)$. Furthermore, the state-estimation error criterion (7) is given by

$$J(A_e, B_e, C_e) = \operatorname{tr} \tilde{Q}\tilde{R}, \tag{30}$$

where the steady-state covariance

$$\tilde{Q} \triangleq \lim_{t\to\infty} \mathbb{E}[\tilde{x}(t)\tilde{x}^{\mathrm{T}}(t)] \tag{31}$$

exists and satisfies the algebraic Lyapunov equation

$$0 = \tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}. \tag{32}$$

Proof. To show that $\tilde{A}$ is asymptotically stable consider the transformation $T \in \mathbb{R}^{(n+n_{es})\times(n+n_{es})}$ given by

$$T \triangleq \begin{bmatrix} 0_{n_s\times n_u} & I_{n_s} & 0_{n_s\times n_{es}} \\ I_{n_u} & 0_{n_u\times n_s} & 0_{n_u\times n_{es}} \\ 0_{n_{es}\times n_u} & 0_{n_{es}\times n_s} & -I_{n_{es}} \end{bmatrix} \tag{33}$$

and define

$$\tilde{x}_0(t) \triangleq T\tilde{x}(t) = \begin{bmatrix} x_s(t) \\ z_u(t) \\ -x_{es}(t) \end{bmatrix}. \tag{34}$$

Using (34) it follows from (27) that

$$\dot{\tilde{x}}_0(t) = \tilde{A}_0\tilde{x}_0(t) + \tilde{w}_0(t), \tag{35}$$

where

$$\tilde{A}_0 \triangleq T\tilde{A}T^{-1} = \begin{bmatrix} A_s & 0_{n_s\times n_e} \\ F_e^{\mathrm{T}}A_{us} - B_eC_s & A_e \end{bmatrix} \tag{36}$$

and

$$\tilde{w}_0(t) \triangleq T\tilde{w}(t). \tag{37}$$

Since A_s is asymptotically stable it follows that $\tilde{A}$ is asymptotically stable if and only if A_e is asymptotically stable. In this case, $\tilde{x}(t) \to 0$ and hence $z_u(t) \to 0$ for arbitrary initial conditions when $w_1(t)$ and $w_2(t)$ are zero. Finally, the second-moment equation (32) is a direct consequence of standard Lyapunov theory (see Ref. 23, p. 104), while (30) is immediate. □

Note that Lemma 2.1 is valid even if A_u is unstable and that the assumption that A_s is stable is used explicitly in the proof.

Finally, to guarantee that $J(A_e, B_e, C_e)$ is finite and to satisfy the observation constraint (14), we define the set of asymptotically stable reduced-order observer-estimators

$$\mathcal{S} \triangleq \{(A_e, B_e, C_e) : A_e \text{ is asymptotically stable and } A_{eu}, A_{esu}, \text{ and } C_{eu} \text{ are given by (21), (22), and (26)}\}.$$

III. Necessary Conditions for the Reduced-Order Observer-Estimator Problem

In this section we obtain necessary conditions which characterize solutions to the Reduced-Order Observer-Estimator Problem. Derivation of these necessary conditions requires additional technical assumptions. Specifically, we further restrict (A_e, B_e, C_e) to the set

$$\mathcal{S}^+ \triangleq \{(A_e, B_e, C_e) \in \mathcal{S} : \quad (A_{es}, B_{es}) \text{ is controllable and } (A_e, C_e) \text{ is observable}\}. \tag{38}$$

As can be seen from the Appendix, the set $\mathcal{S}^+$ constitutes nondegeneracy conditions under which explicit gain expressions can be obtained for the Reduced-Order Observer-Estimator Problem. In order to state the main result we require some additional notation and a lemma concerning a pair of nonnegative-definite matrices.

Lemma 2. Suppose $\hat{Q}, \hat{P}$ are $n\times n$ nonnegative-definite matrices and rank $\hat{Q}\hat{P} = n_{es}$. Then there exist $n_{es} \times n$ matrices G, Γ and an $n_{es}\times n_{es}$ invertible matrix M, unique except for a change of basis in $\mathbb{R}^{n_{es}}$, such that the product $\hat{Q}\hat{P}$ can be factored according to

$$\hat{Q}\hat{P} = G^{\mathrm{T}}M\Gamma, \tag{39}$$

$$\Gamma G^{\mathrm{T}} = I_{n_{es}}. \tag{40}$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^{\mathrm{T}}\Gamma, \quad \tau_\perp \triangleq I_n - \tau \tag{41}$$

are idempotent and have rank n_{es} and $n - n_{es}$, respectively.

Proof. See Ref. 9. □

As shown in Ref. 9, $\hat{Q}\hat{P}$ has a group (Drazin) generalized inverse $(\hat{Q}\hat{P})^{\#} = G^{\mathrm{T}}M^{-1}\Gamma$. Using (40) it follows that the matrix τ is given by (1) since

$$\tau = G^{\mathrm{T}}\Gamma = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}. \tag{42}$$

Note that because of (40), $\tau^2 = G^{\mathrm{T}}\Gamma G^{\mathrm{T}}\Gamma = G^{\mathrm{T}}\Gamma = \tau$, i.e., τ is idempotent.

The following main result gives necessary conditions which characterize solutions to the Reduced-Order Observer-Estimator Problem. For convenience in stating this result define

$$Q_a \triangleq QC^{\mathrm{T}} + V_{12} \tag{43}$$

for arbitrary $Q \in \mathbb{R}^{n\times n}$.

Theorem 1. Suppose $(A_e, B_e, C_e) \in \mathcal{S}^+$ solves the Reduced-Order Observer-Estimator Problem. Then there exist $n \times n$ nonnegative-definite matrices $Q, P, \hat{P}$ and an $n_s \times n_s$ nonnegative-definite matrix $\hat{Q}_s$ such that A_e, B_e, and C_e are given by

$$A_e = \begin{bmatrix} \Phi \\ \Gamma\mu_\perp \end{bmatrix} (A - Q_aV_2^{-1}C) \begin{bmatrix} F \\ G \end{bmatrix}^{\mathrm{T}}, \tag{44}$$

$$B_e = \begin{bmatrix} \Phi \\ \Gamma\mu_\perp \end{bmatrix} Q_aV_2^{-1}, \tag{45}$$

