Optimal Reduced-Order Observer-Estimators

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Abstract

This paper presents a unified approach to designing reduced-order observer-estimators. Specifically, we seek to design a reduced-order estimator satisfying an observation constraint which involves a pre-specified, possibly unstable, subspace of the system dynamics and which also yields reduced-order estimates of the remaining subspace. The results are obtained by merging the optimal projection approach to reduced-order estimation of Bernstein and Hyland with the subspace-observer results of Bernstein and Haddad. A salient feature of this theory is the treatment of unstable dynamics within reduced-order state-estimation theory. In contrast to the standard full-order estimation problem involving a single algebraic Riccati equation, the solution to the reduced-order observer-estimator problem involves an algebraic system of four equations consisting of one modified Riccati equation and three modified Lyapunov equations coupled by two distinct oblique projections.

I. Introduction

As is well known, Kalman filter theory addresses the state-estimation problem in guidance and navigation applications by minimizing a least-squares state-estimation error criterion. However, implementation of the standard Kalman filter is often impractical since it is generally of the same order as the system model. Consequently, designers must often implement reduced-order filters to satisfy real-time processing constraints as well as constraints on filter complexity. A further motivation is the fact that although a system model may have many degrees of freedom (such as coloring filter states and vibrational modes), it is often the case that estimates of only a small number of state variables (e.g., rigid body position and rotational modes) are actually required. The literature on reduced-order estimator design is vast and we note a representative collection of papers\(^{1-23}\) as an indication of longstanding interest in this problem.

Another important issue in estimation theory is the problem of asymptotic observation. As is well-known\(^{23}\), the steady-state Kalman filter is also an asymptotic observer. However, in reduced-order estimation theory the operations of estimation and observation are distinct, i.e., a reduced-order estimator is not necessarily also an observer. In many practical applications, however, it is necessary to design a reduced-order estimator that also observes a specified portion of the system states. Thus, we seek to design reduced-order subspace observers which can asymptotically observe a specified subset of system states.

The contribution of the present paper is a unified approach to reduced-order observer-estimator design. Specifically, we consider a reduced-order estimation problem which also includes a subspace observation constraint. By merging the optimal projection approach to reduced-order state estimation developed by Bernstein and Hyland\(^{6}\) with the subspace-observer result of Bernstein and Haddad\(^{17}\), a reduced-order observer-estimator design theory is developed that includes optimal observation of a pre-specified subspace (e.g., rigid body modes and selected vibrational modes) as well as open reduced-order estimation of the remaining stable subspace (e.g., coloring filter states and remaining vibrational modes).

An additional feature of our approach is that the observed subspace need not be stable, i.e., it may include unstable (for example, neutrally stable) modes. In contrast with the full-order Kalman filter, reduced-order filters for unstable systems may diverge since they may fail to adequately track the unstable modes. The observer-estimator derived in this paper circumvents this problem by including all of the unstable modes within the observed subspace. We note that standard navigational models\(^{26}\) possess neutrally stable modes, while tracking systems typically model targets as having rigid body dynamics. Additional examples include large flexible space structures undergoing open-loop rotational and/or translational motion.

It is important to stress that our results are not intended to provide a basis for feedback control. As is well known, feedback controllers based upon reduced-order filters may exhibit poor performance including instability. The preferred approach is thus to design reduced-order controllers directly\(^{24,25}\).

The starting point for the present paper is the Riccati equation approach developed in Ref. 9. There it was shown that optimal reduced-order, steady-state estimators can be characterized by means of an algebraic system of equations consisting of one modified Riccati equation and two modified Lyapunov equations coupled by a projection matrix \(\Gamma\). Specifically, the order projection \(\Gamma\) is given by

\[
\Gamma = QP(QP)^*.
\]

where \((\cdot)^*\) denotes group (Draxin) generalised inverse and \(Q\) and \(P\) are rank-deficient nonnegative-definite matrices analogous to the controllability and observability Gramians of the estimator. As discussed in Ref. 10, the order projection \(\Gamma\) arises as a direct consequence of optimality and is not the result of an a priori assumption on the internal structure of the reduced-order estimator.

An important point discussed in Ref. 9 is that reduced-order estimators designed by means of either model reduction followed by "full-order" state estimation or full-order state estimation followed by estimator reduction will generally not be optimal for a given order. This point is illustrated by the fact that three matrix equa-
tions characterize the optimal reduced-order state estimator with intrinsic coupling between the "operations" of optimal estimator design and optimal estimator reduction.

The solution presented in Ref. 9, however, did not address the issue of observation of a pre-specified subspace. Consequently, the solution given in Ref. 9 was confined to problems in which the plant is asymptotically stable, while in practice it is often necessary to obtain estimators for plants with unstable modes. Intuitively, it is clear that finite, steady-state state-estimation error for unstable plants is only achievable when the estimator retains, or duplicates in some sense, the unstable modes. The solution given in Ref. 9 is inapplicable to unstable systems for the simple reason that the range of the order projection \( r \) may not fully encompass all of the unstable modes. A partial solution to this problem, given in Ref. 17, involves a new and completely distinct reduced-order solution in which the observation subspace of the estimator is constrained a priori to include all of the unstable modes as well as selected stable modes. Hence the estimator in Ref. 17 effectively serves as an optimal observer for a designated plant subspace.

The subspace observation constraint addressed in Ref. 17 was embedded within the optimization process by fixing the internal structure of the reduced-order estimator. This structure gave rise to a new subspace projection \( \mu \) defined by

\[
\mu = \begin{bmatrix} I_n \otimes P_{\mu} & P_{\mu} \otimes I_n \\ 0_n \otimes 0_n & 0_n \otimes 0_n \end{bmatrix},
\]

where \( P_{\mu} \in \mathbb{R}^{n \times n} \) and \( P_{\mu} \in \mathbb{R}^{n \times n} \) are subblocks of an \( n \times n \) nonnegative-definite matrix \( P \) satisfying a modified algebraic Riccati equation, \( n_d \) is the dimension of the observation subspace of the estimator containing all of the unstable modes and selected stable modes, and \( n_s \) is the dimension of the remaining subspace containing only stable modes. It turns out that the subspace projection \( \mu \), which is completely distinct from the order projection \( r \) defined by (1), plays a crucial role in characterizing the optimal observer gains. Furthermore, it was shown in Ref. 17 that the constrained subspace observer is characterized by one modified Riccati equation and one modified Lyapunov equation coupled by the subspace projection \( \mu \). This subspace observer however, was confined to an \( n_{\mu} \)-dimensional subspace with no estimation of the remaining \( n_s \)-dimensional subspace.

