State Estimation for Linearized MHD Flow

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Abstract—A state estimation problem for linearized magnetohydrodynamic (MHD) flow is considered. The ideal MHD equations governing the flow of plasma in a two-dimensional channel are linearized about an equilibrium flow. Pseudospectral collocation methods are used to spatially discretize the linear partial differential equations and obtain a state-space model of the linearized dynamics. Three different discrete-time Kalman filtering algorithms are used to estimate the state variables, and their performance is analyzed.

I. INTRODUCTION

Plasma, as a distinct state of matter, plays a crucial role in numerous branches of science and engineering. The Earth is constantly bombarded by low energy plasma emitted by the Sun. However, during flares and coronal mass ejections in the Sun, high energy plasma particles are ejected, causing problems with spacecraft and Earth-based systems [1, 2]. Furthermore, plasma is generated for low-thrust, but highly efficient electric propulsion systems for spacecraft [3]. Finally, plasma confinement and control is one of the main challenges of fusion-based power generation [4]. Plasma flow is the subject of magnetohydrodynamics (MHD), which involves both fluid dynamics and electrodynamics. Consequently, MHD is governed by coupled partial differential equations, which include both the Navier-Stokes equations and Maxwell’s equations [5, 6].

The present paper is concerned with modeling and state estimation for uncontrolled plasma flows. Our ultimate goal is to apply state estimation techniques to space weather data in order to estimate the state of the plasma flow throughout a region. As a first step in this direction, we linearize the MHD equations for inviscid, incompressible flow in a 2-dimensional channel about a steady flow condition. To spatially discretize these equations, we consider both Fourier and Chebyshev pseudospectral collocation methods [7, 8].

Next, we analyze the stability of the truncated linear time-invariant model obtained by the Fourier collocation method. Stability tests show that the resulting model is Lyapunov stable. This test is thus inconclusive regarding the stability of the original nonlinear system. The spatially discretized model is then used as the basis for constructing state estimators. We consider the standard Kalman filter in recursive form along with numerically efficient variations, specifically, the reduced rank square root Kalman filter (RRSQRT-KF) and the singular square root Kalman filter (SSQRST-KF). The use of these filters is motivated by the fact that large-scale MHD models of the magnetosphere typically involve several million states [9].

II. IDEAL MHD EQUATIONS

Magnetohydrodynamics (MHD) provides a macroscopic dynamical description of plasma in the presence of electromagnetic fields. We assume that the plasma flow occurs in a non-relativistic regime, and we neglect ionization and recombination, which alter the total number of plasma particles. Also, we assume that the conductivity of the plasma is infinite, that is, \( \sigma = \infty \). The gravitational and Coriolis forces are also ignored. Under these simplifying assumptions the resulting ideal MHD equations are [2, 4, 5]

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \quad (2.1)
\]

\[
\frac{d}{dt} \left( \frac{p}{\rho^\gamma} \right) = 0, \quad (2.2)
\]

\[
\varrho \frac{\partial \vec{u}}{\partial t} + \varrho (\vec{u} \cdot \nabla) \vec{u} = -\nabla p + \frac{1}{\mu_0} \left( \nabla \times \vec{B} \right) \times \vec{B}, \quad (2.3)
\]

\[
\nabla \times (\vec{u} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t}, \quad (2.4)
\]

\[
\nabla \cdot \vec{B} = 0. \quad (2.5)
\]

III. STEADY-STATE FLOW AND PERTURBATIONS IN A 2D CHANNEL

Next, we determine steady-state flow and magnetic field configurations that are consistent with the ideal
Substituting (3.6) into (2.3), and substituting (3.1) and of state (2.2) can be ignored. Let \( \bar{u}_0 \) and \( \bar{B}_0 \) be the steady state solution of the MHD equations (2.1)-(2.5), that is,

\[
\frac{\partial \bar{u}_0}{\partial t} = 0, \quad \frac{\partial \bar{B}_0}{\partial t} = 0. \tag{3.1}
\]

