I. INTRODUCTION

Both first principle (that is, analytical) models and empirical (that is, identified) models are approximate. The required accuracy of a model is application dependent. In this paper, within the context of Hammerstein systems, we assume that an initial model is available and that the fidelity of the initial model is insufficient. For example, the initial model may be erroneous with regard to initial conditions, parameters, or order (due to unmodeled dynamics). Our goal is to apply identification methods and adaptive disturbance rejection methods to improve the accuracy of the initial model. To do this, we combine the initial model with a delta model to obtain an augmented model. This technique is of particular interest when the initial model is a large-scale analytical model or a computer simulation (e.g. CFD or MHD), in which case it is convenient to add a small delta model to it rather than replace the initial model.

A related approach developed in [4] corrects a model of a structure with truncated modes by appending an analytically-derived delta model in parallel. Furthermore, in [3] a method is given for modifying an existing controller based on knowledge of deviations in the plant. Several classes of plant deviations are considered including feedback, parallel and cascade. However, the aim of [3] is not to correct the controller itself, but rather to correct the controller such that it handles deviations in the plant.

A delta model can be combined with the initial model in cascade, parallel, or feedback. In the present paper we consider subspace identification and adaptive disturbance rejection to construct the delta models. For the cascade and parallel augmentation case, we can use subspace identification methods to build the delta models, whereas using adaptive disturbance rejection, delta models for all three cases can be constructed.

For model refinement using subspace identification, we set up an identification problem in which we construct an empirical model of the error between the true system and the initial model, that is, the delta system. This approach assumes the ability to obtain input-output data by experiment. Subspace-based nonlinear identification algorithms [6] are then used to construct a delta model, that is, a model of the delta system. Subspace algorithms are desirable for this application because of their ability to provide an estimate of the delta system’s initial state, and thus correct errors in the initial state. Although, in accordance with subspace algorithms, the state space basis of the identified delta model is unknown, we show that the estimated initial state can nevertheless correct errors in the initial state.

Subspace algorithms [5], [7] are used to identify state space matrices \((A, B, C, D)\) based on the known inputs and outputs of the system. These methods are computationally tractable and naturally applicable to MIMO systems. In this paper the n4sid command in Matlab System Identification Toolbox is used for linear system identification, and the method developed in [6], [2] is used for Hammerstein system identification. With these algorithms, the system order can be manually chosen or automatically set based on numerical criteria.

For model refinement using adaptive disturbance rejection, we formulate the model refinement problem as an adaptive disturbance rejection problem. An adaptive disturbance rejection algorithm is then used to tune the delta model. In this paper, the ARMARKOV algorithm [8] is used for adaptation. Since adaptive disturbance rejection methods for Hammerstein systems are not well developed, we restrict ourselves to linear systems in this case. We also note that the adaptive disturbance rejection algorithm can be used as a system identification tool when the initial model is set to zero in the parallel augmentation framework.

In section 2 we present model refinement using subspace identification algorithms. First model equivalence concepts for Hammerstein systems are examined. These results are then used to suggest procedures for using subspace algorithms for the parallel and cascade augmentation case. In section 3 we discuss model refinement using adaptive disturbance rejection. In section 4 we present a few representative examples to illustrate the above ideas.
II. MODEL REFINEMENT USING SUBSPACE IDENTIFICATION

We first examine model equivalence for Hammerstein systems. This differs from traditional model equivalence concepts in the presence of the static nonlinearity and a non-zero initial conditions. Then we present procedures for cascade model augmentation and parallel model augmentation.

The notation $S \sim \begin{bmatrix} A & B \\ C & D \end{bmatrix} x(0)$ denotes the discrete-time Hammerstein dynamical system

$$
\begin{align*}
x(k+1) &= Ax(k) + Bf(u(k)), \\
y(k) &= Cx(k) + Du(k),
\end{align*}
$$

(II.1)

with initial condition $x(0)$, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $f : \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}^s$. If $(A, B, C)$ is minimal, where $A \in \mathbb{R}^{n \times n}$, then $n$ is the order of $S$. The notation $u_1 \equiv u_2$ means $u_1(k) = u_2(k)$ for all $k \geq 0$.