The purpose of the present paper is to combine the results of Refs. 9 and 17 in order to obtain a general solution to the Reduced-Order Observer-Estimator Problem. Specifically, we seek a reduced-order observer-estimator of order \( n_{\mu} \) satisfying \( n_{\mu} \leq n_s \leq n \), where \( n \) is the dimension of the plant, which includes observation of all of a pre-specified \( n_{\mu} \)-dimensional subspace of the system as well as all of a pre-specified \( n_s \)-dimensional subspace of the system as well as optimal \( n_{\mu} \)-reduced-order estimation of the \( n_s \) states in the residual subspace where \( n_s = n - n_{\mu} \). As shown in Theorem 1, this general solution to the Reduced-Order Observer-Estimator Problem is characterized by four matrix equations including one modified Riccati equation and one modified Lyapunov equation coupled by both the order projection \( r \) and the subspace projection \( \mu \).

Finally, this paper can be extended to several directions. These include the treatment of parameter uncertainties12,16, extensions to nonstrictly proper estimators and singular noise intensity13,21, worst-case frequency-domain design aspects, i.e., an \( H_2 \) constraint on the estimation error19,22, and extensions to the discrete-time setting10,17.

The contents of the paper are as follows. In Section II, we present the Reduced-Order Observer-Estimator Problem. In Section III, we derive the necessary conditions for optimality which characterize solutions to the Reduced-Order Observer-Estimator Problem. To draw connections with the existing literature we specialize Theorem 1 in Section IV to obtain the results of Refs. 9 and 17. We also specialize the results of Theorem 1 to obtain the full-order Kalman filter theory and show that the four matrix equations collapse to the standard observer Riccati equation. To illustrate these results we describe a numerical algorithm in Section V for solving the design equations and apply the algorithm to illustrative numerical examples.

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
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<tbody>
<tr>
<td>( \mathbb{R} )</td>
<td>real numbers, ( r \times s ) real matrices, ( \mathbb{R}^{n \times n} ) expected value</td>
</tr>
<tr>
<td>( L_r(t), s )</td>
<td>identity matrix, transpose, ( r \times s ) zero matrix, ( 0_{r \times s} )</td>
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<tr>
<td>( \text{tr} )</td>
<td>trace</td>
</tr>
<tr>
<td>( \mathcal{N}(Z), \mathcal{E}(Z) )</td>
<td>null space, range of matrix ( Z )</td>
</tr>
<tr>
<td>( p, q, n )</td>
<td>positive integers; ( n \leq n_s \leq n )</td>
</tr>
<tr>
<td>( E )</td>
<td>( n \times \nu, \nu \times q, q \times n ) matrices</td>
</tr>
<tr>
<td>( A, C, L )</td>
<td>( n \times n, \nu \times \nu, \nu \times q ) matrices</td>
</tr>
<tr>
<td>( A_{\text{nl}}, A_{\text{ns}}, A_{\text{nu}}, A_{\text{nu},} )</td>
<td>( n_u \times n_u, n_u \times n_x, n_x \times n_x ) matrices</td>
</tr>
<tr>
<td>( C_{\nu}, C_{\nu})</td>
<td>( \nu \times \nu, q \times q ) matrices</td>
</tr>
<tr>
<td>( R )</td>
<td>( q \times q ) positive-definite matrix</td>
</tr>
<tr>
<td>( v_\nu,v_\nu )</td>
<td>asymptotically stable matrix with eigenvalues in open left half plane</td>
</tr>
<tr>
<td>( A_{\nu}, B_{\nu}, C_{\nu} )</td>
<td>( n_x \times n_x, n_x \times \nu, q \times n_x ) matrices</td>
</tr>
<tr>
<td>( A_{\text{nu},}, A_{\text{nu},} )</td>
<td>( n_x \times n_x, \nu \times \nu, q \times q ) matrices</td>
</tr>
<tr>
<td>( B_{\text{nu},}, B_{\text{nu},} )</td>
<td>( \nu \times \nu, q \times q ) matrices</td>
</tr>
<tr>
<td>( v_\nu(t), v_\nu(t) )</td>
<td>( n \times \nu ) dimensional white noise process with positive-definite intensity ( V_1 )</td>
</tr>
<tr>
<td>( v_\nu(t), v_\nu(t) )</td>
<td>( \nu \times \nu ) dimensional white noise process with positive-definite intensity ( V_2 )</td>
</tr>
<tr>
<td>( V_{12} )</td>
<td>( n \times \nu ) cross intensity of ( v_\nu(t), v_\nu(t) )</td>
</tr>
<tr>
<td>( F, F, H )</td>
<td>( [I_n, 0_n \times n_\nu], [I_n, 0_n \times n_\nu, I_n], [0_n \times n_\nu, I_n] )</td>
</tr>
<tr>
<td>( \tilde{A} )</td>
<td>[ A - F^T B_{\nu} C - F^T A_{\text{nu}} ]</td>
</tr>
<tr>
<td>( \tilde{L} )</td>
<td>[ L - C_{\nu} ]</td>
</tr>
<tr>
<td>( \tilde{R} )</td>
<td>[ F^T R L ]</td>
</tr>
<tr>
<td>( \tilde{a}(t) )</td>
<td>[ v_\nu(t) - F^T B_{\nu} v_\nu(t) ]</td>
</tr>
<tr>
<td>( \tilde{y} )</td>
<td>[ V_1 - V_{12} B_{\nu} F - F^T B_{\nu} V_{12} + F^T B_{\nu} V_{12} F ]</td>
</tr>
<tr>
<td>( \tilde{V}_{12} )</td>
<td>[ B_{\nu} V_{12} - V_{12} B_{\nu}^T F ]</td>
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## II. The Reduced-Order Observer-Estimator Problem

The following problem is addressed.

**Reduced-Order Observer-Estimator Problem**

For the \( n \)-th order system

\[
\dot{z}(t) = Az(t) + v_\nu(t), \quad t \in [0, \infty),
\]

with noisy measurements

\[
y(t) = Cz(t) + v_\nu(t),
\]

design an \( n_{\mu} \)-th order reduced-order strictly proper observer-estimator

\[
\dot{z}_{\mu}(t) = A_z z_{\mu}(t) + B_z y(t),
\]
\( \gamma(t) = C \dot{x}(t) \)

that satisfies the following design criteria:

(i) the observer-estimator (5), (6) is a steady-state asymptotic observer for a specified \( n_e \)-dimensional subspace of the plant (3) where \( n_e \leq n \leq n_s \); and

(ii) the observer-estimator is an optimal estimator which minimizes the least-squares state-estimation error criterion

\[
J(A_s, B_s, C_s) = \lim_{t \to \infty} \text{tr}(LL^T(t) - \gamma(t))^T\text{tr}(\gamma(t))
\]