Hence, it follows from (3.1) and (2.1)-(2.5) that \( \bar{u}_0 \) and \( \bar{B}_0 \) satisfy

\[
\nabla \cdot \bar{u}_0 = 0, \tag{3.2}
\]
\[
\varrho_0 (\bar{u}_0 \cdot \nabla) \bar{u}_0 = \frac{1}{\mu_0} (\nabla \times \bar{B}_0) \times \bar{B}_0, \tag{3.3}
\]
\[
\nabla \times ( \bar{u}_0 \times \bar{B}_0 ) = 0, \tag{3.4}
\]
\[
\nabla \cdot \bar{B}_0 = 0. \tag{3.5}
\]

Next, assume that the plasma flows along the \( \hat{e}_x \) direction with a constant velocity so that \( \bar{u}_0 = u_{0x} \hat{e}_x \) and let the constant magnetic field be prescribed by \( \bar{B}_0 = B_{0y} \hat{e}_y \). Note that the prescribed velocity and magnetic field satisfy (3.2) and (3.5), respectively, and hence \( \bar{u}_0 = u_{0x} \hat{e}_x \) and \( \bar{B}_0 = B_{0y} \hat{e}_y \) are steady-state solutions of the ideal MHD equations.

Next, define the perturbation variables \( \bar{u}_\delta \) and \( \bar{B}_\delta \) by

\[
\bar{u}_\delta \equiv \bar{u} - \bar{u}_0, \quad \bar{B}_\delta \equiv \bar{B} - \bar{B}_0. \tag{3.6}
\]

Substituting \( \theta = \varrho_0 \) and (3.6) into (2.1) yields

\[
\frac{\partial \varrho_0}{\partial t} + \varrho_0 \nabla \cdot \bar{u}_\delta + \varrho_0 \nabla \cdot \bar{u}_\delta = 0. \tag{3.7}
\]

Since the density is constant, substituting (3.2) into (3.7) yields

\[
\nabla \cdot \bar{u}_\delta = 0. \tag{3.8}
\]

Substituting (3.6) into (2.3), and substituting (3.1) and (3.3) into the resulting equation, and ignoring second and higher order perturbation terms yields

\[
\varrho_0 \frac{\partial \bar{u}_\delta}{\partial t} + \varrho_0 (\bar{u}_0 \cdot \nabla) \bar{u}_\delta + \varrho_0 (\bar{u}_\delta \cdot \nabla) \bar{u}_0
\]
\[= \frac{1}{\mu_0} \left[ \left( (\nabla \times \bar{B}_0) \times \bar{B}_3 \right) + \left( (\nabla \times \bar{B}_3) \times \bar{B}_0 \right) \right]. \tag{3.9}
\]

Substituting (3.6) into (2.4) yields

\[
\nabla \times (\bar{u}_0 \times \bar{B}_\delta) + \nabla \times (\bar{u}_\delta \times \bar{B}_0) = \frac{\partial \bar{B}_\delta}{\partial t}. \tag{3.10}
\]

Substituting (3.1) and (3.4) into (3.10) and ignoring second order perturbation terms yields

\[
\nabla \times (\bar{u}_0 \times \bar{B}_\delta) + \nabla \times (\bar{u}_\delta \times \bar{B}_0) = \frac{\partial \bar{B}_\delta}{\partial t}. \tag{3.11}
\]

Substituting (3.6) and (3.5) into (2.5) yields

\[
\nabla \cdot \bar{B}_\delta = 0. \tag{3.12}
\]

Hence, (3.9) yields

\[
\frac{\partial \bar{u}_\delta}{\partial t} + c_1 \frac{\partial \bar{u}_\delta}{\partial x} = c_2 \frac{\partial \bar{B}_\delta}{\partial y}. \tag{3.13}
\]

and it follows from (3.11) that

\[
\frac{\partial \bar{B}_\delta}{\partial t} + c_1 \frac{\partial \bar{B}_\delta}{\partial x} = c_3 \frac{\partial \bar{u}_\delta}{\partial y}. \tag{3.14}
\]

where \( c_1 \equiv u_{0x}, c_2 = B_{0y}/(\varrho_0 \mu_0), \) and \( c_3 \equiv B_{0y}. \) Therefore, (3.8), (3.12), (3.13), and (3.14) are the linearized equations that govern the dynamics of the perturbation variables \( \bar{u}_\delta \) and \( \bar{B}_\delta. \) Note that (3.13) and (3.14) resemble a twodimensional wave equation.

