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A SDRE-Based Asymptotic Observer for Nonlinear Discrete-Time Systems

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***Abstract*— A nonlinear asymptotic observer for a discrete-time nonlinear system is considered. The observer is based on a Kalman filter that uses the state dependent Riccati equation (SDRE) to obtain the filter gain. Unlike the Extended Kalman Filter, the SDRE-based Kalman filter does not involve the evaluation of a Jacobian at every time step. The convergence properties of the SDRE-based Kalman filter when used as an observer in a deterministic setting are analyzed. A few simulation examples are provided to demonstrate the performance and implementation of the SDRE-based observer in both deterministic and stochastic settings.**

I. Introduction

The problem of determining the states of a dynamic system using measurements of the plant output has been widely studied for many decades. For linear time-invariant systems in a deterministic stetting, the Ljungberg observer [1] provides estimates of the state that asymptotically converge to the actual state of the system. In a stochastic setting, the classical Kalman filter [2] provides optimal least-squares estimates of all of the states of a linear time-varying system under process and measurement noise. Although some results on optimal and suboptimal filtering of continuous-time processes do exist, for example, [3] and [4], respectively, the theory of observers and Kalman filtering for estimating the states of a nonlinear system is still not well developed.

Despite the dearth in results on exact optimal nonlinear filters, a wide variety of approximate filters that are not optimal have been developed to estimate the state of nonlinear systems. The extended Kalman filter uses the Jacobian of the nonlinearity in the dynamics and applies linear Kalman filtering techniques to the nonlinear system [5, 6]. In a deterministic setting, the extended Kalman filter can be used as an observer and results that guarantee the convergence of the state estimates to the actual state are available [7, 8]. The extended Kalman filter requires that the covariance of the error be propagated at every time step which is computationally expensive for large scale systems. The ensemble Kalman filter [9] and unscented Kalman filter [10] do not explicitly propagate the error covariance, but instead construct the error covariance by averaging the estimates from a number of state estimators (ensemble) that are initialized with different initial conditions. Furthermore, these methods do not require that the Jacobian of the nonlinearity in the dynamics be known.

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A completely different "adhoc" approach to state estimation of nonlinear systems is the SDRE (state dependent Riccati equation) based Kalman filter [11]. The nonlinear system is viewed as a frozen-in-time linear time varying system, and the Riccati equations used in the state estimation of linear systems are used to determine the Kalman filter gain. In [11], an algebraic Riccati equation is solved at every time step to obtain the Kalman filter gain which can be computationally expensive for large scale systems. Furthermore, if loss of observability occurs during certain time-intervals, then the algebraic Riccati equation may not have a solution and the algebraic Riccati equation based SDRE Kalman filter cannot be used during these time-intervals.

In this paper we consider a SDRE based Kalman filter that uses the Riccati update equation to propagate the error-covariance and determine the filter gain. We apply the SDRE based Kalman filter in a deterministic setting and analyze its performance as an observer. We modify the results obtained in [7] to provide proof of convergence of the error between the state estimates and the actual states of a discrete-time nonlinear system. Finally, we provide simulation examples that demonstrate the use of the SDRE based Kalman filter as an asymptotic observer in a deterministic setting and state estimator in a noisy environment, for nonlinear discrete-time systems.

II. SDRE Based Observer

Consider the discrete-time nonlinear system

$$x_{k+1} \quad = f(x_k), \tag{2.1}$$

with output

$$y_k \quad = C_k x_k, \tag{2.2}$$

where $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^p$. Consider an observer of the form

$$\hat{x}_{k+1} = f(\hat{x}_k^+), \tag{2.3}$$

$$\hat{y}_k = C_k \hat{x}_k, \tag{2.4}$$

where

$$\hat{x}_k^+ = \hat{x}_k + F_k \left(y_k - \hat{y}_k \right). \tag{2.5}$$

The observer gain $F_k \in \mathbb{R}^{n \times p}$ is given by

$$F_k = Q_k C_k^{\mathrm{T}} \left(C_k Q_k C_k^{\mathrm{T}} + V_2 \right)^{-1}, \tag{2.6}$$

where $Q_k \in \mathbb{R}^{n\times n}$ is updated using the state dependent Riccati equation (SDRE)

$$\hat{Q}_k^+ = (I - F_k C_k) Q_k, \tag{2.7}$$

$$Q_{k+1} = A_k \hat{Q}_k^+ A_k^{\mathrm{T}} + V_1, \tag{2.8}$$

and $A_k \triangleq A(\hat{x}_k^+) \in \mathbb{R}^{n\times n}$, where $A(x)$ is chosen such that for all $x \in \mathbb{R}^n$

$$A(x)x = f(x). \tag{2.9}$$

The input u_k has been ignored in (2.1)-(2.4) for simplicity.