To make condition (i) more precise, partition (3), (4) according to

\[
x(t) = \begin{bmatrix} x_e(t) \\ x_s(t) \end{bmatrix} \in \mathbb{R}^{n_e} + \mathbb{R}^{n_s}, \quad x_e(t) \in \mathbb{R}^{n_e}, \quad x_s(t) \in \mathbb{R}^{n_s},
\]

where

\[
\begin{bmatrix} \dot{x}_e(t) \\ \dot{x}_s(t) \end{bmatrix} = \begin{bmatrix} A_{se} & A_{se} \\ A_{se} & A \end{bmatrix} \begin{bmatrix} x_e(t) \\ x_s(t) \end{bmatrix} + \begin{bmatrix} w_{ue}(t) \\ w_{ux}(t) \end{bmatrix},
\]

\[
y(t) = [C_e C_s] \begin{bmatrix} x_e(t) \\ x_s(t) \end{bmatrix} + \omega_u(t)
\]

and (5), (6) as

\[
x_e(t) = \begin{bmatrix} x_{e1}(t) \\ x_{e2}(t) \end{bmatrix} \in \mathbb{R}^{n_e}, \quad x_{s1}(t) \in \mathbb{R}^{n_s}, \quad n = n_e + n_s,
\]

\[
\begin{bmatrix} \dot{x}_{e1}(t) \\ \dot{x}_{e2}(t) \end{bmatrix} = \begin{bmatrix} A_{se1} & A_{se1} \\ A_{se2} & A \end{bmatrix} \begin{bmatrix} x_{e1}(t) \\ x_{e2}(t) \end{bmatrix} + \begin{bmatrix} B_{ue1}(t) \\ B_{ue2}(t) \end{bmatrix} y(t),
\]

\[
y_e(t) = [C_{e1} C_{e2}] x_{e1}(t).
\]

We note that the partitioned form of the matrix \( A \) appearing in (9) allows us to characterize the two subspaces corresponding to \( x_e(t) \) and \( x_s(t) \). The \( n_s \times n_e \) zero matrix in the (2,1)-block of \( A \) is needed in order to achieve asymptotic observation of \( x_e(t) \) independently of \( x_s(t) \). If necessary, the matrix \( A \) can be recast in the form (9) by utilizing a similarity transformation to a modal basis. Of course, the coupling matrix \( A_{se} \) may be either zero or nonzero.

Furthermore, in (8)-(13) we implicitly assume that \( 0 < n_e < n_s \). The special cases \( n_s = 0 \) and \( n_s = n_e \) will be discussed later in this section and in Section IV. The observation condition (i) is captured by imposing the additional constraint

\[
\lim_{t \to \infty} [x_e(t) - x_s(t)] = 0,
\]

for all \( x(0) \) and \( x_s(0) \) when \( w_{ux}(t) \equiv 0 \) and \( w_u(t) \equiv 0 \). The requirement (14) implies that zero asymptotic observation error for a specified \( n_e \)-dimensional subspace is achieved under zero external disturbances and arbitrary initial conditions.

To require that the observer-estimator is also an optimal reduced-order estimator, the matrix \( L \) identifies the states or linear combinations of states whose estimates are desired. In accordance with the partitioning given in (8), \( L \) is partitioned as

\[
L \triangleq [L_e \quad L_s].
\]

Thus, the goal of the Reduced-Order Observer-Estimator Problem is to design a reduced-order observer-estimator of order \( n_e \) which observes a specified plant subspace and provides optimal estimates of specified linear combinations of plant states. Since the observer-estimator (5), (6) serves as a reduced-order observer for an \( n_e \)-dimensional subspace of the plant (3), its order \( n_e \) must satisfy \( n_e \leq n_s \leq n \).
\[ A_{\text{err}} \triangleq A_n - B_nC_n, \quad (21) \]
\[ A_{\text{err}} \triangleq -B_nC_n, \quad (22) \]

With (21) and (22), \( A_n \) becomes
\[ A_n = \begin{bmatrix} A_n - B_nC_n & A_{\text{err}} \\ -B_nC_n & A_n \end{bmatrix}, \quad (23) \]

Now the error states \( z_e(t) \) satisfy
\[ \dot{z}_e = A_{\text{err}}z_e(t) + (A_n - B_nC_n)z_x(t) - A_{\text{err}}z_x(t) \]
\[ + wz(t) - B_nw_2(t), \quad (24) \]

where \( A_{\text{err}} \) is given by (21).

Next, note that the least-squares state-estimation error criterion (7) is given by
\[ J(A_n, B_n, C_n) = \lim_{t \to \infty} \mathbb{E}[L_n z_e(t) + L_s z_s(t)] \]
\[ + (L_n - C_nz_x(t) - C_nz_s(t))^T R[L_n z_e(t)] (25) \]
\[ + L_s z_s(t) + (L_n - C_nz_x(t) - C_nz_s(t)]. \]

Now, to eliminate the explicit dependence of the estimation error (25) on \( z_x(t) \) in favor of \( z_e(t) \), we constrain
\[ C_{\text{err}} \triangleq L_n. \quad (26) \]

The constraints (21), (22), and (26) on the reduced-order observer-estimator gains \( A_n, A_{\text{err}}, \) and \( C_{\text{err}} \) are thus imposed in order for the reduced-order observer-estimator to asymptotically observe \( z_x(t) \) subspace of the plant (9). Note that constraints (21) and (22) are consistent with the full-order Kalman filter result (16) in which \( A_n \) and \( A_{\text{err}} \) are given by the constraints (21) and (22).

Next, using constraints (21) and (22) to eliminate the dependence on \( z_x(t) \), it follows that the augmented system (20) has the form
\[ \dot{\tilde{z}}(t) = \tilde{A}\tilde{z}(t) + \tilde{u}(t), \quad t \geq 0, \quad (27) \]

where
\[ \tilde{z}(t) \triangleq \begin{bmatrix} z_e(t) \\ z_x(t) \end{bmatrix} \in \mathbb{R}^{n+n_s}, \]
\[ \tilde{u}(t) \triangleq \begin{bmatrix} w_1(t) - B_nw_2(t) \\ w_2(t) \end{bmatrix}, \quad (28) \]

and
\[ \tilde{A} \triangleq \begin{bmatrix} A_n - B_nC_n & A_{\text{err}} - B_nC_n & -A_{\text{err}} \\ 0_n & A_n & 0_n \\ B_nC_n & B_nC_n & A_n \end{bmatrix} \quad (29) \]

We now show that the stability of \( \tilde{A} \) is equivalent to the stability of \( A_n \).