Taking the partial derivative of (3.13) with respect to \( t \) and \( x \) yields

\[
\partial_{tt} \bar{u}_\delta + c_1 \partial_{tx} \bar{u}_\delta = c_2 \partial_{ty} \bar{B}_\delta \tag{3.15}
\]

and

\[
\partial_{tx} \bar{u}_\delta + c_1 \partial_{xx} \bar{u}_\delta = c_2 \partial_{xy} \bar{B}_\delta, \tag{3.16}
\]

respectively. Taking the partial derivative of (3.14) with respect to \( y \) yields

\[
\partial_{ty} \bar{B}_\delta + c_1 \partial_{yx} \bar{B}_\delta = c_3 \partial_{yy} \bar{u}_\delta. \tag{3.17}
\]

Dividing (3.15) by \( c_2, \) and multiplying (3.16) by \( \frac{c_2}{c_1}, \) and adding the resulting equation with (3.17) yields

\[
\partial_{tt} \bar{u}_\delta - 2c_1 \partial_{tx} \bar{u}_\delta - c_1^2 \partial_{xx} \bar{u}_\delta + c_2c_3 \partial_{xy} \bar{u}_\delta. \tag{3.18}
\]

Note the symmetry in (3.13) and (3.14) with respect to \( \bar{u}_\delta \) and \( \bar{B}_\delta. \) Hence, a similar procedure yields

\[
\partial_{tt} \bar{B}_\delta - 2c_1 \partial_{tx} \bar{B}_\delta - c_1^2 \partial_{xx} \bar{B}_\delta + c_2c_3 \partial_{xy} \bar{B}_\delta. \tag{3.19}
\]

Now, the equations governing the perturbations in the velocity field \( \bar{u}_\delta \) and the magnetic field \( \bar{B}_\delta, \) have been decoupled.

### IV. State-Space Modeling Using Spatial Discretization Methods

The partial differentiation equation (3.18) involves the \( \partial_{xx} \) operator and hence a separation of variable technique cannot be used to obtain an equivalent ordinary differential equation representation. Hence, we use spatial
discretization methods to obtain an ODE model of the linearized perturbation dynamics. Let \( \hat{u}_x = u_q \hat{e}_x + u_s \hat{e}_y \). It follows from (3.8) that there exists a scalar potential function \( \psi(x, y, t) \) such that

\[
\begin{align*}
\hat{u}_{\hat{x}} &= \frac{\partial \psi}{\partial y}(x, y, t), & \hat{u}_{\hat{y}} &= -\frac{\partial \psi}{\partial x}(x, y, t).
\end{align*}
\]

Hence, it follows from (3.18) that

\[
\partial_{(ty)} \psi + 2c_1 \partial_{xy} \psi + c_2^2 \partial_{xx} \psi = c_2 c_3 \partial_{yyy} \psi.
\]

(4.2)

Assume \( \psi(x, y, t) \) has a solution of the form

\[
\psi(x, y, t) = U_m(x, t) W_m(y),
\]

(4.3)

where for all \( m = 0, 1, \ldots, n \), \( U_m(x, t) \) and \( W_m(y) \) are periodic and given by

\[
\begin{align*}
U_m(x, t) &= \sum_{k=0}^{m-1} \hat{q}_{mk} e^{i(kx + i)}.
\end{align*}
\]

(4.11)

\[
\text{Next, define } Q_m \in \mathbb{R}^n \text{ by}
\]

\[
Q_m \triangleq \begin{bmatrix} q_{m1} & \cdots & q_{mn} \end{bmatrix}^T.
\]

(4.12)

Using the Fourier collocation differentiation matrix \( \mathbb{M} \) in (4.8) yields

\[
\begin{align*}
\hat{Q}_m + 2c_1 D F_n \hat{Q}_m + c_2^2 D F_n Q_m + k_0, m Q_m &= 0, & \text{where } D F_n \in \mathbb{R}^{n \times n} \text{ and the } (i, j) \text{th element of } D F_n \text{ is given by}
\end{align*}
\]

\[
\begin{align*}
D F_n(i, j) &= \begin{cases} \frac{1}{2} \text{ if } i = j, \\ 0, \end{cases}
\]

(4.13)

A state-space representation of (4.13) is

\[
\begin{bmatrix} \dot{Q}_m \\ \dot{Q}_m \end{bmatrix} = A_m \begin{bmatrix} Q_m \\ Q_m \end{bmatrix},
\]