Note that the parametrization $A(x)$ is not unique. If $A_1(x)$ and $A_2(x)$, are two distinct parametrizations of $f(x)$, that is,

$$A_1(x)x = f(x), \;\; A_2(x)x = f(x), \tag{2.10}$$

then, for all matrix function $M(x) \in \mathbb{R}^{n\times n}$, $\tilde{A}(x)$ is also a parametrization of $f(x)$, where

$$\tilde{A}(x) = M(x)A_1(x) + (I - M(x))A_2(x). \tag{2.11}$$

Note that there are two common formulations of the extended Kalman filter, namely, the two-step recursive update and the one-step recursive update, respectively. Similarly, the SDRE-based observer also has a two-step recursive formulation and a one-step recursive formulation. The two-step recursive formulation of the SDRE-based observer is given by (2.3)-(2.8). The one-step recursive update formulation of the SDRE-based observer is given by

$$\hat{x}_{k+1} = f(\hat{x}_k) + \overline{F}_k(y_k - \hat{y}_k), \tag{2.12}$$

$$\hat{y}_k = C_k \hat{x}_k. \tag{2.13}$$

The observer gain for the one-step formulation is given by

$$\overline{F}_k = A_k Q_k C_k^{\mathrm{T}} \left(C_k Q_k C_k^{\mathrm{T}} + V_2 \right)^{-1}, \tag{2.14}$$

where Q_k is updated using the state dependent Riccati equation

$$Q_{k+1} = A_k Q_k A_k^{\mathrm{T}} - A_k Q_k C_k^{\mathrm{T}} \hat{V}_{2,k}^{-1} C_k Q_k A_k^{\mathrm{T}} + V_1, \tag{2.15}$$

and $\hat{V}_{2,k} \triangleq C_k Q_k C_k + V_2$.

III. Bounds on the Riccati Equation

The nonlinear discrete-time equation (2.1) can be viewed as a frozen-in-time linear equation

$$x_{k+1} = A(x_k)x_k. \tag{3.1}$$

Note that (3.1) is not a linear time-varying system. However, we view $A_k \triangleq A(\hat{x}_k^+)$ appearing in the discrete-time Riccati update equation (2.8) as a time varying matrix and ignore the fact that A_k was obtained using the state $\hat{x}_k^+$. Next, we use the results presented in [7] and [12] to obtain *a priori* bounds on $\hat{Q}_k^+$ and Q_k that are updated using the state dependent Riccati equations (2.7) and (2.8). We denote the Euclidean norm of a vector by $|\cdot|$, and the induced norm of a matrix (maximum singular value) by $\|\cdot\|$.

Lemma 3.1: Assume there exists $\nu > 0$ such that for all $x \in \mathbb{R}^n$, the parametrization $A(x)$ that satisfies (2.9) is chosen such that $A(x)^{\mathrm{T}}A(x) \geqslant \nu I$. Furthermore, assume that V_2 is positive definite. If there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, and an integer $M > 0$ such that, for all $k \geq M$,

$$\alpha_1 I \leqslant \sum_{i=k-M}^{k-1} \Phi(k, i+1) V_1 \Phi^{\mathrm{T}}(k, i+1) \leqslant \alpha_2 I, \tag{3.2}$$

$$\beta_1 I \leqslant \sum_{i=k-M}^{k} \Phi^{-\mathrm{T}}(k, i) C_k^{\mathrm{T}} V_2^{-1} C_k \Phi(k, i)^{-1} \leqslant \beta_2 I, \tag{3.3}$$

then

$$\frac{1}{\beta_2 + \frac{1}{\alpha_1}} I \leqslant \hat{Q}_k^+ \leqslant \left(M \frac{\beta_2^2 \alpha_2}{\beta_1^2} + \frac{1}{\beta_1} \right) I, \tag{3.4}$$

where

$$\Phi(k, i) \triangleq A_{k-1} A_{k-2} \cdots A_i. \tag{3.5}$$

Proof of this lemma can be found in [7].