**Lemma 1.** \( \tilde{A} \) is asymptotically stable if and only if \( A_n \) is asymptotically stable. In this case, \( \lim_{t \to \infty} z_e(t) = 0 \) for \( w_1(t) \equiv 0, w_2(t) \equiv 0, \) and for all initial conditions \( z(0), z_x(0) \). Furthermore, the state-estimation error criterion (7) is given by
\[ J(A_n, B_n, C_n) = \text{tr} \Phi \tilde{R}, \quad (30) \]

where the steady-state covariance
\[ \Phi \triangleq \lim_{t \to \infty} \mathbb{E}[\tilde{z}(t)\tilde{z}^T(t)] \quad (31) \]

exists and satisfies the algebraic Lyapunov equation
\[ 0 = \tilde{\Phi} + \tilde{\Phi}^T + \tilde{V}. \quad (32) \]

**Proof.** To show that \( \tilde{A} \) is asymptotically stable consider the transformation
\[ T \in \mathbb{R}^{(n+n_s) \times (n+n_s)} \]

given by
\[ T \triangleq \begin{bmatrix} 0 & I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & -I_{n_s} & 0 \\ 0 & 0 & 0 & -I_{n_s} \end{bmatrix} \quad (33) \]

and define
\[ \tilde{z}_0(t) \triangleq T\tilde{z}(t) = \begin{bmatrix} z_e(t) \\ z_x(t) \\ -z_e(t) \\ -z_x(t) \end{bmatrix}. \quad (34) \]

Using (34) it follows from (27) that
\[ \dot{\tilde{z}}_0(t) = \tilde{A}_0\tilde{z}_0(t) + \tilde{u}_0(t), \]

where
\[ \tilde{A}_0 \triangleq \tilde{T}\tilde{A}\tilde{T}^{-1} = \begin{bmatrix} A_n & A_{\text{err}} & 0_n \times_n \n_s & A_n \end{bmatrix} \quad (35) \]

and
\[ \tilde{u}_0(t) \triangleq \tilde{T}\tilde{u}(t). \quad (36) \]

Since \( \tilde{A}_0 \) is asymptotically stable it follows that \( \tilde{A} \) is asymptotically stable if and only if \( A_n \) is asymptotically stable. In this case, \( \tilde{z}(t) \to 0 \) and hence \( z_x(t) \to 0 \) for arbitrary initial conditions when \( w_1(t) \) and \( w_2(t) \) are zero.

Finally, to guarantee that \( J(A_n, B_n, C_n) \) is finite and to satisfy the observation constraint (14), we define the set of asymptotically stable reduced-order observer-estimators
\[ S \triangleq \{(A_n, B_n, C_n) : A_n \text{ is asymptotically stable and } A_{\text{err}}, C_{\text{err}} \text{ are given by (21), (22), and (26).} \} \quad (38) \]

**III. Necessary Conditions for the Reduced-Order Observer-Estimator Problem**

In this section we obtain necessary conditions which characterize solutions to the Reduced-Order Observer-Estimator Problem. Derivation of these necessary conditions requires additional technical assumptions. Specifically, we further restrict \( (A_n, B_n, C_n) \) to the set
\[ S^+ \triangleq \{(A_n, B_n, C_n) : (A_n, B_n) \text{ is controllable and } (A_n, C_n) \text{ is observable.}\} \quad (39) \]

As can be seen from the Appendix, the set \( S^+ \) constitutes nondegeneracy conditions under which explicit gain expressions can be obtained for the Reduced-Order Observer-Estimator Problem. In order to state the main result we require some additional notation and a lemma concerning a pair of nonnegative-definite matrices.

**Lemma 2.** Suppose \( \tilde{Q}, \tilde{P} \) are \( n \times n \) nonnegative-definite matrices and rank \( \tilde{Q}\tilde{P} = n_s. \) Then there exist \( n_s \times n \) matrices \( C, \Gamma \) and an \( n_s \times n_s \) invertible matrix \( M, \) unique except for a change of basis in \( \mathbb{R}^{n_s}, \) such that the product \( \tilde{Q}\tilde{P} \) can be factored according to
\[ \tilde{Q}\tilde{P} = M^T\Gamma M; \quad (40) \]
\[ \tilde{Q}\tilde{P} = \Gamma \tilde{Q}\tilde{P} = I_{n_s}. \quad (41) \]

Furthermore, the \( n \times n \) matrices
\[ \tau \triangleq \tilde{Q}\tilde{P}; \quad (42) \]

are idempotent and have rank \( n_s \) and \( n - n_s, \) respectively.

**Proof.** See Ref. 9. \( \Box \)

As shown in Ref. 9, \( \tilde{Q}\tilde{P} \) has a group (Drasin) generalized inverse \( \tilde{Q}^+, \tilde{P}^+ = G^T M^{-1} \Gamma. \) Using (40) it follows that the matrix \( \tau \) is given by (1) since
\[ \tau = \tilde{Q}\tilde{P} = \tilde{Q}^+ \tilde{P}^+ = \tilde{Q}\tilde{P}. \quad (42) \]
Note that because of (40), $r^2 = G^T G C T = G C T = \tau$, i.e., $\tau$ is idempotent.

The following main result gives necessary conditions which characterize solutions to the Reduced-Order Observer-Estimator Problem. For convenience in stating this result define

$$ Q_\tau \triangleq Q C^T + V_{12} $$

for arbitrary $Q \in \mathbb{R}^{n \times n}$.

**Theorem 1.** Suppose $(A_\tau, B_\tau, C_\tau) \in S^+$ solves the Reduced-Order Observer-Estimator Problem. Then there exist $n \times n$ nonnegative-definite matrices $Q, P, \tilde{P}$ and an $n \times n$ nonnegative-definite matrix $Q_r$ such that $A_\tau, B_\tau$, and $C_\tau$ are given by

$$ A_\tau = \begin{bmatrix} \Phi & \mu \mu_1 \end{bmatrix} \left[ (A - Q_r V_2^{-1} C) \begin{bmatrix} F \end{bmatrix}^T \right], \quad (44) $$

$$ B_\tau = \begin{bmatrix} \Phi & \mu \mu_1 \end{bmatrix} Q_r V_2^{-1}, \quad (45) $$

$$ C_\tau = L \begin{bmatrix} F \end{bmatrix}^T G, \quad (46) $$

and such that $Q, P, Q_r$, and $\tilde{P}$ satisfy

$$ 0 = AQ + QA^T + V_1 - Q_r V_2^{-1} Q_r^T $$

$$ + r_1 \mu_1 Q_r V_2^{-1} Q_r^T \mu_1^T, \quad (47) $$

$$ 0 = (A - \mu_1 A T - \mu_1 Q_r V_2^{-1} C T) \tilde{P} $$

$$ + P(A - \mu_1 A T - \mu_1 Q_r V_2^{-1} C T) $$

$$ + \tau_1 L^T R L \tau_1, \quad (48) $$

$$ 0 = A_\tau Q_r + Q_r A_\tau^T + H(Q_r V_2^{-1} Q_r^T) $$

$$ - r_1 \mu_1 Q_r V_2^{-1} \mu_1^T H, \quad (49) $$

$$ 0 = (A - Q_r V_2^{-1} C) \tilde{P} + P(A - Q_r V_2^{-1} C) $$

$$ + \tau_1 L^T R L \tau_1 + [\mu(A - Q_r V_2^{-1} C) R] \tilde{P} $$

$$ + P[\mu(A - Q_r V_2^{-1} C) R], \quad (50) $$

$$ \text{rank } \tilde{Q} = \text{rank } \tilde{P} = \text{rank } \tilde{Q} \tilde{P} = n, \quad (51) $$

where

$$ P = \begin{bmatrix} P_u & P_{ru} \\ P_{ur}^T & P_r \end{bmatrix} \in \mathbb{R}^{(n+\nu) \times (n+\nu)}, $$

$$ P_u > 0, \quad (52) $$

$$ F \triangleq [I_n \quad 0_{n \times n}], \quad \Phi \triangleq [I_n \quad \mu \mu_1 P_u^T], \quad (53) $$

$$ \mu \triangleq F^T \Phi, \quad (54) $$

$$ \mu \triangleq F^T \Phi, \quad (55) $$

$$ Q \triangleq \mu \mu_1 \begin{bmatrix} I_n & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} \end{bmatrix} \mu_1^T. \quad (56) $$

Furthermore, the minimal value of the least-squares state-estimation error criterion (7) is given by

$$ J(A_\tau, B_\tau, C_\tau) = \text{tr } Q L^T R L. \quad (57) $$

Next, we present a partial converse of the necessary conditions which guarantees that the observation constraint (14) is enforced.