$$C_e = L\begin{bmatrix} F \\ G \end{bmatrix}^{\mathrm{T}}, \tag{46}$$

and such that $Q, P, \hat{Q}_s$, and $\hat{P}$ satisfy

$$\begin{aligned} 0 &= AQ + QA^{\mathrm{T}} + V_1 - Q_aV_2^{-1}Q_a^{\mathrm{T}} \\ &\quad + \tau_\perp\mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}\tau_\perp^{\mathrm{T}}, \end{aligned} \tag{47}$$

$$\begin{aligned} 0 &= (A - \mu A\tau - \mu Q_aV_2^{-1}C\tau_\perp)^{\mathrm{T}}P \\ &\quad + P(A - \mu A\tau - \mu Q_aV_2^{-1}C\tau_\perp) \\ &\quad + \tau_\perp^{\mathrm{T}}L^{\mathrm{T}}RL\tau_\perp, \end{aligned} \tag{48}$$

$$\begin{aligned} 0 &= A_s\hat{Q}_s + \hat{Q}_sA_s^{\mathrm{T}} + H(Q_aV_2^{-1}Q_a^{\mathrm{T}} \\ &\quad - \tau_\perp\mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}\tau_\perp^{\mathrm{T}})H^{\mathrm{T}}, \end{aligned} \tag{49}$$

$$\begin{aligned} 0 &= (A - Q_aV_2^{-1}C)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q_aV_2^{-1}C) \\ &\quad + L^{\mathrm{T}}RL - \tau_\perp^{\mathrm{T}}L^{\mathrm{T}}RL\tau_\perp + [\mu(A - Q_aV_2^{-1}C)\tau]^{\mathrm{T}}P \\ &\quad + P[\mu(A - Q_aV_2^{-1}C)\tau], \end{aligned} \tag{50}$$

$$\operatorname{rank}\hat{Q} = \operatorname{rank}\hat{P} = \operatorname{rank}\hat{Q}\hat{P} = n_{es}, \tag{51}$$

where

$$P = \begin{bmatrix} P_u & P_{us} \\ P_{us}^{\mathrm{T}} & P_s \end{bmatrix} \in \mathbb{R}^{(n_u+n_s)\times(n_u+n_s)}, \tag{52}$$

$$P_u > 0, \tag{53}$$

$$F \triangleq [I_{n_u} \quad 0_{n_u\times n_s}], \quad \Phi \triangleq [I_{n_u} \quad P_u^{-1}P_{us}], \tag{54}$$

$$\mu \triangleq F^{\mathrm{T}}\Phi = \begin{bmatrix} I_{n_u} & P_u^{-1}P_{us} \\ 0_{n_s\times n_u} & 0_{n_s} \end{bmatrix}, \quad \mu_\perp \triangleq I_n - \mu, \tag{55}$$

$$\hat{Q} \triangleq \mu_\perp \begin{bmatrix} 0_{n_u} & 0_{n_u\times n_s} \\ 0_{n_s\times n_u} & \hat{Q}_s \end{bmatrix} \mu_\perp^{\mathrm{T}}. \tag{56}$$

Furthermore, the minimal value of the least-squares state-estimation error criterion (7) is given by

$$J(A_e, B_e, C_e) = \operatorname{tr} QL^{\mathrm{T}}RL. \tag{57}$$

Next, we present a partial converse of the necessary conditions which guarantees that the observation constraint (14) is enforced.

Theorem 2. Suppose there exist $n \times n$ nonnegative-definite matrices $Q, P, \hat{P}$ and an $n_s \times n_s$ nonnegative-definite matrix $\hat{Q}_s$ satisfying (47)–(56). Then, with $\hat{Q}$ given by (56), the matrix

$$\tilde{Q} = \begin{bmatrix} Q + \hat{Q} & \hat{Q}\Gamma^{\mathrm{T}} \\ \Gamma\hat{Q} & \Gamma\hat{Q}\Gamma^{\mathrm{T}} \end{bmatrix} \tag{58}$$

satisfies (32) with (A_e, B_e, C_e) given by (44)–(46). Furthermore, $(\tilde{A}, \tilde{V}^{\frac{1}{2}})$ is stabilizable if and only if A_e is asymptotically stable. In this case, (A_{es}, B_{es}) is controllable, (A_e, C_e) is observable, the observation constraint (14) holds for all arbitrary initial conditions $x(0), x_e(0)$ when $w_1(t) \equiv 0, w_2(t) \equiv 0$, and the least-squares state-estimation error criterion is given by (57).

The proofs of Theorems 1 and 2 are given in the Appendix .

Theorem 1 presents necessary conditions for the Reduced-Order Observer-Estimator Problem. These necessary conditions consist of a system of one modified Riccati equation and three modified Lyapunov equations coupled by two distinct oblique (not necessarily orthogonal) projections τ and μ. Note that τ and μ are idempotent since $\tau^2 = \tau$ and $\mu^2 = \mu$. As discussed earlier, the fixed-order constraint on the estimator order gives rise to the order projection τ, while the observation constraint (14) gives rise to the subspace projection μ. It is easy to see that rank $\mu = n_u$ and it can be shown[9] using Sylvester's inequality and (40) that rank $\tau = n_{es}$.

Remark 1. Note that with B_e given by (45), the expressions (44) and (46) for A_{eu}, A_{esu}, and C_{eu} are equivalent to the constraints (21), (22), and (26).

Remark 2. By defining the $n_e \times n$ matrices

$$\tilde{G} \triangleq \begin{bmatrix} F \\ G \end{bmatrix}, \quad \tilde{\Gamma} \triangleq \begin{bmatrix} \Phi \\ \Gamma\mu_\perp \end{bmatrix}, \tag{59}$$

it can be shown that

$$\tilde{\Gamma}\tilde{G}^{\mathrm{T}} = \begin{bmatrix} I_{n_u} & 0_{n_u\times n_{es}} \\ 0_{n_{es}\times n_u} & I_{n_{es}} \end{bmatrix} = I_{n_e}. \tag{60}$$

Using (60) one can thus define a third *composite* projection

$$\tilde{\tau} \triangleq \tilde{G}^{\mathrm{T}}\tilde{\Gamma} = \mu + \tau\mu_\perp = \mu + \tau - \tau\mu, \tag{61}$$

where rank $\tilde{\tau} = n_e$. Using (59), the gains (44)–(46) can be written as

$$A_e = \tilde{\Gamma}(A - Q_aV_2^{-1}C)\tilde{G}^{\mathrm{T}} = \tilde{\Gamma}A\tilde{G}^{\mathrm{T}} - B_eC\tilde{G}^{\mathrm{T}}, \tag{62}$$

$$B_e = \tilde{\Gamma}Q_aV_2^{-1}, \tag{63}$$

$$C_e = L\tilde{G}^{\mathrm{T}}. \tag{64}$$

Remark 3. It follows from (42) and (56) that

$$\mu\tau = \mu\hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#} = \mu\mu_\perp \begin{bmatrix} 0_{n_u} & 0_{n_u\times n_s} \\ 0_{n_s\times n_u} & \hat{Q}_s \end{bmatrix} \mu_\perp^{\mathrm{T}}\hat{P}(\hat{Q}\hat{P})^{\#}. \tag{65}$$