It follows from (2.7) and (3.4) that, for all $k > 0$

$$\|Q_k\| \leqslant \left(M \frac{\beta_2^2 \alpha_2}{\beta_1^2} + \frac{1}{\beta_1} \right) \|A_{k-1}\|^2 + \|V_1\|. \tag{3.6}$$

Note that (3.2) holds if there exists $\alpha \in \mathbb{R}$ such that for all $x_k \in \mathbb{R}$, $\|A(x_k)\| \leqslant \alpha$ and V_1 is positive definite. Furthermore, it can be shown that (3.3) holds if the following observability condition is satisfied, namely, there exist $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 I \leqslant \mathcal{O}^{\mathrm{T}}(k-M, k) \mathcal{O}(k-M, k) \leqslant \gamma_2 I, \tag{3.7}$$

where

$$\mathcal{O}(k-M, k) \triangleq \begin{bmatrix} C_{k-M} \\ C_{k-M+1} A_{k-M} \\ \vdots \\ C_k A_{k-1} \cdots A_{k-M} \end{bmatrix}. \tag{3.8}$$

IV. Convergence of the Error in the Estimates

Next, we use the bounds on Q_k and $\hat{Q}_k^+$ in (3.4) and (3.6) to determine sufficient conditions that guarantee asymptotic stability of the error dynamics. First, we make the following assumptions for setting up the error analysis.

Assumption 4.1:

i.) V_1 is positive definite

ii.) Q_k and $\hat{Q}_k^+$, propagated using (2.7) and (2.8), are uniformly bounded, that is, there exist $q, q^+ > 0$ such that, for all $k \geqslant 0$, $\|Q_k\| \leqslant q$ and $\|(\hat{Q}_k^+)^{-1}\| \leqslant q^+$.

iii.) $\alpha \triangleq \sup \|A_k\| < \infty$.

iv.) C_k is bounded, that is, there exists $c > 0$ such that, for all $k \geqslant 0$, $\|C\| \leqslant c$.

v.) The parametrization $A(x)$ of $f(x)$ is chosen such that, there exists $L > 0$, such that, for all $x_1, x_2 \in \mathbb{R}^n$, $\|A(x_1) - A(x_2)\| \leqslant L|x_1 - x_2|$.

vi.) Assume that there exists $\sigma > 0$, such that, for all $k \geqslant 0$, $|x_k| \leqslant \sigma$.

Note that Lemma 3.1 provides a condition when ii.) of Assumption 4.1 is satisfied. Define $P_k \in \mathbb{R}^{n\times n}$ by

$$P_k \triangleq Q_k^{-1}. \tag{4.1}$$

It follows from (2.6) and (2.7) that

$$P_k = \left(\hat{Q}_k^+\right)^{-1} - C_k^{\mathrm{T}} V_2^{-1} C_k. \tag{4.2}$$

Hence, ii.) of Assumption 4.1, (4.1) and (4.2) imply that

$$\frac{1}{q} \leqslant \|P_k\| \leqslant p, \tag{4.3}$$

where $p \triangleq q^+ + c^2\|V_2^{-1}\|$. Furthermore, it follows from i.), ii.), and iii.) of Assumption 4.1 that

$$\|\left(Q_k^+\right)^{-1} + A_k^{\mathrm{T}} V_1^{-1} A_k\| \leqslant r, \tag{4.4}$$

where $r \triangleq q^+ + \alpha^2\|V_1^{-1}\|$. Finally, it follows from (2.7) and (4.1) that

$$(I - F_k C_k) = \hat{Q}_k^+ Q_k^{-1} = \hat{Q}_k^+ P_k \tag{4.5}$$

which implies that

$$\|I - F_k C_k\| \leqslant p q^+ \tag{4.6}$$

Theorem 4.1: Consider the nonlinear discrete-time system (2.1)-(2.2) and the associated SDRE based observer (2.3)-(2.9). Suppose that Assumption 4.1 holds, then, if

$$\sigma L r p^3 (q^+)^2 (2\alpha + \sigma L) < \frac{1}{q^2}, \tag{4.7}$$

then the SDRE based observer is an asymptotic observer for the deterministic system (2.1)-(2.2), that is, $x_k - \hat{x}_k \to 0$ asymptotically as $k \to \infty$.