(4.15)

where \( A_m \in \mathbb{R}^{2n \times 2n} \) is defined by

\[
A_m \triangleq \begin{bmatrix} 0_n & I_n \\ -c_2^2 D F_n + k_0, m I_n & -2c_1 D F_n \end{bmatrix}.
\]

(4.16)

Note that \( A_m \) can be factored as

\[
A_m = P \begin{bmatrix} -c_1 D F_n + j k_0, m I_n \\ -c_1 D F_n - j k_0, m I_n \end{bmatrix} p^{-1},
\]

(4.17)

where \( S \in \mathbb{R}^{n \times n} \) is defined by \( S \triangleq \begin{bmatrix} c_2 D F_n + k_0, m I_n \end{bmatrix} \), \( T \in \mathbb{C}^{n \times n} \) is defined by \( T \triangleq \begin{bmatrix} c_1 D F_n - j k_0, m I_n \end{bmatrix} S^{-1} \), and \( P \in \mathbb{R}^{2n \times 2n} \) is defined by

\[
\begin{bmatrix} I_n \\ 0_n \end{bmatrix}.
\]

(4.18)

which implies that

\[
\text{spec}(A_m) = \text{spec}(-c_1 D F_n + j k_0, m I_n) \cup \text{spec}(-c_1 D F_n - j k_0, m I_n).
\]

(4.19)

It follows from (4.14) that \( D F_n \) is skew symmetric and hence all its eigenvalues lie on the imaginary axis. Hence, (4.19) implies that the eigenvalues of \( A_m \) can be confined to the imaginary axis, that is, for all \( \lambda \in \text{spec}(A_m) \), \( \text{Re}(\lambda) = 0 \). Note that (4.13) is a second order system and can be expressed as

\[
M \ddot{Q}_m + G \dot{Q}_m + K Q_m = 0,
\]

(4.20)

where \( M = I_n, G = 2c_1 D F_n, \) and \( K = c_2^2 D F_n + k_0, m I_n \). Since \( GK = KG + \frac{1}{2} G G^T \) is positive definite, it follows from Proposition 3 of [10] that (4.13) is Lyapunov stable.

Note that (4.13) represents the dynamics of the \( n \)th mode and it follows from the principle of superposition that the solution to (4.2) is given by

\[
\psi(x, y, t) = \sum_{m=1}^{\infty} U_m(x, t) W_m(y).
\]

Retaining \( m \) modes and defining the modal state vector \( \bar{Q} \in \mathbb{R}^{2nr} \) by

\[
\bar{Q} \triangleq \begin{bmatrix} Q_1^T & \cdots & Q_r^T & \cdots & Q_r^T \end{bmatrix}^T,
\]

(4.15)
and (4.16) that
\[ \hat{Q} = A\hat{Q}, \]
where \( A \in \mathbb{R}^{2nr \times 2nr} \) is the block-diagonal matrix
\[
A \triangleq \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_r
\end{bmatrix}.
\]
Hence, \( \text{spec}(A) = \text{spec}(A_1) \cup \cdots \cup \text{spec}(A_r) \) and, for all \( \lambda \in \text{spec}(A), \text{Re}(\lambda) = 0 \).