**Definition II.1.** Let $S_1$ and $S_2$ be systems with inputs $u_1$ and $u_2$ and outputs $y_1$ and $y_2$, respectively. Then $S_1$ and $S_2$ are equivalent if $y_1 \equiv y_2$ whenever $u_1 \equiv u_2$.

**Proposition II.1.** Consider the $n$th order systems

$$
S_1 \sim \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} f_1 x(0), \quad S_2 \sim \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} f_2 x(0)
$$

with $f_1(0) = f_2(0) = g_1(0) = g_2(0) = 0$. Then $S_1$ and $S_2$ equivalent if and only if $D_1g_1(u) = D_2g_2(u)$ for all $u$, and there exists a nonsingular matrix $S \in \mathbb{R}^{n \times n}$ such that

$$
A_1 = SA_2S^{-1}, \quad C_1 = C_2S^{-1}, \quad x_1(0) = Sx_2(0),
$$

(II.3)

and

$$
B_1f_1(u) = SB_2f_2(u) \quad \text{for all } u.
$$

(II.4)

**Proof.** Sufficiency is immediate. To prove necessity, by setting $u_1 = u_2 = 0$, and using equivalence, we can show that $x_1(0) = Sx_2(0)$.


A. Cascade Delta-Model Augmentation

Consider the system $S_0 \sim \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} f_0 x(0)$ and the initial model $S_m \sim \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} f_m y_m x_m(0)$. The system with input $y_m$ and output $y_\Delta$ is the cascade delta system $S_\Delta$, which is connected in series with the initial model to obtain the cascade-augmented model $S_n \triangleq S_\Delta S_m$ shown in Figure 1.

**Proposition II.2.** Suppose that $S_\Delta S_m$ and $S_0$ are equivalent. Then $y_\Delta \equiv y_0$ for all inputs $u$.

In view of Proposition II.2, we use subspace-based nonlinear identification algorithms [6] to construct a cascade delta model $S_\Delta$ with input $u_\Delta = y_m$, where $y_m$ is the initial model output, and output $y_\Delta = y_0$, where $y_0$ is the true system output. Then, a cascade-augmented model $S_n \triangleq S_\Delta S_m$ can be constructed as shown in Figure 1 to approximate the true system $S_0$.

B. Parallel Delta-Model Augmentation

Consider the system $S_0 \sim \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} f_0 x(0)$ and the initial model $S_m \sim \begin{bmatrix} A_m & B_m \\ C_m & D_m \end{bmatrix} f_m y_m x_m(0)$. The system with input $u$ and output $y_\Delta = y_0 - y_m$ is the parallel delta system $S_0 - S_m$, which is illustrated in Figure 2.

**Proposition II.3.** Consider two Hammerstein systems $S_0$ and $S_m$, then the parallel delta system $S_0 - S_m$ is also a Hammerstein system.

**Proof.** Let $u \in \mathbb{R}^m$ be the inputs to both the systems and let $y_0, y_m \in \mathbb{R}^l$ be the outputs of the two Hammerstein systems respectively. Now, the parallel delta system output is given by

$$
\begin{align*}
y_\Delta(k) &= y_0(k) - y_m(k) \\
&= C_0x_0(k) + D_0g_0(u(k)) - C_mx_m(k) \\
&= -D_0g_0(u(k)) \\
&= \begin{bmatrix} C_0 & -C_m \end{bmatrix} \begin{bmatrix} x_0(k) \\
x_m(k) \end{bmatrix} \\
&+ \begin{bmatrix} D_0 & -D_m \end{bmatrix} \begin{bmatrix} g_0(u(k)) \\
g_m(u(k)) \end{bmatrix}.
\end{align*}
$$