**Theorem 2.** Suppose there exist $n \times n$ nonnegative-definite matrices $Q, P, \tilde{P}$ and an $n \times n$ nonnegative-definite matrix $Q_r$ satisfying (47)-(56). Then, with $\tilde{Q}$ given by (56), the matrix

$$ \tilde{Q} = \begin{bmatrix} Q + \tilde{Q} & \tilde{Q} \tilde{P}^T \\ \tilde{P} Q & \tilde{P} \tilde{P}^T \end{bmatrix} $$

satisfies (32) with $(A_\tau, B_\tau, C_\tau)$ given by (44)-(46). Furthermore, $(\hat{A}, \hat{V}, \hat{F})$ is stabilisable if and only if $A_\tau$ is asymptotically stable. In this case, $(A_\tau, C_\tau)$ is controllable, $(A_\tau, C_\tau)$ is observable, the observation constraint (14) holds for all arbitrary initial conditions $z(0), x_r(0)$ when $w_1(t) \equiv 0, w_2(t) \equiv 0$, and the least-squares state-estimation error criterion is given by (57).

The proofs of Theorems 1 and 2 are given in the Appendix.

Theorem 1 presents necessary conditions for the Reduced-Order Observer-Estimator Problem. These necessary conditions consist of a system of one modified Riccati equation and three modified Lyapunov equations coupled by two distinct oblique (not necessarily orthogonal) projections $\tau$ and $\mu$. Note that $\tau$ and $\mu$ are idempotent since $r^2 = \tau$ and $\mu^2 = \mu$. As discussed earlier, the fixed-order constraint on the estimator order gives rise to the order projection $r$, while the observation constraint (14) gives rise to the subspace projection $\mu$. It is easy to see that $\text{rank } \mu = n$ and it can be shown using Sylvester's inequality and (40) that $\text{rank } \tau = n_r$.

**Remark 1.** Note that with $B_\tau$ given by (45), the expressions (44) and (46) for $A_\tau, \mu \mu_1$, and $C_\tau$ are equivalent to the constraints (21), (22), and (26).

**Remark 2.** By defining the $n \times n$ matrices

$$ \tilde{Q} \triangleq \begin{bmatrix} F \\ \mu \mu_1 \end{bmatrix}, \quad \tilde{P} \triangleq \begin{bmatrix} \Phi \end{bmatrix}, \quad (59) $$

it can be shown that

$$ \tilde{F} \tilde{Q}^T = \begin{bmatrix} I_n & 0_{n \times n_r} \\ 0_{n \times n_r} & 0_{n \times n_r} \end{bmatrix} = I_r. \quad (60) $$

Using (60) one can thus define a third composite projection

$$ \tau \triangleq \tilde{Q}^T \tilde{P} = \mu + r_\mu = \mu + \tau - \tau_\mu, \quad (61) $$

where $\text{rank } \tau = n_r$. Using (59), the gains (44)-(46) can be written as

$$ A_\tau = \hat{F}(A - Q_r V_2^{-1} C) \hat{G}^T = \hat{F} A \hat{G}^T - B_r C \hat{G}^T, \quad (62) $$

$$ B_\tau = \hat{F} Q_r V_2^{-1}, \quad (63) $$

$$ C_\tau = L \hat{G}^T. \quad (64) $$

**Remark 3.** It follows from (42) and (56) that

$$ \mu \tau = \mu Q \tilde{P}(Q \tilde{P})^* \mu \mu_1 \begin{bmatrix} 0_n & 0_{n \times n_r} \\ 0_{n \times n_r} & Q_r \end{bmatrix} \mu_1^T \tilde{P}(Q \tilde{P})^*. \quad (65) $$

Since $\mu \mu_1 = 0$, we obtain

$$ 0 = \mu \tau \quad (66) $$

as a consequence of optimality. Partitioning

$$ \tau = \begin{bmatrix} r_u \quad r_{ru} \\ r_u \quad r_{ru} \end{bmatrix} \in \mathbb{R}^{(n+\nu) \times (n+\nu)}, $$

(67) implies

$$ r_u = -P_u^T P_{ru} r_{ru}, \quad r_{ru} = -P_{ru}^T P_{ru} r_{ru}. \quad (68) $$

**Remark 4.** Note that for $(A_\tau, B_\tau, C_\tau)$ given by (44)-(46), the observer-estimator (5) or, equivalently (12), assumes the innovations form

$$ \tilde{x}(t) = \hat{F} A \hat{G}^T \tilde{x}(t) + \hat{F} Q_r V_2^{-1} [v(t) - C \hat{G}^T \tilde{x}(t)]. \quad (69) $$

**Remark 5.** By introducing the quasi-full-state estimate $\hat{z}(t) \triangleq \hat{Q}^T \tilde{x}(t) \in \mathbb{R}^{n_r}$ so that $\hat{z}(t) = \tilde{z}(t)$ and $x_r(t) = \hat{z}(t) \in \mathbb{R}^{n_r}$, (69) can be written as

$$ \dot{\hat{z}}(t) = \tau \hat{A} \hat{z}(t) + \hat{F} Q_r V_2^{-1} [v(t) - C \hat{z}(t)], \quad (70) $$

or, equivalently,

$$ \dot{\hat{z}}(t) = (\mu + r_\mu) A(\mu + r_\mu) \hat{z}(t) $$

$$ + (\mu + r_\mu) Q_r V_2^{-1} [v(t) - C \hat{z}(t)]. \quad (71) $$
Note that although the implemented observer-estimator (69) has the reduced-order state $x_\mu(t) \in \mathbb{R}^{n\mu}$, (71) can be viewed as a quasi-full-order observer-estimator whose geometric structure is dictated by the projections $\tau$ and $\mu$. Specifically, error inputs $Q_\nu V_\nu^{-1}g(t) - C(t)$ are annihilated unless they are contained in $\mathcal{K}(\mu + r\mu)^T$. Hence, the observation subspace of the observer-estimator is precisely $\mathcal{K}(\mu + r\mu)^T$.