Since $\mu\mu_\perp = 0$, we obtain

$$0 = \mu\tau \tag{66}$$

as a consequence of optimality. Partitioning

$$\tau = \begin{bmatrix} \tau_u & \tau_{us} \\ \tau_{su} & \tau_s \end{bmatrix} \in \mathbb{R}^{(n_u+n_s)\times(n_u+n_s)}, \tag{67}$$

(66) implies

$$\tau_u = -P_u^{-1}P_{us}\tau_{su}, \quad \tau_{us} = -P_u^{-1}P_{us}\tau_s. \tag{68}$$

Remark 4. Note that for (A_e, B_e, C_e) given by (44)–(46), the observer-estimator (5) or, equivalently (12), assumes the innovations form

$$\dot{x}_e(t) = \tilde{\Gamma}A\tilde{G}^{\mathrm{T}}x_e(t) + \tilde{\Gamma}Q_aV_2^{-1}[y(t) - C\tilde{G}^{\mathrm{T}}x_e(t)]. \tag{69}$$

Remark 5. By introducing the quasi-full-state estimate $\hat{x}(t) \triangleq \tilde{G}^{\mathrm{T}}x_e(t) \in \mathbb{R}^n$ so that $\tilde{\tau}\hat{x}(t) = \hat{x}(t)$ and $x_e(t) = \tilde{\Gamma}\hat{x}(t) \in \mathbb{R}^{n_e}$, (69) can be written as

$$\dot{\hat{x}}(t) = \tilde{\tau}A\tilde{\tau}\hat{x}(t) + \tilde{\tau}Q_aV_2^{-1}[y(t) - C\hat{x}(t)], \tag{70}$$

or, equivalently,

$$\begin{aligned} \dot{\hat{x}} &= (\mu + \tau\mu_\perp)A(\mu + \tau\mu_\perp)\hat{x}(t) \\ &\quad + (\mu + \tau\mu_\perp)Q_aV_2^{-1}[y(t) - C\hat{x}(t)]. \end{aligned} \tag{71}$$

Note that although the *implemented* observer-estimator (69) has the reduced-order state $x_e(t) \in \mathbb{R}^{n_e}$, (71) can be viewed as a quasi-full-order observer-estimator whose geometric structure is dictated by the projections τ and μ. Specifically, error inputs $Q_a V_2^{-1}[y(t) - C\hat{x}(t)]$ are annihilated unless they are contained in $[\mathcal{N}(\mu + \tau\mu_\perp)]^\perp = \mathcal{R}[(\mu + \tau\mu_\perp)^\mathrm{T}]$. Hence, the observation subspace of the observer-estimator is precisely $\mathcal{R}[(\mu + \tau\mu_\perp)^\mathrm{T}]$.

Remark 6. In the full-order Kalman filter case it is well known that an orthogonality condition

$$\mathbb{E}\{[x(t) - x_e(t)]x_e^\mathrm{T}(t)\} = 0 \tag{72}$$

is satisfied. For the observer-estimator problem an analogous condition[20] is

$$\mathbb{E}\{[x_u(t) - x_{eu}(t)]x_{es}^\mathrm{T}(t)\} = 0. \tag{73}$$

This condition does not hold automatically, however, but must be imposed as an additional side constraint. It can be shown that requiring (73) leads to

$$0 = FG^\mathrm{T} \tag{74}$$

and, consequently,

$$0 = F\tau, \quad 0 = \mu^\mathrm{T}\tau. \tag{75}$$

Using (75), it follows that τ has the structure

$$\tau = \begin{bmatrix} 0_{n_u} & 0_{n_u \times n_s} \\ \tau_{su} & \tau_s \end{bmatrix} \tag{76}$$

so that the composite projection $\tilde{\tau}$ has the form

$$\tilde{\tau} = \begin{bmatrix} I_{n_u} & P_u^{-1}P_{us} \\ 0_{n_s \times n_u} & \tau_s - \tau_{su}P_u^{-1}P_{us} \end{bmatrix}. \tag{77}$$

IV. Specializations of Theorem 1

To draw connections with the previous literature, a series of specializations of Theorem 1 is now given. Specifically, to recover the full-order steady-state Kalman filter from Theorem 1 take $n_{es} = n_s$ or, equivalently, $n_e = n$. Since $\tilde{\Gamma}\tilde{G}^\mathrm{T} = I_n$, let $S = \tilde{\Gamma} \in \mathbb{R}^{n \times n}$ and $S^{-1} = \tilde{G}^\mathrm{T} \in \mathbb{R}^{n \times n}$. In this case the optimal gains (44)–(46) become

$$A_e = S(A - Q_a V_2^{-1}C)S^{-1}, \tag{78}$$
$$B_e = SQ_a V_2^{-1}, \tag{79}$$
$$C_e = LS^{-1}. \tag{80}$$

Furthermore, in this case since

$$\tau_\perp\mu_\perp = I_n - \mu - \tau\mu_\perp = I_n - \tilde{G}^\mathrm{T}\tilde{\Gamma} = I_n - S^{-1}S = 0, \tag{81}$$

the modified Riccati equation (47) specializes to the standard observer Riccati equation

$$0 = AQ + QA^\mathrm{T} + V_1 - Q_a V_2^{-1} Q_a^\mathrm{T} \tag{82}$$

and (48)–(50) are superfluous. Note that (78)–(80) are precisely the standard steady-state Kalman filter gains in an alternative basis specified by the basis transformation S. Since $J(A_e, B_e, C_e) = J(SA_eS^{-1}, SB_e, C_eS^{-1})$, however, this change of basis leaves the estimation error unchanged.