(II.6)

So the parallel system can be written as

$$
\begin{align*}
x_\Delta(k+1) &= A_\Delta x_\Delta(k) + B_\Delta f_\Delta(u(k)), \\
y_\Delta(k) &= C_\Delta x_\Delta(k) + D_\Delta g_\Delta(u(k)),
\end{align*}
$$

(II.7)

where, $x_\Delta(k) = \begin{bmatrix} x_0(k) \\
x_m(k) \end{bmatrix}$, $A_\Delta = \begin{bmatrix} A_0 & 0 \\
0 & A_m \end{bmatrix}$, $B_\Delta = \begin{bmatrix} B_0 & 0 \\
0 & B_m \end{bmatrix}$, $f_\Delta(u) = \begin{bmatrix} f_0(u) \\
f_m(u) \end{bmatrix}$, $C_\Delta = \begin{bmatrix} C_0 & -C_m \end{bmatrix}$, $D_\Delta = \begin{bmatrix} D_0 & -D_m \end{bmatrix}$, $g_\Delta(u) = \begin{bmatrix} g_0(u) \\
g_m(u) \end{bmatrix}$ and $x_\Delta(0) = \begin{bmatrix} x_0(0) \\
x_m(0) \end{bmatrix}$.
Thus the resulting system is again a Hammerstein system.

\[
\begin{bmatrix}
  x_0(0) \\
  x_m(0)
\end{bmatrix}
\]. Thus the resulting system is again a Hammerstein system.

**Proposition II.4.** Suppose \(S_0 - S_m\) and \(S_\Delta\) are equivalent with outputs \(y_0 - y_m\) and \(y_\Delta\), respectively. Then the output \(y_0 - y_m - y_\Delta\) of the system \(S_0 - S_m - S_\Delta\) is identically zero for all inputs \(u\).

**Proof.** Since \(S_0 - S_m\) and \(S_\Delta\) are equivalent it follows that \(y_\Delta \equiv y_0 - y_m\). Therefore \(y_0 \equiv y_m + y_\Delta\) and hence \(S_0\) and \(S_m + S_\Delta\) are equivalent.

To construct a parallel delta model, consider the initial model \(S_m\) of the true system \(S_0\). From Prop. II.3, the delta system \(S_\Delta = S_0 - S_m\) is a Hammerstein system. The subspace based nonlinear identification algorithm [6] can thus be used to construct a model of the delta system based

\[
\begin{bmatrix}
  x_0(0) \\
  x_m(0)
\end{bmatrix}
\]. Thus the resulting system is again a Hammerstein system.

**Fig. 2.** Parallel Delta System

\[
\begin{bmatrix}
  x_0(0) \\
  x_m(0)
\end{bmatrix}
\]. Thus the resulting system is again a Hammerstein system.

**Fig. 3.** Delta-Model Identification for Parallel Augmentation

\[
\begin{bmatrix}
  x_0(0) \\
  x_m(0)
\end{bmatrix}
\]. Thus the resulting system is again a Hammerstein system.

**Fig. 4.** Parallel Model Augmentation

**Proposition II.4.** Suppose \(S_0 - S_m\) and \(S_\Delta\) are equivalent with outputs \(y_0 - y_m\) and \(y_\Delta\), respectively. Then the output \(y_0 - y_m - y_\Delta\) of the system \(S_0 - S_m - S_\Delta\) is identically zero for all inputs \(u\).

**Proposition II.5.** Suppose that \(S_0 \equiv S_m\) and \(S_\Delta\) are equivalent. Then \(S_\Delta\) and \(S_m + S_\Delta\) are equivalent.

**Proof.** Since \(S_0 - S_m\) and \(S_\Delta\) are equivalent it follows that \(y_\Delta \equiv y_0 - y_m\). Therefore \(y_0 \equiv y_m + y_\Delta\) and hence \(S_0\) and \(S_m + S_\Delta\) are equivalent.