Proof Let $e_k \triangleq x_k - \hat{x}_k$. Then

$$e_{k+1} = f(x_k) - f(\hat{x}_k^+). \tag{4.8}$$

Next, define $\tilde{e}_k \in \mathbb{R}^n$ by

$$\tilde{e}_k \triangleq \hat{x}_k - \hat{x}_k^+. \tag{4.9}$$

Substituting (2.5) into (4.9) and using (2.2) and (2.4) yields

$$\begin{aligned} \tilde{e}_k &= x_k - \hat{x}_k - F_k(y_k - \hat{y}_k), \\ &= x_k - \hat{x}_k - F_k C_k \left(x_k - \hat{x}_k\right), \\ &= (I - F_k C_k) e_k. \end{aligned} \tag{4.10}$$

Note that (4.8) can be expressed as

$$e_{k+1} \quad = f(x_k) - A_k \hat{x}_k^+. \tag{4.11}$$

Adding and subtracting $A_k x_k$ to the right hand side of (4.11) yields

$$\begin{aligned} e_{k+1} &= A_k \left(x_k - \hat{x}_k^+\right) + f(x_k) - A_k x_k, \\ &= A_k \tilde{e}_k + \left[f(x_k) - A_k x_k\right], \\ &= A_k \left(I - F_k C_k\right) e_k + b_k, \end{aligned} \tag{4.12}$$

where

$$b_k \triangleq f(x_k) - A_k x_k. \tag{4.13}$$

Define $V(P_k, e_k) \in \mathbb{R}$ by

$$V(P_k, e_k) \triangleq e_k^{\mathrm{T}} P_k e_k, \tag{4.14}$$

where P_k is defined in (4.1). Define $\Delta V(P_k, e_k)$ by

$$\Delta V(P_k, e_k) \triangleq e_{k+1}^{\mathrm{T}} P_{k+1} e_{k+1} - e_k^{\mathrm{T}} P_k e_k. \tag{4.15}$$

Substituting (4.12) into (4.15) yields

$$\begin{aligned} V(P_k, e_k) = e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k \\ + b_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k + b_k P_{k+1} b_k^{\mathrm{T}} \\ + e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} b_k - e_k^{\mathrm{T}} P_k e_k. \end{aligned} \tag{4.16}$$

It follows from [7] that

$$\begin{aligned} e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k - e_k^{\mathrm{T}} P_k e_k \\ \leqslant -e_k^{\mathrm{T}} Q_k^{-1} \left(\left(Q_k^+\right)^{-1} + A_k^{\mathrm{T}} V_1^{-1} A_k\right)^{-1} Q_k^{-1} e_k. \end{aligned} \tag{4.17}$$

Note that (4.16) and (4.17) imply that

$$\begin{aligned} \Delta V(P_k, e_k) \leqslant &- e_k^{\mathrm{T}} P_k \left(\left(Q_k^+\right)^{-1} + A_k^{\mathrm{T}} V_1^{-1} A_k\right)^{-1} P_k e_k \\ &+ b_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k \\ &+ e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} b_k + b_k P_{k+1} b_k^{\mathrm{T}}. \end{aligned} \tag{4.18}$$

It follows from Assumption 4.1 and (4.6) that

$$\begin{aligned} & b_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k \\ &\quad + e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} b_k + b_k P_{k+1} b_k^{\mathrm{T}} \\ &\quad \leqslant \|P_{k+1}\| |b_k| \left(2\|A_k\| \|I - F_k C_k\| |e_k| + |b_k|\right) \\ &\quad \leqslant p |b_k| \left(2\alpha p q^+ |e_k| + |b_k|\right). \end{aligned} \tag{4.19}$$

Note that b_k defined by (4.13) can be expressed as

$$\begin{aligned} b_k &= f(x_k) - A_k x_k \\ &= A(x_k) x_k - A(\hat{x}_k^+) x_k \\ &= \left[A(x_k) - A(\hat{x}_k^+)\right] x_k. \end{aligned} \tag{4.20}$$

Hence, it follows from Assumption 4.1 and (4.6) that

$$\begin{aligned} |b_k| &\leqslant \|A(x_k) - A(\hat{x}_k^+)\| |x_k| \\ &\leqslant L\sigma |\tilde{e}_k| \\ &\leqslant L\sigma \|I - F_k C_k\| |e_k| \\ &\leqslant L\sigma p q^+ |e_k|. \end{aligned} \tag{4.21}$$