Next, to recover the optimal projection results of Ref. 9 involving reduced-order estimators for stable plants without a subspace observation constraint, let $n_u = 0, n_s = n, n_{es} = n_e, A_s = A$, and $n_e < n$, set $\mu = 0$ so that $\mu_\perp = I_n$, and replace $\begin{bmatrix} \Phi \\ \Gamma\mu_\perp \end{bmatrix}$ and $\begin{bmatrix} F \\ G \end{bmatrix}^\mathrm{T}$ by Γ and G^T, respectively. Then the optimal gains (44)–(46) become

$$A_e = \Gamma(A - Q_a V_2^{-1}C)G^\mathrm{T}, \tag{83}$$
$$B_e = \Gamma Q_a V_2^{-1}, \tag{84}$$
$$C_e = LG^\mathrm{T}, \tag{85}$$

and equations (47)–(50) specialize to

$$0 = AQ + QA^\mathrm{T} + V_1 - Q_a V_2^{-1} Q_a^\mathrm{T} + \tau_\perp Q_a V_2^{-1} Q_a^\mathrm{T} \tau_\perp^\mathrm{T}, \tag{86}$$
$$0 = A\hat{Q} + \hat{Q}A^\mathrm{T} + Q_a V_2^{-1} Q_a^\mathrm{T} - \tau_\perp Q_a V_2^{-1} Q_a^\mathrm{T} \tau_\perp^\mathrm{T}, \tag{87}$$
$$0 = (A - Q_a V_2^{-1}C)^\mathrm{T}\hat{P} + \hat{P}(A - Q_a V_2^{-1}C) + L^\mathrm{T}RL - \tau_\perp^\mathrm{T} L^\mathrm{T} R L \tau_\perp. \tag{88}$$

These are equations (2.10)–(2.12) of Ref. 9.

Finally, we can also recover the results of Ref. 17 where the reduced-order observer is constrained to observe an n_u-dimensional plant subspace without estimating the remaining n_s-dimensional subspace. In this case let $n_e = n_u, n_{es} = 0$, and $\tau = 0$ so that $\tau_\perp = I_n$. Furthermore, let $\begin{bmatrix} \Phi \\ \Gamma\mu_\perp \end{bmatrix}$ and $\begin{bmatrix} F \\ G \end{bmatrix}^\mathrm{T}$ be replaced by Φ and F^T respectively so that the gain expressions (44)–(46) become

$$A_e = \Phi(A - Q_a V_2^{-1}C)F^\mathrm{T}, \tag{89}$$
$$B_e = \Phi Q_a V_2^{-1}, \tag{90}$$
$$C_e = LF^\mathrm{T}, \tag{91}$$

and equations (47)–(50) specialize to

$$0 = AQ + QA^\mathrm{T} + V_1 - Q_a V_2^{-1} Q_a^\mathrm{T} + \mu_\perp Q_a V_2^{-1} Q_a^\mathrm{T} \mu_\perp^\mathrm{T}, \tag{92}$$

$$0 = (A - \mu Q_a V_2^{-1}C)^\mathrm{T}P + P(A - \mu Q_a V_2^{-1}C) + L^\mathrm{T}RL. \tag{93}$$

These are equations (2.17) and (2.18) of Ref. 17.

V. Numerical Algorithm and Illustrative Numerical Examples

In this section we present a numerical algorithm for solving the optimality conditions for the Reduced-Order Observer-Estimator Problem and consider two illustrative numerical examples.

Algorithm 1. To solve (47)–(50), carry out the following steps:

Step 1. Initialize $k = 1$, $\mu^{(1)} = I_n$, $\tau^{(1)} = I_n$;

Step 2. With $\mu = \mu^{(k)}$ and $\tau = \tau^{(k)}$, solve (47) for $Q^{(k)} = Q$;

Step 3. With $Q = Q^{(k)}, \mu = \mu^{(k)}$, and $\tau = \tau^{(k)}$, solve (48) and (49) for $P^{(k)} = P$ and $\hat{Q}_s^{(k)} = \hat{Q}_s$;

Step 4. With $Q = Q^{(k)}, P = P^{(k)}, \mu = \mu^{(k)}$, and $\tau = \tau^{(k)}$, solve (50) for $\hat{P}^{(k)} = \hat{P}$;

Step 5. If convergence of $Q^{(k)}$ and $P^{(k)}$ has been attained then evaluate A_e, B_e, C_e using (44)–(46) and stop; else continue;

Step 6. Use $P = P^{(k)}, \hat{Q}_s = \hat{Q}_s^{(k)}$, and $\hat{P} = \hat{P}^{(k)}$ to define $\mu^{(k+1)} = \mu$ and $\tau^{(k+1)} = \tau$ using (39)–(41),(55), (56);

Step 7. Replace k by $k+1$ and go to Step 1.

The above algorithm is a straightfoward iterative scheme which is fairly easy to implement. More sophisticated algorithms can be developed by utilizing homotopic continuation techniques[27]. For the examples discussed below, however, Algorithm 1 proved to be adequate.

Our first example, adopted from Ref. 28, pp. 99–101, involves a satellite in circular orbit. The linearized error equations representing the deviation from a perfect circular orbit are given by

$$
\begin{bmatrix} \dot{r} \\ \ddot{r} \\ \dot{\theta} \\ \ddot{\theta} \\ \dot{\phi} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\omega/r_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega^2 & -\varepsilon \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_0} \\ 0 \\ \frac{1}{m_0} \\ 0 \\ \frac{1}{m_0} \end{bmatrix} W_0, \tag{94}
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} W_2^{(1)} \\ W_2^{(2)} \\ W_2^{(3)} \end{bmatrix} \tag{95}
$$

where r, θ, ϕ are spherical coordinates, r_0 is the orbit radius, ω denotes orbital frequency, and $\varepsilon > 0$.

Here the state vector represents the deviation from a circular equatorial orbit and is expressed in spherical coordinates. We note that $\varepsilon = 0$ was assumed in Ref. 28, although $\varepsilon > 0$ is assumed here to reflect dissipation in this coordinate due possibly to on-board forces. Furthermore, stochastic disturbance models are utilized here in place of deterministic inputs appearing in Ref. 28. To reflect a plausible mission we assume the following data:

$$\omega = 2\pi \text{ rad/day},\ m_0 = 50\text{kg},\ r_0 = 42.2 \times 10^6\text{m}, \tag{96}$$

$$\sigma^2(W_0)/m_0^2 = 384 \text{ Nt}^2 - \text{day}, \tag{97}$$

$$\sigma^2(W_2^{(1)}) = 8.9 \times 10^6 \text{ m}^2 - \text{day}, \tag{98}$$

$$\sigma^2(W_2^{(2)}) = \sigma^2(W_2^{(3)}) = 7.84 \times 10^{-7} \text{ rad}^2 - \text{day}, \tag{99}$$

where $\sigma^2(\cdot)$ denotes noise intensity.