To construct a parallel delta model, consider the initial model \(S_m\) of the true system \(S_0\). From Prop. II.3, the delta system \(S_\Delta = S_0 - S_m\) is a Hammerstein system. The subspace based nonlinear identification algorithm [6] can thus be used to construct a model of the delta system based on the input \(u\) and the output \(y_0 - y_m\). This is illustrated in Figure 3. The identified model \(\hat{S}_\Delta\) is an approximation of \(S_0 - S_m\). Next, a Hammerstein parallel-augmented model \(\hat{S}_\Delta \equiv S_m + \hat{S}_\Delta\) can be constructed as shown in Figure 4 to approximate the true system \(S_0\).

Since linear systems are special cases of Hammerstein systems, with \(f(u) = u\), all the above arguments apply even if we have a linear initial model \(L_m\) for a Hammerstein system \(S_0\).

**III. Model Refinement Using Adaptive Disturbance Rejection**

Adaptive disturbance theory for Hammerstein systems is not well developed, so in this section we restrict our discussion to linear systems only. We discuss model refinement using adaptive disturbance rejection with cascade, parallel, feedback and combined cascade-feedback architectures.

Consider a linear system \(L_0\) described by the discrete-time state-space equations

\[
x_0(k+1) = A_0 x_0(k) + D_{10} \left[ w_1(k) \right], \quad (\text{III.1})
\]

\[
y_0(k) = C_0 x_0(k) + D_{20} w_2(k), \quad (\text{III.2})
\]

and an initial model \(L_m\) with the equations

\[
x_m(k+1) = A_m x_m(k) + D_{1m} w_1(k) + B_m u(k), \quad (\text{III.3})
\]

\[
y_m(k) = C_m x_m(k) + D_{2m} w_2(k) + D_m u(k), \quad (\text{III.4})
\]

where \(w_1(k)\) is a known input signal, \(u(k)\) is the output from the delta model, and \(w_2(k)\) is an unmeasured disturbance in the true system \(L_0\).

The objective of delta modeling is to construct a delta model in cascade, parallel, or feedback with the initial model, so that the resulting augmented model matches the true system. In the case of a parallel delta model, \(B_m\) and \(D_m\) are zero since the output from the delta model is directly added to the output of the initial model. In Section 2 subspace identification was used to construct the delta model. In the case of feedback interconnection, however, subspace identification cannot be used since the required \(u(k)\) to achieve model matching is unknown. Instead, the problem is recast as an adaptive disturbance rejection problem.

Consider again the true system (III.1) and (III.2), and the initial model (III.3) and (III.4). To achieve model matching we require that the error \(y_0(k) - y_m(k)\) be small. Hence the control inputs \(u(k)\) have to be modified in a way that makes \(y_m(k)\) equal to \(y_0(k)\). This problem can be viewed as a disturbance rejection problem where the performance variable \(z = y_0 - y_m\) is minimized in the presence of external disturbances \(w_1(k)\) and \(w_2(k)\).

Four different interconnection structures are considered for the delta model, represented by \(\zeta_0 = \frac{A_m}{B_m} \frac{A_0}{B_0}\). In all four cases the signal used to tune the controller parameters is the error between the outputs of the true system and the model.
\( z = y_0 - y_m \). Defining \( \tilde{x}(k) \triangleq \begin{bmatrix} x_0(k) \\ x_m(k) \end{bmatrix} \) and \( w(k) \triangleq \begin{bmatrix} w_1(k) \\ w_2(k) \end{bmatrix} \), it will be shown that the delta modeling problem can be written in the standard control architecture form shown in Figure 5. In the context of adaptive disturbance rejection, the delta model \( \mathcal{L}_c \) is the adaptive controller, the initial model is the plant \( \mathcal{L}_m \), and knowledge of \( w_2(k) \) or the true system \( \mathcal{L}_0 \) is not required. Thus, adaptively controlling the initial model \( \mathcal{L}_m \) to minimize the performance variable \( z \), achieves the objective of constructing an augmented model to match the true system \( \mathcal{L}_0 \). So, once the model refinement problem is recast in the form of Figure 5, a standard adaptive disturbance rejection algorithm like ARMARKOV [8] can be used to tune the delta model.