Let \( y_{\text{out},i,j} \triangleq u_{\delta,i}(x_i, y_j, t) \) be the measured perturbation in the flow velocity \( u_x \) from \( u_{0\delta} \) at \( (x_i, y_j) \), where \( x_i \) is one of the grid points. It follows from (4.1), (4.3), and (4.10) that
\[ y_{\text{out},i,j} = \partial_x \psi(x, y, t) \bigg|_{(x=x_i, y=y_j)} = C_{x,i,j} \hat{Q}, \]
where \( C_{x,i,j} \in \mathbb{R}^{1 \times 2nr} \) has entries \( C_{x,i,j} = [ C_1^{x,i,j} \cdots C_n^{x,i,j} ] \), and, for all \( m = 1, \ldots, r, C_m^{x,i,j} \in \mathbb{R}^{2n} \) is defined by
\[
C_m^{x,i,j} \triangleq [ 0_{2n-1} W_m^c(y_j) 0_{2n-1} ].
\]
Next, we consider the case when the measurement \( \tilde{y}_{\text{out},i,j} \triangleq u_{\delta,i}(x_i, y_j, t) \) is the perturbation in the flow velocity \( u_y \) from \( u_{0\delta} \) at \( (x_i, y_j) \). It follows from (4.1) that
\[ \tilde{y}_{\text{out},i,j} = -\partial_y \psi(x, y, t) \bigg|_{(x=x_i, y=y_j)}. \]
Using (4.3), (4.10), and the Fourier collocation differentiation matrix \( D_{F_n} \) in (4.25) yields
\[ \tilde{y}_{\text{out},i,j} = -C_{y,i,j} D_{F_n} \hat{Q}, \]
where \( C_{y,i,j} \in \mathbb{R}^{1 \times 2nr} \) has entries \( C_{y,i,j} = [ C_1^{y,i,j} \cdots C_n^{y,i,j} ] \), and, for all \( m = 1, \ldots, r, C_m^{y,i,j} \in \mathbb{R}^{2n} \) is defined by
\[
C_m^{y,i,j} \triangleq [ 0_{2n-1} W_m^c(y_j) 0_{2n-1} ].
\]
Note that (3.18) and (3.19) are similar and hence the solution to \( \hat{B}_i \) is similar to that of \( \hat{u}_i \), and the constants are determined by the initial and boundary values of the magnetic field instead of the velocity field.

### B. Chebyshev Collocation Method

Next, we express \( U_m(x, t) \) as a Chebyshev series in \( x \) with time varying coefficients. Let \( L_1 = -1 \) and \( L_2 = 1 \), and, for all \( i = 1, \ldots, n, \) let \( x_i = -\cos \left[ \frac{(i-1)n}{n+1} \right] \) be the \( n \) Gauss-Lobatto grid points in the interval \([-1, 1]\) (see [7, 8]). Consider a solution of the form (4.3) and define \( q_m^{x,i} \triangleq U_m(x_i, t) \). The truncated Chebyshev series expansion for the solution \( U_m(x, t) \) is (see [7])
\[
U_m(x, t) = \sum_{k=0}^{n-1} \tilde{q}_{mk} \phi_k(x),
\]
where \( \phi_k(x) \triangleq \cos(k \cos^{-1}(x)) \), and, for all \( k = 0, \ldots, n-1, \) \( \tilde{q}_{mk} \) is defined by
\[
\tilde{q}_{mk} \triangleq \frac{1}{\gamma_k} \sum_{i=1}^{n} q_m^{x,i} \phi_k(x_i) w_i,
\]
where
\[
\gamma_k = \left\{ \pi, \begin{array}{ll}
\pi k = 0 & \text{or } k = n-1, \\
\frac{\pi}{2(n-k)}, & 0 < k < n-1,
\end{array} \right. \]
and \( c_i \) is defined by
\[
c_i \triangleq \begin{cases}
\frac{2}{\pi}, & i = 1 \text{ or } i = n, \\
1, & 1 < i < n.
\end{cases}
\]
The state-space model is then given by (4.21), where \( A_m \) is defined by (4.16) with \( D_{F_n} \) replaced by \( \tilde{D}_{F_n} \) and using the Chebyshev collocation differentiation matrix (see [8]) in (4.8) yields (4.13) with \( D_{F_n} \) replaced by \( \tilde{D}_{C_n} \), where the \((i,j)\)th entry of \( \tilde{D}_{C_n} \) is defined by
\[
\tilde{D}_{C_n(i,j)} = \begin{cases}
\frac{\pi}{2(n-i)}, & i \neq j, \\
\frac{\pi}{2(n-i+1)}, & 1 < i = j \leq n, \\
\frac{2(1-i)\pi}{2(n-i)\gamma_1}, & i = j = 1, \\
\frac{2(n-1)^2 \pi}{8}, & i = j = n,
\end{cases}
\]
and the singular square root Kalman filter (SSQRT-KF) [15], namely, the standard Kalman filter (KF) [13], the reduced rank square root Kalman filter (RRSQRT-KF) [14], and the singular square root Kalman filter (SSQRT-KF) [15], are used for state estimation of linearized MHD flow in a two dimensional channel, under different measurement noise conditions.