To treat this problem within our formulation, we note that the upper left 4×4 block of (94) has neutrally stable eigenvalues $0, 0, j\omega$, and $-j\omega$. Hence we set $n_u = 4$ and $n_s = 2$ and seek to design an optimal 4th-order observer for the unstable subspace. In this case $n_s = 0$ and thus we need only solve the subspace observer equations (92), (93). As inputs to the estimator design process we chose to weight the angular position coordinates by r_0 in the interest of dimensional compatibility, i.e.,

$$R = 1, \quad L = [1 \quad 0 \quad r_0 \quad 0 \quad r_0 \quad 0]. \tag{100}$$

A study was conducted to assess the performance of the optimal subspace observer compared to a full-order steady-state Kalman filter as well as a reduced-order Kalman filter obtained using a truncated model consisting of only the first $n_u = 4$ states. The study involved a series of designs for decreasing magnitudes of the parameter ε, i.e., decreasing stability of the ϕ and $\dot{\phi}$ states. The results of the study are summarized in Figure 1.

To further illustrate the algorithm we consider an example reminiscent of a rigid body with flexible appendages. Hence define

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -.01 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 & -.02 \end{bmatrix},
$$

$$C = [1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0],$$

$$L = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0], \qquad R = 1,$$

$$
V_1 = DD^{\mathrm{T}}, \qquad D = \begin{bmatrix} .1 & 0 \\ 0 & 1 \\ .1 & 0 \\ 0 & 1 \\ .1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad V_{12} = 0, \qquad V_2 = 1.
$$

Note that the dynamic model involves one rigid body mode and two flexible modes at frequencies 1 and 2 rad/sec with .5% damping ratios. The matrix C captures the fact that the rigid body position measurement is corrupted by the flexible modes (i.e., observation spillover), the matrix L expresses the desire to estimate the rigid body position, and the matrix V_1 was chosen to capture the type of noise correlation which arises when the dynamics are transformed into a modal basis.

For the full-order steady-state Kalman filter the optimal estimation error was $J = 1.533$. We then truncated the higher frequency flexible mode and obtained a suboptimal 4th-order observer as a "full-order" estimator for the truncated system. The performance of this suboptimal estimator evaluated for the 6th-order plant was $J = 3.537$. By applying Algorithm 1 an optimal 4th-order subspace observer was obtained. The performance of this optimal estimator was $J = 1.572$.

A second-order suboptimal filter was also obtained as a "full-order" estimator for a truncated plant consisting of the rigid body mode only. The performance of this suboptimal estimator was $J = 78.74$. In contrast, the optimal 2nd-order subspace observer constrained to observe only the rigid body mode had performance $J = 2.328$.

VI. Conclusion

Optimality conditions have been obtained for the problem of designing reduced-order observer-estimators. The principal feature of the theory presented herein is the ability of the reduced-order observer-estimator to observe a possibly unstable subspace of the plant while providing optimal estimates of specified linear combinations of the remaining plant states. The necessary conditions for optimality comprise a system of *four* matrix equations coupled by *two* oblique projections which determine the optimal estimator gains. The results given herein generalize previous results obtained for the stable plant case.

Appendix: Proofs of Theorem 1 and Theorem 2

To optimize (30) over the open set $\mathcal{S}^+$ subject to the constraint (32), form the Lagrangian

$$
\begin{aligned} \mathcal{L}(A_{es}, A_{eus}, B_e, C_{es}, \tilde{Q}, \tilde{P}, \lambda) \\ \triangleq \operatorname{tr}\{\lambda \tilde{Q}\tilde{R} + [\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^{\mathrm{T}} + \tilde{V}]\tilde{P}\}, \end{aligned} \tag{101}
$$

where the Lagrange multipliers $\lambda \geq 0$ and $\tilde{P} \in \mathbb{R}^{(n+n_{es})\times(n+n_{es})}$ are not both zero. We thus obtain

$$\frac{\partial \mathcal{L}}{\partial \tilde{Q}} = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \lambda\tilde{R}. \tag{102}$$

Setting $\frac{\partial \mathcal{L}}{\partial \tilde{Q}} = 0$ yields

$$0 = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \lambda\tilde{R}. \tag{103}$$

Since $\tilde{A}$ is assumed to be stable, $\lambda = 0$ implies $\tilde{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\tilde{P}$ is nonnegative definite.

Now partition $(n+n_{es}) \times (n+n_{es})$ $\tilde{Q}, \tilde{P}$ into $n \times n$, $n \times n_{es}$, and $n_{es} \times n_{es}$ subblocks as

$$\tilde{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2 \end{bmatrix}, \quad \tilde{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^{\mathrm{T}} & P_2 \end{bmatrix}. \tag{104}$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial \mathcal{L}}{\partial \tilde{Q}} = \tilde{A}^{\mathrm{T}}\tilde{P} + \tilde{P}\tilde{A} + \tilde{R} = 0, \tag{105}$$

$$\frac{\partial \mathcal{L}}{\partial A_{es}} = P_{12}^{\mathrm{T}} Q_{12} + P_2 Q_2 = 0, \tag{106}$$

$$\frac{\partial \mathcal{L}}{\partial A_{eus}} = F(P_1 Q_{12} + P_{12} Q_2) = 0, \tag{107}$$

$$\frac{\partial \mathcal{L}}{\partial B_{eu}} = FP_1F^{\mathrm{T}}B_{eu} - FP_1Q_aV_2^{-1} + FP_{12}B_{es} = 0, \tag{108}$$

$$\frac{\partial \mathcal{L}}{\partial B_{es}} = P_2 B_{es} V_2 - P_{12}^{\mathrm{T}} Q_a + P_{12}^{\mathrm{T}} F^{\mathrm{T}} B_{eu} V_2 = 0, \tag{109}$$

$$\frac{\partial \mathcal{L}}{\partial C_{es}} = -RLQ_{12} + RC_{es}Q_2 = 0. \tag{110}$$