**A. Cascade Interconnection**

In the cascade case, the input to the delta model \( \mathcal{L}_c \) is \( w_1(k) \), and thus we define the controller inputs \( y(k) \triangleq w_1(k) \). This scheme shown in Figure 6 can be represented in the standard form as

\[
\begin{bmatrix}
  z(k) \\
  y(k)
\end{bmatrix} = \begin{bmatrix}
  \mathcal{L}_0 - \begin{bmatrix}
    \mathcal{L}_m & 0 \\
    I & 0
  \end{bmatrix} & -\mathcal{L}_m \\
\end{bmatrix} \begin{bmatrix}
  w(k) \\
  u(k)
\end{bmatrix}.
\]

(III.5)

with the state space realization

\[
\begin{bmatrix}
  \tilde{x}(k+1) \\
  z(k) \\
  y(k)
\end{bmatrix} =
\begin{bmatrix}
  A_0 & 0 & 0 \\
  0 & A_m & D_{10} \\
  C_0 & -C_m & D_{20}
\end{bmatrix}
\begin{bmatrix}
  D_{1m} & 0 \\
  I & 0 \\
  0 & -D_m
\end{bmatrix}
\begin{bmatrix}
  \tilde{x}(k) \\
  w(k) \\
  u(k)
\end{bmatrix}.
\]

(III.6)

**B. Parallel Interconnection**

In parallel interconnection, the delta model \( \mathcal{L}_c \) is connected as shown in Figure 7. The parallel architecture represented in the standard form has the state space realization

\[
\begin{bmatrix}
  \tilde{x}(k+1) \\
  z(k) \\
  y(k)
\end{bmatrix} =
\begin{bmatrix}
  A_0 & 0 & 0 \\
  0 & A_m & D_{10} \\
  C_0 & -C_m & D_{2m}
\end{bmatrix}
\begin{bmatrix}
  D_{1m} & 0 \\
  I & 0 \\
  0 & -D_m
\end{bmatrix}
\begin{bmatrix}
  \tilde{x}(k) \\
  w(k) \\
  u(k)
\end{bmatrix}.
\]

(III.7)

**C. Feedback Interconnection**

The feedback interconnection problem is shown in Figure 8. In this case the controller inputs \( y(k) \triangleq y_m(k) \). Thus the standard form state space representation is

\[
\begin{bmatrix}
  \tilde{x}(k+1) \\
  z(k) \\
  y(k)
\end{bmatrix} =
\begin{bmatrix}
  A_0 & 0 & 0 \\
  0 & A_m & D_{10} \\
  C_0 & -C_m & D_{20}
\end{bmatrix}
\begin{bmatrix}
  D_{1m} & 0 \\
  I & 0 \\
  0 & -D_m
\end{bmatrix}
\begin{bmatrix}
  \tilde{x}(k) \\
  w(k) \\
  u(k)
\end{bmatrix}.
\]

(III.8)
D. Cascade and Feedback Interconnection

In the feedback interconnection an additional feedforward path can be included to obtain a combined feedback and cascade interconnection, as illustrated in Figure 9. Noting that \( y(k) \triangleq \begin{bmatrix} y_m(k) \\ w_1(k) \end{bmatrix} \), the standard problem can be written in the state space form as

\[
\begin{bmatrix}
\dot{x}(k+1) \\
y(k)
\end{bmatrix} = 
\begin{bmatrix}
A_0 & 0 \\
0 & A_m
\end{bmatrix}
\begin{bmatrix}
x(k) \\
y_m(k)
\end{bmatrix}
+ 
\begin{bmatrix}
D_{10} \\ D_{20}
\end{bmatrix}
\begin{bmatrix}
\tilde{w}(k) \\
w(k)
\end{bmatrix}
- 
\begin{bmatrix}
B_{1m} \\ B_{2m}
\end{bmatrix}
\begin{bmatrix}
y_m(k) \\
y(k)
\end{bmatrix},
\]

where

\[
C_0 = -C_m, \quad D_{20} = \begin{bmatrix} D_{2m} & 0 \end{bmatrix}, \quad -D_m
\]

in this form, an adaptive disturbance rejection algorithm can be used to perform system identification.