Substituting (4.21) into (4.19) yields

$$\begin{aligned} & b_k^{\mathrm{T}} P_{k+1} A_k \left(I - F_k C_k\right) e_k \\ &\quad + e_k^{\mathrm{T}} \left(I - F_k C_k\right)^{\mathrm{T}} A_k^{\mathrm{T}} P_{k+1} b_k + b_k P_{k+1} b_k^{\mathrm{T}} \\ &\quad \leqslant \|P_{k+1}\| |b_k| \left(2\|A_k\| \|I - F_k C_k\| |e_k| + |b_k|\right) \\ &\quad \leqslant \sigma L p^3 (q^+)^2 (2\alpha + \sigma L) |e_k|^2. \end{aligned} \tag{4.22}$$

Furthermore, it follows from (4.3) and (4.4) that

$$\begin{aligned} e_k^{\mathrm{T}} P_k \left(\left(Q_k^+\right)^{-1} + A_k^{\mathrm{T}} V_1^{-1} A_k\right)^{-1} P_k e_k &\geqslant \frac{1}{r} |P_k e_k|^2 \\ &\geqslant \frac{1}{q^2 r} |e_k|^2. \end{aligned} \tag{4.23}$$

Hence, it follows from (4.18), (4.22) and (4.23) that

$$\Delta V(P_k, e_k) \leqslant -\phi(|e_k|^2), \tag{4.24}$$

where

$$\phi(|e_k|^2) \triangleq \left[\frac{1}{q^2 r} - \sigma L p^3 (q^+)^2 (2\alpha + \sigma L)\right] |e_k|^2. \tag{4.25}$$

Note that since P_k is positive definite $V(P_k, e_k)$ is positive definite and hence if $\phi(|e_k|^2)$ is a class-$\mathcal{K}$ function, it then follows from (4.24) that $e_k \to 0$ asymptotically as $k \to \infty$. Furthermore, (4.7) implies that $\phi(|e_k|^2)$ is a class-$\mathcal{K}$ function. Hence, if (4.7) is satisfied then $e_k \to 0$ asymptotically as $k \to \infty$ and (2.3) is an asymptotic observer for (2.1). □

V. Simulation Example

Consider the following discrete-time nonlinear system with state $x \triangleq \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$ and dynamics

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} 0.01x_{1,k} - x_{2,k} \\ x_{1,k} - 0.003x_{2,k}^2 \end{bmatrix} \tag{5.1}$$

and output

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k}, \end{bmatrix} \tag{5.2}$$

so that for all $k \geqslant 0$, $C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$. Consider the following parametrization $A(x)$

$$A(x) = \begin{bmatrix} 0.01 & -1 \\ 1 & -0.003x_2 \end{bmatrix} \tag{5.3}$$

so that (5.1) can be expressed as $x_{k+1} = A(x_k)x_k$. Hence, it follows from (5.3) that for all $x, \tilde{x} \in \mathbb{R}^n$,

$$A(x) - A(\tilde{x}) = \begin{bmatrix} 0 & 0 \\ 0 & -0.003(x_2 - \tilde{x}_2) \end{bmatrix}. \tag{5.4}$$

Hence, (5.4) implies that

$$\|A(x) - A(\tilde{x})\| \leqslant L|x - \tilde{x}|, \tag{5.5}$$

where $L = 0.003$. The system (5.1) is simulated from an initial condition x_0. The SDRE based observer is simulated using (2.3)-(2.7) with initial condition $\hat{x}_0 \neq x_0$. We choose $V_1 = 10I$ and $V_2 = I$. The values of p, q^+, r, q, and σ are evaluated after the entire simulation has been performed and are listed in Table 1. Hence, it can be seen from Table

Bound	Value
p	0.06
q	70.67
q^+	1.06
r	1.11
α	1
σ	0.66
L	0.003
$\sigma L r p^3 (q^+)^2 (2\alpha + \sigma L) - \frac{1}{q^2}$	-0.0001

TABLE I

Values of various bounds used in Theorem 4.1.