Expanding (32) and (105) yields

$$\begin{aligned}
0 &= AQ_1 - F^{\mathrm{T}}B_{eu}CQ_1 - F^{\mathrm{T}}A_{eus}Q_{12}^{\mathrm{T}} + Q_1A^{\mathrm{T}} \\
&\quad - Q_1C^{\mathrm{T}}B_{eu}^{\mathrm{T}}F - Q_{12}A_{eus}^{\mathrm{T}}F + V_1 - V_{12}B_{eu}^{\mathrm{T}}F \\
&\quad - F^{\mathrm{T}}B_{eu}V_{12}^{\mathrm{T}} + F^{\mathrm{T}}B_{eu}V_2B_{eu}^{\mathrm{T}}F, && (111)\\
0 &= AQ_{12} - F^{\mathrm{T}}B_{eu}CQ_{12} - F^{\mathrm{T}}A_{eus}Q_2 + Q_1C^{\mathrm{T}}B_{es}^{\mathrm{T}} \\
&\quad + Q_{12}A_{es}^{\mathrm{T}} + V_{12}B_{es}^{\mathrm{T}} - F^{\mathrm{T}}B_{eu}V_2B_{es}^{\mathrm{T}}, && (112)\\
0 &= A_{es}Q_2 + Q_2A_{es}^{\mathrm{T}} + B_{es}CQ_{12} + Q_{12}^{\mathrm{T}}C^{\mathrm{T}}B_{es}^{\mathrm{T}} \\
&\quad + B_{es}V_2B_{es}^{\mathrm{T}}, && (113)\\
0 &= A^{\mathrm{T}}P_1 - C^{\mathrm{T}}B_{eu}^{\mathrm{T}}FP_1 + C^{\mathrm{T}}B_{es}^{\mathrm{T}}P_{12}^{\mathrm{T}} + P_1A \\
&\quad - P_1F^{\mathrm{T}}B_{eu}C + P_{12}B_{es}C + L^{\mathrm{T}}RL, && (114)\\
0 &= A^{\mathrm{T}}P_{12} - C^{\mathrm{T}}B_{eu}^{\mathrm{T}}FP_{12} + C^{\mathrm{T}}B_{es}^{\mathrm{T}}P_2 - P_1F^{\mathrm{T}}A_{eus} \\
&\quad + P_{12}A_{es} - L^{\mathrm{T}}RC_{es}, && (115)\\
0 &= A_{es}^{\mathrm{T}}P_2 + P_2A_{es} - A_{eus}^{\mathrm{T}}FP_{12} - P_{12}^{\mathrm{T}}F^{\mathrm{T}}A_{eus} \\
&\quad + C_{es}^{\mathrm{T}}RC_{es}. && (116)
\end{aligned}$$

Lemma 3. Q_2, P_2, and $P_u \triangleq FP_1F^{\mathrm{T}} - FP_{12}P_2^{-1}P_{12}^{\mathrm{T}}F^{\mathrm{T}}$ are positive definite.

Proof. By a minor extension of the results from Ref. 29, (113) can be rewritten as

$$\begin{aligned} 0 &= (A_{es} + B_{es}CQ_{12}Q_2^{+})Q_2 \\ &\quad + Q_2(A_{es} + B_{es}CQ_{12}Q_2^{+})^{\mathrm{T}} + B_{es}V_2B_{es}^{\mathrm{T}}, \end{aligned} \tag{117}$$

where Q_2^{+} is the Moore-Penrose or Drazin generalized inverse of Q_2. Next note that since (A_{es}, B_{es}) is controllable it follows from Lemma 2.1 and Theorem 3.6 of Ref. 30 that $(A_{es}+B_{es}CQ_{12}Q_2^{+}, B_{es}V_2^{\frac{1}{2}})$ is controllable. Now, since Q_2 and $B_{es}V_2B_{es}^{\mathrm{T}}$ are nonnegative definite, Lemma 12.2 of Ref. 30 implies that Q_2 is positive definite. To show that P_2 and P_u are positive definite, consider the transformation T given by (33) such that $\tilde{x}_0(t) = T\tilde{x}(t)$ where $\tilde{x}_0(t)$ is given by (34). Using this transformation (105) becomes

$$0 = \tilde{A}_0^{\mathrm{T}}T^{-\mathrm{T}}\tilde{P}T^{-1} + T^{-\mathrm{T}}\tilde{P}T^{-1}\tilde{A}_0 + T^{-\mathrm{T}}\tilde{R}T^{-1}, \tag{118}$$

where $\tilde{A}_0$ is given by (36). Noting that $T^{-\mathrm{T}} = T$ and that

$$T^{-\mathrm{T}}\tilde{P}T^{-1} = \begin{bmatrix} HP_1H^{\mathrm{T}} & HP_1F^{\mathrm{T}} & HP_{12} \\ FP_1H^{\mathrm{T}} & FP_1F^{\mathrm{T}} & FP_{12} \\ P_{12}^{\mathrm{T}}H^{\mathrm{T}} & P_{12}^{\mathrm{T}}F^{\mathrm{T}} & P_2 \end{bmatrix}, \tag{119}$$

the (2,2) block of the above Lyapunov equation is

$$0 = A_e^{\mathrm{T}}P_e + P_eA_e + C_e^{\mathrm{T}}RC_e, \tag{120}$$

where

$$P_e \triangleq \begin{bmatrix} FP_1F^{\mathrm{T}} & FP_{12} \\ P_{12}^{\mathrm{T}}F^{\mathrm{T}} & P_2 \end{bmatrix}. \tag{121}$$

Using (120) and the fact that (A_e, C_e) is observable, it follows that P_e is positive definite. Hence, it follows from Ref. 29 that P_2 and $P_u \triangleq FP_1F^{\mathrm{T}} - FP_{12}P_2^{-1}P_{12}^{\mathrm{T}}F^{\mathrm{T}}$ are positive definite. □

Since Q_2 and P_2 are invertible, (106) and (107) can be written as

$$-P_2^{-1}P_{12}^{\mathrm{T}}Q_{12}Q_2^{-1} = I_{n_{es}}, \tag{122}$$

$$0 = F(P_1Q_{12}Q_2^{-1} + P_{12}). \tag{123}$$

Now define the $n \times n$ matrices

$$Q \triangleq Q_1 - Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad P \triangleq P_1 - P_{12}P_2^{-1}P_{12}^{\mathrm{T}}, \tag{124}$$

$$\hat{Q} \triangleq Q_{12}Q_2^{-1}Q_{12}^{\mathrm{T}}, \qquad \hat{P} \triangleq P_{12}P_2^{-1}P_{12}^{\mathrm{T}}, \tag{125}$$

$$\tau \triangleq -Q_{12}Q_2^{-1}P_2^{-1}P_{12}^{\mathrm{T}}, \tag{126}$$

and the $n_{es} \times n, n_{es} \times n_{es}$, and $n_{es} \times n$ matrices

$$G \triangleq Q_2^{-1}Q_{12}^{\mathrm{T}}, \quad M \triangleq Q_2P_2, \quad \Gamma \triangleq -P_2^{-1}P_{12}^{\mathrm{T}}. \tag{127}$$