IV. EXAMPLES

A. Model Refinement Using Subspace Identification

1) Unmodeled Dynamics Example: Here we consider the equations of an acoustic duct. The equations and state space realizations are given in [1], where the speed of acoustic waves is 343 m/s, the density of air is 1.21kg/m³, length of the duct is \( L = 2m \), and the duct model includes four modes.

To emulate unmodeled dynamics the initial model \( S_m \) of the acoustic duct is obtained by deleting the states associated with the second mode. Hence \( S_0 \) is \( 8^{th} \) order and \( S_m \) is \( 6^{th} \) order. The system parameters and initial conditions of the retained states are assumed to be known. The frequency responses of the true system and the initial model are shown in Figure 10.

The fit errors using cascade augmentation for the various orders of the cascade delta model are shown in Table I, from which it is clear that a \( 6^{th} \)-order cascade delta model is needed for an accurate fit of the forced and free responses.

For parallel augmentation, an accurate fit for the forced and free responses is obtained with a \( 2^{nd} \)-order parallel delta model as shown in Figure 11 and Table II.

2) Parametric Error and Initial Conditions Error Example: We consider a spring mass damper system with an input nonlinearity. Defining the states as \( x \triangleq [ q \ \dot{q} ]^T \), where \( q \) is the position of the mass and \( \dot{q} \) is the velocity, the state space matrices are

\[
A = \begin{bmatrix}
0 & -k/m \\
-c/m & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1/m
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1
\end{bmatrix},
\]

where \( m \) is the mass, \( k \) is the spring stiffness and \( c \) is the damping constant. The input nonlinearity is chosen to be \( f(u) = u^2 + u^3 \). We use \( k = 2 \), \( m = 5 \), and \( c = 3 \) and a square wave as the input sequence \( u \).

The N4SID command in Matlab is used to identify a linear model of the Hammerstein system described above. This identified model is then used as the initial model, and Hammerstein delta-model augmentation is performed on this initial model. The initial model is erroneous in the nonlinearity and the linear dynamics. Table III shows the error in the forced response for both the initial and the parallel augmented model.

B. Model Refinement Using Adaptive Disturbance Rejection

Consider the same spring mass system used in the previous example. The initial model has a parametric error in the
spring stiffness and the damping coefficient. The true values are chosen to be $m = 8$, $k = 20$, $b = 1$ and the erroneous values in the initial model are $m = 8$, $k = 44$, $b = 4$. A white noise input drives the true system and the initial model. The frequency response curves of the true system, initial model and the augmented model is shown in Figure 12. The combined feedback and cascade architecture is employed in this example.

For identification of the spring mass damper system, the initial model is set to zero in the parallel interconnection framework. The frequency response curves of the true system and the identified model are shown in Figure 13.

V. Conclusion

In this paper we developed and illustrated two approaches to improve model accuracy by using a delta model combined in cascade, parallel or feedback with an initial model. In the case of model refinement using subspace algorithms, by identifying the initial fit error system, the identified model was combined with the initial model to construct an augmented model. This technique was shown to be only partially effective for correcting initial conditions and parametric errors. However, the method is more effective for correcting unmodeled dynamics. In the case of model refinement using adaptive disturbance rejection, the problem was recast as a disturbance rejection problem and the AR-MARKOV adaptive disturbance rejection algorithm was used to tune the delta model. This technique was found to be more effective for correcting parametric errors.

References