### V. Kalman Filtering Estimator

In this section, three state space observers, namely, the standard Kalman filter (KF) [13], the reduced rank square root Kalman filter (RRSQRT-KF) [14], and the singular square root Kalman filter (SSQRT-KF) [15], are used for state estimation of linearized MHD flow in a two dimensional channel, under different measurement noise conditions.

Consider the following state space representation of the linearized MHD system
\[
x_{k+1} = Ax_k + w_k \quad (5.1)
\]
\[
y_k = Cx_k + v_k, \quad (5.2)
\]
where \( x_{k+1} = \hat{Q}_k x_k + C_{x,i,j} \), \( y_k = y_{\text{out},i,j} \), \( w_k \in \mathbb{R}^{2nr} \) is the process noise, and \( v_k \in \mathbb{R}^p \) the measurement noise. Furthermore, assume that \( w_k \) and \( v_k \) are uncorrelated white Gaussian noise with zero mean and covariance matrices \( Q \) and \( R \), respectively.
A. Discrete Time Kalman Filter

Consider the discrete-time dynamical system described by (5.1) and (5.2). For this system, we consider a state estimator of the form

\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - \hat{y}_{k|k-1}), \quad k \geq 0,
\]

where \( L_k \in \mathbb{R}^{n \times m} \) and \( \hat{x}_{k|k} \) is the estimate of \( x_k \) based on observations up to time \( k \), with output

\[
\hat{y}_{k|k-1} = C\hat{x}_{k|k-1}.
\]

Kalman [16] found a recursive solution to obtain the optimal \( L_k \) which minimizes the estimation error defined by

\[
e_k[k] \triangleq x_k - \hat{x}_{k|k}.
\]

Define the error covariance matrix \( P_{k|k} \) by

\[
P_{k|k} \triangleq E[e_k[k]e_k[k]^T],
\]

where \( E[\cdot] \) denotes the expected value operator. The solution can be summarized as follows:

1) Compute the prior error covariance matrix and the estimated states

\[
P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^T + Q_k-1.
\]

\[
\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1}.
\]

2) Compute the Kalman gain

\[
L_k = P_{k|k-1}C_k^T(R_k + C_kP_{k|k-1}C_k^T)^{-1},
\]

3) Update the estimated states using (5.3) and (5.4)

\[
P_{k|k} = (I_n - L_kC_k)P_{k|k-1}.
\]

B. SSQRT-KF Algorithm

SSQRT-KF is based on the assumption that the process noise is negligible or zero, that is \( Q = 0 \). If \( Q = 0 \) in (5.10), the Schur complement of (5.10) is

\[
M = \begin{pmatrix}
R + CP_{k-1}^TCT & CP_{k-1}

P_{k-1} & P_{k-1}^T
\end{pmatrix},
\]

where

\[
R \triangleq LR_k^T, \quad P_{k|k-1} \triangleq S_{k|k-1}S_{k|k-1}^T.
\]

A QR decomposition of \( M \) in (5.11) yields

\[
\begin{pmatrix}
L_R & CS_{k|k-1}
0 & S_{k|k-1}
\end{pmatrix}U_k = \begin{pmatrix}
\tilde{F}_k & 0
K_k & S_{k|k-1}
\end{pmatrix},
\]

where \( U_k \) is orthogonal and \( S_{k|k} \) is \( n \times l \), with \( l \) chosen larger than the number of unstable eigenvalues of \( A \), and \( P_{k|k} = S_{k|k}S_{k|k}^T \). Note that the QR decomposition is performed on a small matrix of size \( (p+l) \times p \) and hence is cheap to compute. Assuming \( \tilde{F}_k \) is invertible, the Kalman filter gain is given by

\[
L_k = \tilde{K}_k\tilde{F}_k^{-1},
\]

where \( \tilde{K}_k \) is the matrix of factors. If the measurements are uncorrelated, that is, if \( R \) is diagonal, then \( F_k \) will be diagonally dominant and hence, instead of computing \( F_k^{-1} \), we invert its diagonal entries to obtain a diagonal approximation of \( F_k^{-1} \). Moreover, if \( A \) and \( C \) are sparse, the construction of the left factor in the left hand side of (5.13) is cheap as well.