**Remark 6.** In the full-order Kalman filter case it is well known that an orthogonality condition

$$\mathbf{E}\{[x(t) - x(t)]x(t)^T\} = 0$$

is satisfied. For the observer-estimator problem an analogous condition is

$$\mathbf{E}\{[p(t) - p(t)]p(t)^T\} = 0.$$  

(73)

This condition does not hold automatically, however, but must be imposed as an additional side constraint. It can be shown that requiring (73) leads to

$$0 = F\mathbf{G}^T$$

and, consequently,

$$0 = F\mathbf{r}, \quad 0 = \mu \mathbf{r}.$$  

(75)

Using (75), it follows that $\tau$ has the structure

$$\tau = \begin{bmatrix} n_n & 0_{n_n \times n_n} \\ 0_{n_n} & \tau_o \end{bmatrix}$$

(76)

so that the composite projection $\tilde{T}$ has the form

$$\tilde{T} = \begin{bmatrix} I_{n_n} & P_{\mu}^{-1}P_{\nu} \\ 0_{n_n \times n_n} & \tau_o - \tau_o P_{\mu}^{-1}P_{\nu} \end{bmatrix}.$$  

(77)

**IV. Specializations of Theorem 1**

To draw connections with the previous literature, a series of specialisations of Theorem 1 is now given. Specifically, to recover the full-order steady-state Kalman filter from Theorem 1 take $n_n = n_o$ or, equivalently, $n_n = n$. Since $\mathbf{F}\mathbf{G}T = I_n$, let $S = \tilde{T} \in \mathbb{R}^{n \times n}$ and $S^{-1} = \tilde{T}^T \in \mathbb{R}^{n \times n}$. In this case the optimal gains (44)-(46) become

$$A_s = S(A - Q_\nu V_\nu^{-1}C)S^{-1},$$  

(78)

$$B_s = SQ_\nu V_\nu^{-1},$$  

(79)

$$C_s = LS^{-1}.$$  

(80)

Furthermore, in this case since

$$\tau_o \mu = I_n - \mu - r\mu = I_n - \tilde{T}^T \tilde{T} = I_n - S^{-1}S = 0,$$  

(81)

the modified Riccati equation (47) specializes to the standard observer Riccati equation

$$0 = AQ + QA^T + V_1 - Q_\nu V_\nu^{-1}Q_\nu^T$$

and (48)-(50) are superfluous. Note that (78)-(80) are precisely the standard steady-state Kalman filter gains in an alternative basis specified by the basis transformation $S$. Since $J(A_s, B_s, C_s) = J(SA, S^{-1}, SB, SC, S^{-1})$, however, this change of basis leaves the estimation error unchanged.

Next, to recover the optimal projection results of Ref. 9 involving reduced-order estimators for stable plants without a subspace observation constraint, let $n_o = 0, n_n = n, n_n = n_n$, $A_s = A$, and $n_n < n$, set $\mu = 0$ so that $\mu \mu = I_n$, and replace $[\Phi \mu]$ and $[\mathbf{F} \mathbf{G}^T]$ by $\Phi$ and $\mathbf{F}^T$, respectively. Then the optimal gains (44)-(46) become

$$A_s = \Gamma(A - Q_\nu V_\nu^{-1}C)\Gamma^T,$$

(83)

$$B_s = \Gamma Q_\nu V_\nu^{-1},$$

(84)

$$C_s = \Gamma L^T,$$  

(85)

and equations (47)-(50) specialize to

$$0 = AQ + QA^T + V_1 - Q_\nu V_\nu^{-1}Q_\nu^T$$

$$+ r_\nu Q_\nu V_\nu^{-1}Q_\nu^T,$$  

(86)

$$0 = AQ + QA^T + Q_\nu V_\nu^{-1}Q_\nu^T$$

$$- r_\nu Q_\nu V_\nu^{-1}Q_\nu^T,$$  

(87)

$$0 = (A - Q_\nu V_\nu^{-1}C)T + \hat{P}(A - Q_\nu V_\nu^{-1}C)$$

$$+ L^T RL - \Gamma L^T RL \Gamma,$$  

(88)

These are equations (2.10)-(2.12) of Ref. 9.

Finally, we can also recover the results of Ref. 17 where the reduced-order observer is constrained to observe an $n_n$-dimensional plant subspace without estimating the remaining $n_u$-dimensional subspace. In this case let $n_n = n_n$, $n_u = 0$, and $\tau = 0$ so that $\tau_o = I_n$. Furthermore, let $[\Phi \mu]$ and $[\mathbf{F} \mathbf{G}^T]$ be replaced by $\Phi$ and $\mathbf{F}^T$ respectively so that the gain expressions (44)-(46) become

$$A_s = \Phi(A - Q_\nu V_\nu^{-1}C)\Phi^T,$$  

(89)

$$B_s = \Phi Q_\nu V_\nu^{-1},$$  

(90)

$$C_s = L\Phi^T,$$  

(91)

and equations (47)-(50) specialize to

$$0 = AQ + QA^T + V_1 - Q_\nu V_\nu^{-1}Q_\nu^T$$

$$+ X_\nu Q_\nu V_\nu^{-1}Q_\nu^T,$$  

(92)

$$0 = (A - Q_\nu V_\nu^{-1}C)T + X(A - Q_\nu V_\nu^{-1}C) + L^T RL,$$  

(93)

These are equations (2.17) and (2.18) of Ref. 17.

**V. Numerical Algorithm and Illustrative Numerical Examples**

In this section we present a numerical algorithm for solving the optimality conditions for the Reduced-Order Observer-Estimator Problem and consider two illustrative numerical examples.

**Algorithm 1.** To solve (47)-(50), carry out the following steps:

- **Step 1.** Initialize $k = 0$, $P(k) = I_n$, $r(0) = I_n$;
- **Step 2.** With $\mu = \mu(0)$ and $\tau = r(0)$, solve (47) for $Q(k) = Q$;
- **Step 3.** With $Q = Q(k), \mu = \mu(k)$, and $\tau = r(k)$, solve (48) and (49) for $P(k) = P$ and $Q(k) = Q$;
- **Step 4.** With $Q = Q(k), P = P(k), \mu = \mu(k)$, and $\tau = r(k)$, solve (50) for $P(k) = P(k)$;
- **Step 5.** If convergence of $Q(k)$ and $P(k)$ has been attained then evaluate $A_s, B_s, C_s$ using (44)-(46) and stop; else continue;
- **Step 6.** Use $P = P(k)$, $Q = Q(k)$, and $P = P(k)$ to define $\mu(k+1) = \mu$ and $r(k+1) = r$ using (39)-(41), (55), (56);
- **Step 7.** Replace $k$ by $k + 1$ and go to Step 1.

The above algorithm is a straightforward iterative scheme which is fairly easy to implement. More sophisticated algorithms can be developed by utilizing homotopic continuation techniques. For the examples discussed below, however, Algorithm 1 proved to be adequate.