Note that $Q, P, \hat{Q}, \hat{P}$ are nonnegative definite and that $FPF^{\mathrm{T}} = P_u$. Next partition $n \times n$ $P, \hat{Q}$ into $n_u \times n_u, n_u \times n_s$, and $n_s \times n_s$ subblocks as

$$P = \begin{bmatrix} P_u & P_{us} \\ P_{us}^{\mathrm{T}} & P_s \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} \hat{Q}_u & \hat{Q}_{us} \\ \hat{Q}_{us}^{\mathrm{T}} & \hat{Q}_s \end{bmatrix}. \tag{128}$$

Since P_u is invertible (see Lemma 3) define the $n_u \times n$ matrices

$$F \triangleq [I_{n_u} \ 0_{n_u \times n_s}], \qquad \Phi \triangleq [I_{n_u} \ P_u^{-1}P_{us}], \tag{129}$$

and $n \times n$ matrix

$$\mu \triangleq F^{\mathrm{T}}\Phi. \tag{130}$$

Next note that with the above definitions (122) is equivalent to (40) and that (39) holds. Hence $\tau = G^{\mathrm{T}}\Gamma$ is idempotent, i.e., $\tau^2 = \tau$. Similarly, since $\Phi F^{\mathrm{T}} = I_{n_u}$, μ is also idempotent.

It is helpful to note the identities

$$\begin{aligned} \hat{Q} &= Q_{12}G = G^{\mathrm{T}}Q_{12}^{\mathrm{T}} = G^{\mathrm{T}}Q_2G, \\ \hat{P} &= -P_{12}\Gamma = -\Gamma^{\mathrm{T}}P_{12}^{\mathrm{T}} = \Gamma^{\mathrm{T}}P_2\Gamma, \end{aligned} \tag{131}$$

$$\tau G^{\mathrm{T}} = G^{\mathrm{T}}, \qquad \Gamma\tau = \Gamma, \tag{132}$$

$$\hat{Q} = \tau\hat{Q}, \qquad \hat{P} = \hat{P}\tau, \tag{133}$$

$$\hat{Q}\hat{P} = -Q_{12}P_{12}^{\mathrm{T}}. \tag{134}$$

Using (122) and Sylvester's inequality, it follows that

$$\operatorname{rank} G = \operatorname{rank} \Gamma = \operatorname{rank} Q_{12} = \operatorname{rank} P_{12} = n_{es}. \tag{135}$$

Now using (131) and Sylvester's inequality yields

$$\begin{aligned} n_{es} &= \operatorname{rank} Q_{12} \\ &\quad + \operatorname{rank} G - n_{es} \le \operatorname{rank} \hat{Q} \le \operatorname{rank} Q_{12} = n_{es}, \end{aligned} \tag{136}$$

which implies that rank $\hat{Q} = n_{es}$. Similarly, rank $\hat{P} = n_{es}$, and rank $\hat{Q}\hat{P} = n_{es}$ follows from (134).

Next, using (134) and the above identities, it follows from (123) that

$$0 = FP\hat{Q}. \tag{137}$$

Using the partitioned form (128) of P and $\hat{Q}$, (137) implies

$$\hat{Q} = \mu_{\perp} \begin{bmatrix} 0_{n_u} & 0_{n_u \times n_s} \\ 0_{n_s \times n_u} & \hat{Q}_s \end{bmatrix} \mu_{\perp}^{\mathrm{T}}. \tag{138}$$

The components of $\tilde{Q}$ and $\tilde{P}$ can be written in terms of $Q, P, \hat{Q}, \hat{P}, G$, and Γ as

$$Q_1 = Q + \hat{Q}, \qquad P_1 = P + \hat{P}, \tag{139}$$

$$Q_{12} = \hat{Q}\Gamma^{\mathrm{T}}, \qquad P_{12} = -\hat{P}G^{\mathrm{T}}, \tag{140}$$

$$Q_2 = \Gamma\hat{Q}\Gamma^{\mathrm{T}}, \qquad P_2 = G\hat{P}G^{\mathrm{T}}. \tag{141}$$

Furthermore, it is useful to note that

$$F\Phi^{\mathrm{T}} = F, \quad 0 = \Phi G^{\mathrm{T}}, \quad F^{\mathrm{T}} = \mu F^{\mathrm{T}}, \quad 0 = FA^{\mathrm{T}}PG^{\mathrm{T}}, \tag{142}$$

$$0 = GP\mu, \qquad I_{n_{e*}} = \Gamma\mu_\perp G^{\mathrm{T}}, \qquad \Phi = F\mu, \tag{143}$$

$$0 = \mu\tau, \quad \tau = \mu_\perp\tau, \quad \mu = \mu\tau_\perp, \quad \tau_\perp\mu_\perp = \mu_\perp\tau_\perp\mu_\perp, \tag{144}$$

which follow from (137) and (138).

The expressions for (45) and (46) follow from (108)–(110) by using the above identities. Next, computing G (115) + (116) along with (116) yields (44). Substituting (139)–(141) into (111)–(116) along with the expression for A_e it follows that (113) = Γ (112) and (116) = G (115). Thus (113) and (116) are superfluous and can be omitted. Thus, (111)–(116) reduce to

$$0 = AQ + QA^{\mathrm{T}} + \mu_\perp A\hat{Q} + \hat{Q}A^{\mathrm{T}}\mu_\perp^{\mathrm{T}} + V_1 - Q_aV_2^{-1}Q_a^{\mathrm{T}} + \mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}, \tag{145}$$

$$0 = [\mu_\perp A\hat{Q} + \hat{Q}A^{\mathrm{T}}\mu_\perp^{\mathrm{T}} + \mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}]\Gamma^{\mathrm{T}}, \tag{146}$$

$$0 = (A - \mu Q_aV_2^{-1}C)^{\mathrm{T}}P + P(A - \mu Q_aV_2^{-1}C) + (A - Q_aV_2^{-1}C)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q_aV_2^{-1}C) + L^{\mathrm{T}}RL, \tag{147}$$

$$0 = [(A - Q_aV_2^{-1}C)^{\mathrm{T}}\hat{P} + \hat{P}(A - Q_aV_2^{-1}C) + P\mu(A - Q_aV_2^{-1}C) + L^{\mathrm{T}}RL]G^{\mathrm{T}}. \tag{148}$$