1 that (4.7) is satisfied which implies that $\phi(|e_k|^2)$ defined in (4.25) is a class-$\mathcal{K}$ function for this example and hence $e_k \to 0$ asymptotically as $k \to \infty$. Figure 1 shows the actual state x_k and the estimates obtained from the SDRE based observer. Figure 2 shows the norm of the error in the estimates. The norm of the error in the estimates when $F_k \equiv 0$ is also shown by dotted lines in Figure 2. Figure 3 shows a plot of $V(P_k, e_k)$ when the SDRE-based estimator is used.

VI. Simulation Example : Van der Pol Oscillator

Next, consider the following discrete-time model of the Van der Pol Oscillator

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + hx_{2,k} \\ x_{2,k} + h\left[(1 - x_{1,k}^2)x_{2,k} - x_{1,k}\right] \end{bmatrix}, \tag{6.1}$$

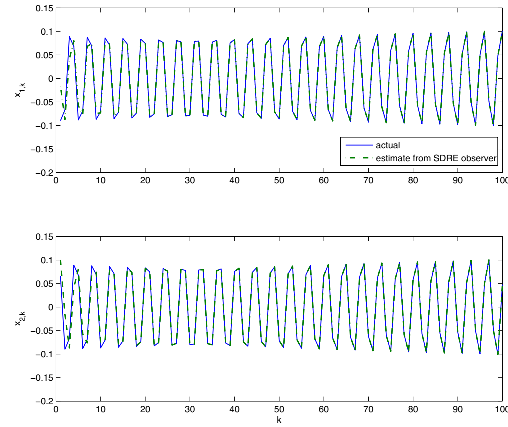

Fig. 1. The actual state x_k is shown as solid lines. The state estimates obtained from the SDRE based observer are plotted using dashed lines.

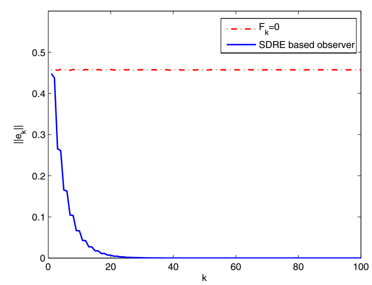

Fig. 2. The norm of the error in the estimates is plotted in this figure. Note that $e_k \to 0$ as k becomes large. The norm of the error in the estimates when $F_k = 0$ for all $k \geqslant 0$ is also shown in this figure.

where $h = 0.15$ is the sampling time. Let the output y_k be given by

$$y_k = x_{1,k}, \tag{6.2}$$

so that $C_k = \begin{bmatrix} 1 & 0 \end{bmatrix}$ for all $k \geqslant 0$. Consider the following parametrization of $A(x)$

$$A(x) = \begin{bmatrix} 1 & h \\ -h(1 + x_1x_2) & 1 + h \end{bmatrix}, \tag{6.3}$$

so that $x_{k+1} = A(x_k)x_k$. Note that it is difficult to analytically determine L that satisfies v.) of Assumption 4.1. However, we still use the SDRE-based observer (2.3)-(2.8) to obtain the estimate $\hat{x}_k$ of the state x_k in (6.1).

We choose $V_1 = 10I$ and $V_2 = I$, and choose initial conditions so that $\hat{x}_0 \neq x_0$. The state estimates and the actual state of the Van der Pol Oscillator are shown in Figure 4 and the norm of error between the state estimates and the actual state is shown in Figure 5. It can be seen from Figure 5 that the error $e_k \to 0$ asymptotically, as $k \to \infty$. The result in this paper provides sufficient conditions to guarantee the asymptotic convergence of the error to 0, and as seen in Figure 5, they are not necessary conditions. Furthermore, since the bounds obtained using the norm

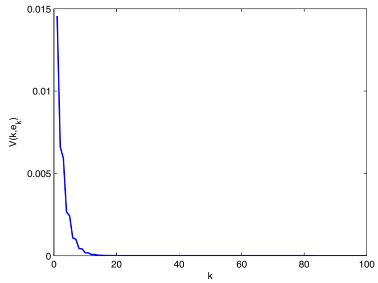

Fig. 3. Potential function $V(k, e_k)$ for the example in Section 5. Note that $V(P_k, e_k)$ is always decreasing and thus for all $k > 0$, $\Delta V(P_k, e_k) < 0$.