Our first example, adopted from Ref. 28, pp. 99-101, involves a satellite in circular orbit. The linearized error equations representing the deviation from a perfect circular orbit are given by
where $r, \theta, \phi$ are spherical coordinates, $r_0$ is the orbit radius, $\omega$ denotes orbital frequency, and $\varepsilon > 0$.

Here the state vector represents the deviation from a circular equatorial orbit and is expressed in spherical coordinates. We note that $\varepsilon = 0$ was assumed in Ref. 28, although $\varepsilon > 0$ is assumed here to reflect dissipation in this coordinate due possibly to on-board forces. Furthermore, stochastic disturbance models are utilized here in place of deterministic inputs appearing in Ref. 28.

To treat this problem within our formulation, we note that the upper left $4 \times 4$ block of (94) has a nearly stable eigenvalues 0, $\omega$, and $-\omega$. Hence we set $n_a = 4$ and $n_r = 2$ and seek to design an optimal 4th-order observer for the rigid body mode only. The performance of this suboptimal estimator was $J = 78.74$. In contrast, the optimal 2nd-order subspace observer constrained to observe only the rigid body mode had performance $J = 2.328$.

### Appendix: Proofs of Theorem 1 and Theorem 2

To optimise (30) over the open set $S^+$ subject to the constraint (32), form the Lagrangian

$$L(A_{xx}, A_{xx}, B, C, \bar{Q}, \bar{P}, \lambda) = \text{tr}(\lambda \bar{A} \bar{A}^T + [\bar{A} \bar{Q}] + \bar{Q} \bar{A}^T + \bar{V} | \bar{P}),$$

where the Lagrange multipliers $\lambda \geq 0$ and $\bar{P} \in \mathbb{R}^{[n+n_a] \times [n+n_a]}$ are not both zero. We thus obtain

$$\frac{\partial L}{\partial \bar{Q}} = \bar{A} \bar{A}^T + \bar{P} \bar{A} + \lambda \bar{R}. \tag{102}$$

Setting $\frac{\partial L}{\partial \bar{Q}} = 0$ yields

$$0 = \bar{A} \bar{A}^T + \bar{P} \bar{A} + \lambda \bar{R}. \tag{103}$$

Since $\bar{A}$ is assumed to be stable, $\lambda = 0$ implies $\bar{P} = 0$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\bar{P}$ is nonnegative definite.

Now partition $(n + n_{xx}) \times (n + n_{xx})$ $\bar{Q}, \bar{P}$ into $n \times n$, $n \times n_{xx}$, and $n_{xx} \times n_{xx}$ subblocks as

$$\bar{Q} = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}, \quad \bar{P} = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_2 \end{bmatrix}. \tag{104}$$

Thus, with $\lambda = 1$ the stationarity conditions are given by

$$\frac{\partial L}{\partial \bar{Q}} = \bar{A} \bar{A}^T + \bar{P} \bar{A} + \bar{R} = 0, \tag{105}$$

Note that the dynamic model involves one rigid body mode and two flexible modes at frequencies 1 and 2 rad/sec with 5% damping ratios. The matrix $C$ captures the fact that the rigid body position measurement is corrupted by the flexible modes (i.e., observation spillover), the matrix $A$ expresses the desire to estimate the rigid body position, and the matrix $V_1$ was chosen to capture the type of noise correlation which arises when the dynamics are transformed into a modal basis.

For the full-order steady-state Kalman filter the optimal estimation error was $J = 1.533$. We then truncated the higher frequency flexible mode and obtained a suboptimal 4th-order observer as a "full-order" estimator for the truncated system. The performance of this suboptimal estimator evaluated for the 6th-order plant was $J = 3.537$. By applying Algorithm 1 an optimal 4th-order subspace observer was obtained. The performance of this optimal estimator was $J = 1.572$.

A second-order suboptimal filter was also obtained as a "full-order" estimator for a truncated plant consisting of the rigid body mode only. The performance of this suboptimal estimator was $J = 78.74$. In contrast, the optimal 2nd-order subspace observer constrained to observe only the rigid body mode had performance $J = 2.328$.
Expanding (32) and (105) yields

\[
\begin{align*}
\frac{\partial L}{\partial A_{ee}} &= P_{12}Q_{12} + P_{22}, \\
\frac{\partial L}{\partial A_{eu}} &= F(P_{12}Q_{12} + P_{12}Q_{2}), \\
\frac{\partial L}{\partial B_{eu}} &= FP_{12}B_{eu} - FP_{12}Q_{12} + FP_{12}B_{eu} = 0, \\
\frac{\partial L}{\partial B_{se}} &= P_{2}B_{se}V_{2} - P_{12}Q_{se} + P_{12}B_{se}B_{eu} = 0, \\
\frac{\partial L}{\partial C_{se}} &= -BR_{12}Q_{se} + RC_{se}Q_{2} = 0.
\end{align*}
\]

Expanding (32) and (105) yields

\[
\begin{align*}
0 &= A_{s}Q_{1} - F^{T}B_{eu}Q_{12} + F^{T}A_{eu}Q_{12} + Q_{1}A^{T}T
- Q_{1}C^{2}B_{en}^{T}F - Q_{12}A_{eu}F + V_{1} - V_{12}B_{en}^{T}F
- F^{T}B_{eu}V_{12}^{T}F + F^{T}B_{eu}V_{2}B_{en}^{T}F, \\
0 &= A_{s}Q_{2} - F^{T}B_{eu}Q_{12} - F^{T}A_{eu}Q_{12} + Q_{1}C^{2}B_{en}^{T}F
+ Q_{12}A_{eu}F + V_{12}B_{en}^{T}F - F^{T}B_{eu}V_{2}B_{en}^{T}F, \\
0 &= A_{s}Q_{2} + Q_{2}A^{T}T + B_{eu}Q_{12} + Q_{1}C^{2}B_{en}^{T}F
+ B_{eu}B_{en}^{T}F, \\
0 &= A_{s}P_{1} - C^{2}B_{en}^{T}F + C^{2}B_{en}^{T}P_{12} + P_{1}A
- P_{1}F^{T}B_{eu}C + P_{12}B_{eu}C + L^{T}B_{RL}, \\
0 &= A_{s}P_{2} - C^{2}B_{en}^{T}F + P_{12} + C^{2}B_{en}^{T}P_{12} - P_{1}F^{T}A_{eu}
+ P_{12}A_{eu} - L^{T}R^{T}C_{eu}, \\
0 &= A_{s}P_{2} + P_{2}A_{eu} - A_{eu}^{T}P_{2} + P_{12}F^{T}A_{eu}
+ C_{eu}^{T}RC_{eu},
\end{align*}
\]