A key characteristic of the SSQRT-KF algorithm is that the spectrum of the state space observer dynamics matrix \( A - L_kC \) is constructed by reflecting the eigenvalues of \( A \) with \( |\lambda| > 1 \) to their unit circle mirror images \( 1/|\lambda| \), and leaving the eigenvalues with \( |\lambda| < 1 \) unchanged.

C. RRSSQRT-KF Algorithm

In the RRSSQRT-KF estimator the square root factors are based on an eigendecomposition. Let \( P_{k|k-1} = V_kA_kV_k^T \) be the eigendecomposition of the error covariance matrix \( P_{k|k-1} \), so that \( S_{k|k-1} = V_kA_k^{1/2} \) is a square root factor of \( P_{k|k-1} \). The error covariance matrix is now approximated by using only \( q \) leading eigenvalues. With the ordering \( |\lambda_1| \geq \ldots \geq |\lambda_q| \geq 0 \), an approximation is obtained by truncating \( S_{k|k-1} \) after the first \( q \) columns. The algorithm is as follows:

1. Update \( \hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1} \)

2. Update \( S_{k|k-1} = [A_{k-1}S_{k-1|k-1}LQ_{k-1}] \)

3. Perform rank reduction of the error covariance:

\[
S_{k|k-1}^* = [S_{k|k-1}V_k]\text{i}m_1:n,1:q,
\]

where \( S_{k|k-1}^* = 0 \).

4. Compute \( L_k \) and \( S_{k|k} \) using the scalar update of Potter [14] for independent measurements as follows, \( S_{k|k} = S_{k|k-1}^* \), for \( i = 1 \) to \( p \),

\[
H = S_{k|k-1}C_k(i,i)^T
\]

\[
F = (H^TH + R_k(i,i))^{-1}
\]

\[
L_k(i,i) = S_{k|k-1}^*H
\]

\[
S_{k|k} = S_{k|k} - L_kH^T(1 + (FR_k(i,i))^{1/2})^{-1}
\]

end.

5. Compute \( \hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C_k\hat{x}_{k|k-1}) \)

The procedure when the measurements are correlated is given in [14].

VI. STATE ESTIMATION FOR THE LINEARIZED MHD MODEL

We consider a \( 20 \times 20 \) grid with equidistant points (the grid points along the \( \vec{e}_x \) direction are the Fourier collocation points), where \( 0 < x < 2\pi \), \( 0 < y < 1 \), and sample time \( T_s = 10^{-3} \) seconds. The number of modes retained is \( m = 5 \) and hence, \( A_q \in \mathbb{R}^{200 \times 200} \), and \( C_q \in \mathbb{R}^{400 \times 200} \). Although a system may be fully observable with just one measurement output, the discrete-time linearized system turns out to be marginally observable, because all the poles are clustered on the unit circle, which entails numerical round-off error. Figure 2 shows that at
least 10 output measurements are needed to guarantee the full observability of the system. Hence, we use 50 measurements to ensure that the system is observable. Since the linearized MHD system is marginally observable as shown in Figure 3, all the three methods converge very slowly (see Figure 4).

The classical KF has a better performance compared to the others, because the error dynamics of the Kalman filter (Figure 3 top-right) is stable, whereas the error dynamics of the other filters are not. The KF gain approximation for the suboptimal techniques use only the most important modes, which is not possible for the linearized MHD system because all the modes are equally important. Hence, the dynamics of suboptimal KF estimators are oscillatory as shown in Figure 4.

VII. CONCLUSION

A discrete-time linearized model for the flow of plasma in a two-dimensional channel was obtained. The obtained model is marginally stable and has oscillatory dynamics which makes the system difficult to observe. Three different state space observers were studied, namely, Kalman filter, RRSQRT-KF filter, and SSQRT-KF filter.

Although the Kalman filter performs well, and is very reliable and robust as shown in Figure 4, it converges slowly. Another drawback of the Kalman filter is that it is prohibitive to compute for large scale systems. On the other hand, RRSQRT-KF and SSQRT-KF exhibit numerical problems due to the fact that the system is marginally observable making it difficult to find a stable suboptimal KF.

Moreover, these methods are based on the square root algorithm, and hence their convergence rate is low when the eigenvalues of $A_2$ are located very close to the unit circle (see Figure 3). Therefore, we cannot exploit the advantages of the proposed suboptimal Kalman filters for the linearized MHD model.

REFERENCES