Next, using (145) $+G^{\mathrm{T}}\Gamma$ (146) $G-$ (146) $G - [$ (146) $G]^{\mathrm{T}}$ yields (47). Similarly, using (147) $+\Gamma^{\mathrm{T}}G$ (148) $\Gamma-$ (148) $\Gamma - [$ (148) $\Gamma]^{\mathrm{T}}$ and $\Gamma^{\mathrm{T}}G$ (148) $\Gamma-$ (148) $\Gamma - [$ (148) $\Gamma]^{\mathrm{T}}$ yields (48) and (50). Now using $G^{\mathrm{T}}\Gamma$ (146) $G-$ (146) $G - [$ (146) $G]^{\mathrm{T}}$ yields

$$0 = \mu_\perp A\hat{Q} + \hat{Q}A^{\mathrm{T}}\mu_\perp^{\mathrm{T}} + \mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}} - \tau_\perp\mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}\tau_\perp^{\mathrm{T}}. \tag{149}$$

Using (138), (149) becomes

$$0 = \mu_\perp \begin{bmatrix} 0_{n_u} & 0_{n_u \times n_s} \\ 0_{n_s \times n_u} & A_s\hat{Q}_s + \hat{Q}_sA_s^{\mathrm{T}} \end{bmatrix} \mu_\perp^{\mathrm{T}} + \mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}} - \tau_\perp\mu_\perp Q_aV_2^{-1}Q_a^{\mathrm{T}}\mu_\perp^{\mathrm{T}}\tau_\perp^{\mathrm{T}}. \tag{150}$$

Next, computing $H(150)H^{\mathrm{T}}$ yields (49). Note conversely that if (49) is satisfied, then (A.36) holds since $\mu_\perp\tau_\perp\mu_\perp = \tau_\perp\mu_\perp$.

Finally, to prove Theorem 2 we use (44)–(50) to obtain (32) and (105)–(110). Let $A_e, B_e, C_e, G, \Gamma, F, \Phi, \tau, \mu, Q, P, \hat{Q}, \hat{P}, \hat{Q}_s, \tilde{Q}$ be as in the statement of Theorem 1 and define $Q_1, Q_{12}, Q_2, P_1, P_{12}, P_2$ by (108)–(110). Using (40), $\Phi F^{\mathrm{T}} = I_{n_u}$, (45) and (46) it is easy to verify (139)–(141). Next substitute the definitions of $Q, P, \hat{Q}, \hat{P}, G, \Gamma, F, \Phi, \tau, \mu$ into (47)–(50) using (40), (41), and (133) to obtain (32) and (105). Finally, note that

$$\tilde{Q} = \begin{bmatrix} Q & 0_{n \times n_{e*}} \\ 0_{n_{e*} \times n} & 0_{n_{e*}} \end{bmatrix} + \begin{bmatrix} I_n \\ \Gamma \end{bmatrix} \hat{Q}[I_n \ \Gamma^{\mathrm{T}}],$$

which shows that $\tilde{Q} \geq 0$. Now using the assumed existence of a nonnegative-definite solution to (32) and the stabilizability condition $(\tilde{A}, \tilde{V}^{\frac{1}{2}})$, it follows from the dual of Lemma 12.2 of Ref. 30, that $\tilde{A}$ is asymptotically stable. Since $\tilde{A}_0$ is upper block triangular, A_e is also asymptotically stable. Conversely, since A_s is assumed to be asymptotically stable A_e stable implies $(\tilde{A}, \tilde{V}^{\frac{1}{2}})$ stabilizable. ◻

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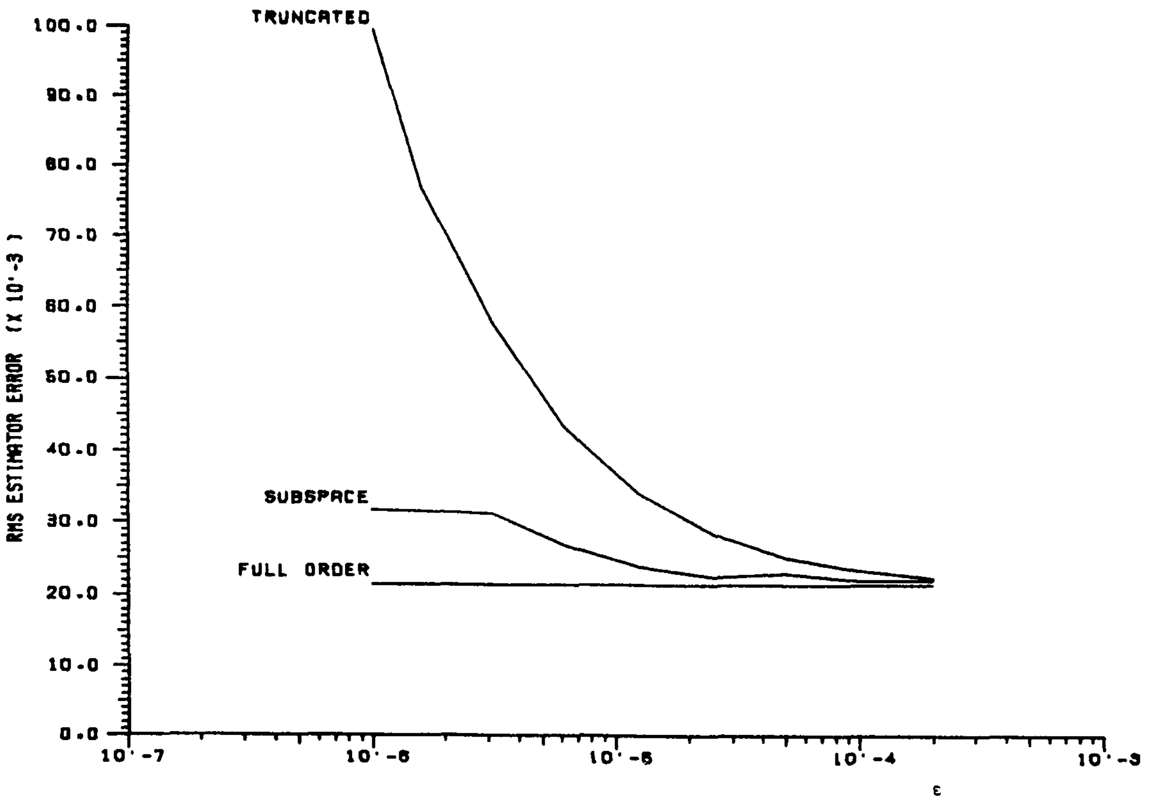

ESTIMATOR PERFORMANCE COMPARISON

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