Now define the \( n \times n \) matrices

\[
\begin{align*}
Q &= Q_{12}Q_{2} - Q_{12}Q_{2}^{T}, \\
P &= P_{12}P_{22} - P_{12}P_{2}^{T}, \\
\tau &= -Q_{12}P_{22}^{T} - P_{12}P_{2}^{T}.
\end{align*}
\]

and the \( n_{se} \times n, n_{se} \times n_{se}, \) and \( n_{se} \times n_{se} \) matrices

\[
G = Q_{22}^{T}, \\
M = Q_{22}, \\
G = -P_{12}P_{2}^{T}.
\]

Note that \( Q, P, \hat{Q}, \hat{P} \) are nonnegative definite and that \( FP^{T}F = P_{u} \). Next partition \( n \times n \) \( P, \hat{Q} \) into \( n_{u} \times n_{u}, n_{u} \times n_{u}, \) and \( n_{u} \times n_{se} \) subblocks as

\[
\begin{align*}
P &= \begin{bmatrix} P_{u} & P_{u} \\
\end{bmatrix}, \\
\hat{Q} &= \begin{bmatrix} \hat{Q}_{u} & \hat{Q}_{se} \\
\end{bmatrix}.
\end{align*}
\]

Since \( P_{u} \) is invertible (see Lemma 3) define the \( n_{u} \times n \) matrices

\[
F = [I_{n_{u}} \ 0_{n_{u} \times n_{se}}], \\
\Phi = [I_{n_{u}} \ P_{u}^{-1}P_{u}]
\]

and \( n \times n \) matrix

\[
\mu = FT\Phi.
\]

Next note that with the above definitions \( 122 \) is equivalent to \( 40 \) and that \( 30 \) holds. Hence \( \tau = G^{T}T \) is idempotent, i.e., \( \tau^{2} = \tau \). Similarly, since \( \Phi^{T}T = I_{n_{u}} \), \( \mu \) is also idempotent.

It is helpful to note the identities

\[
\begin{align*}
\hat{Q} &= Q_{12}G = G^{T}Q_{12}, \\
\hat{P} &= -P_{12}F = -T^{T}P_{22} = T^{T}P_{2}, \\
\tau G &= \Gamma, \\
\tau &= \hat{Q}, \\
\hat{P} &= \hat{P}.
\end{align*}
\]

\[
\hat{Q} = -Q_{12}^{T}P_{22},
\]

Using \( 122 \) and Sylvester's inequality, it follows that

\[
\begin{align*}
\text{rank} \ G &= \text{rank} \ F = \text{rank} \ Q_{12} = \text{rank} \ P_{u} = n_{se}.
\end{align*}
\]

Now using \( 131 \) and Sylvester's inequality yields

\[
\begin{align*}
n_{se} &= \text{rank} \ Q_{12} \\
+ \text{rank} \ G - n_{se} &\leq \text{rank} \ \hat{Q} \leq \text{rank} \ Q_{12} = n_{se},
\end{align*}
\]

which implies that \( \text{rank} \ \hat{Q} = n_{se} \). Similarly, \( \text{rank} \ \hat{P} = n_{se} \), and \( \text{rank} \ \hat{Q} \hat{P} = n_{se} \) follows from \( 134 \). Now using \( 134 \) and the above identities, it follows from \( 123 \) that

\[
\begin{align*}
0 &= FP_{u}\hat{Q},
\end{align*}
\]

Using the partitioned form \( 128 \) of \( P \) and \( \hat{Q} \), \( 137 \) implies

\[
\hat{Q} = \mu I_{n_{u}}
\]

The components of \( \hat{Q} \) and \( \hat{P} \) can be written in terms of \( Q, P, \phi, \hat{Q}, \hat{P}, G, \) and \( \Gamma \) as

\[
\begin{align*}
Q_{1} &= Q + \hat{Q}, \\
P_{1} = P + \hat{P}, \\
Q_{12} &= \hat{Q} \Gamma, \\
P_{2} = -P_{u} \Gamma, \\
Q_{2} &= \Gamma \hat{Q} \Gamma, \\
P_{2} = \hat{P} \Gamma \Gamma.
\end{align*}
\]

Furthermore, it is useful to note that

\[
\begin{align*}
F^{T} = F, \\
0 = \phi \Gamma, \\
F^{T} = \mu F^{T}, \\
0 = FA^{T}P_{u}G^{T}.
\end{align*}
\]
which follow from (137) and (138).

The expressions for (45) and (46) follow from (108)–(110) by using the above identities. Next, computing $G$ (115) along with (116) yields (44). Substituting (139)–(141) into (111)–(116) along with the expression for $A_2$ it follows that (113) = $F$ (112) and (116) = $G$ (118). Thus (113) and (116) are superfluous and can be omitted. Thus, (111)–(116) reduce to

$$0 = A Q + Q A^T + \mu_1 A Q + \hat{Q} A T \mu_1^T + V_1 - Q_2 V_2^{-1} Q_2^T$$

$$+ \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T, \quad (145)$$

$$0 = [\mu_1 A Q + \hat{Q} A T \mu_1^T + \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T]^T \Phi,$$  

$$0 = (A - \mu Q_2 V_2^{-1} C)^T P + P (A - \mu Q_2 V_2^{-1} C)^T \quad (147)$$


$$0 = \mu_1 A Q + \hat{Q} A T \mu_1^T + \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T$$

$$- \tau_1 \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T. \quad (149)$$

Using (138), (149) becomes

$$0 = \mu_1 \left[ \begin{array}{cc} 0 & 0 \\ 0 & x_{n,n} \end{array} \right] A_2 Q_1 + \hat{Q} A_1^T \mu_1^T + \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T$$

$$- \tau_1 \mu_1 Q_2 V_2^{-1} Q_2 \mu_1^T. \quad (150)$$

Next, computing $H (150) H^T$ yields (49). Note conversely that if (49) is satisfied, then (A.36) holds since $\mu_1, \tau_1, \mu_1 = \tau_1 \mu_1^T$.

Finally, to prove Theorem 2 we use (44)–(50) to obtain (32) and (105)–(110). Let $A_1, B_1, C_1, G, D, F, \Phi, \tau, \mu, Q, P, \bar{Q}, F^T$ be as in the statement of Theorem 1 and define $Q_1, Q_2, Q_3, P_1, P_2, P_3$ by (108)–(110). Using (40), $\Phi F^T = I_{n,n}$, (45) and (46) it is easy to verify (139)–(141).

Next substitute the definitions of $Q, P, \bar{Q}, \bar{F}, \Phi, \tau, \mu$ into (47)–(50) using (40), (41), and (133) to obtain (32) and (105). Finally, note that

$$\hat{Q} = \left[ \begin{array}{cc} Q & 0_{n,n} \\ 0_{n,n} & x_{n,n} \end{array} \right] + \left[ \begin{array}{c} 0_n \end{array} \right] \hat{Q} [I_n, G^T],$$

which shows that $\hat{Q} \geq 0$. Now using the assumed existence of a nonnegative-definite solution to (32) and the stabilisability condition ($\bar{A}, \bar{Q}^T$) it follows from the dual of Lemma 12.2 of Ref. 30, that $\bar{A}$ is asymptotically stable.

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