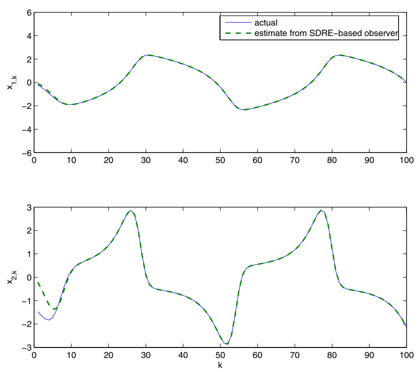

Fig. 4. The actual state and the estimates obtained using the SDRE-based observer are plotted for the Van der Pol oscillator. Although Theorem 4.1 cannot be used to guarantee the asymptotic convergence of the estimates to the actual state, it can be seen from the figure that the state estimates do converge to the actual states.

operators are conservative, the sufficient conditions cannot be satisfied easily.

VII. Simulation Example : One-Dimensional Hydrodynamic Flow

Consider a compressible and inviscid fluid flowing across a one-dimensional channel. The flow dynamics are given by Euler's equations which contains coupled PDE's. However, a finite-volume discrete-time model of the hydrodynamic flow can be obtained using the upwind Roe's scheme. If Neumman boundary conditions are used at the first cell and Dirichlet boundary condition are used at the last cell, it then follows from [13] that the state update equation is

$$x_{k+1} = f(x_k, u_{\mathrm{BC},k} + w_{\mathrm{BC},k}), \tag{7.1}$$

where $x \in \mathbb{R}^{3(n-2)}$ and $u_{\mathrm{BC}} \in \mathbb{R}^3$ is defined by

$$\begin{aligned} x &\triangleq \begin{bmatrix} \varrho_2 & m_2 & \mathcal{E}_1 & \cdots & \varrho_n & m_n & \mathcal{E}_n \end{bmatrix}^{\mathrm{T}}, \\ u_{\mathrm{BC}} &\triangleq \begin{bmatrix} \varrho_1 & m_1 & \mathcal{E}_1 \end{bmatrix}^{\mathrm{T}} \end{aligned} \tag{7.2}$$

and for $i = i, \ldots, n$, ϱ_i, m_i, and $\mathcal{E}_i \in \mathbb{R}$ are the density, momentum, and energy, at the center of the ith cell (indicated by black dots in Figure 6), respectively. The entries and structure of $f(\cdot)$ in (7.1) depend on the flow

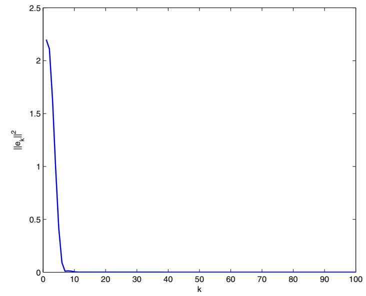

Fig. 5. The norm of the error in the estimates is plotted for the Van der Pol oscillator. Even though the assumptions in theorem 4.1 are not satisfied, the error does converge to 0 asymptotically.

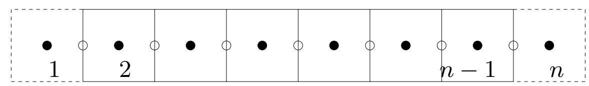

Fig. 6. One-dimensional grid used in the finite volume scheme

variables at the cell interface (indicated by circles in Figure 4) and are defined in [13]. Note that $u_{\mathrm{BC},k}$ is the boundary condition at the first cell and is assumed to be known, however $w_{\mathrm{BC},k} \in \mathbb{R}^3$ represents the unmodeled drivers and is unavailable for all $k \geqslant 0$. The nonlinear discrete-time update equation (7.1) can be expressed as

$$x_{k+1} = A(x_k)x_k + B(x_k, u_{\mathrm{BC},k} + w_{\mathrm{BC},k}), \tag{7.3}$$

so that (7.3) resembles a frozen-in-time state dependent linear equation. Note that the parametrization of $A(x_k)$ and $B(x_k, u_{\mathrm{BC},k} + w_{\mathrm{BC},k})$ is not unique. Let y_k be the measurement of density, momentum and energy at certain cells so that

$$y_k = Cx_k + D_2 w_k, \tag{7.4}$$

where w_k is the sensor noise with zero-mean and unit covariance. Note that entries of C are either 1's or 0's depending on the cells where measurements are available.

Let $n = 20$ so that $x \in \mathbb{R}^{54}$. For all $k \geqslant 0$, the boundary condition at the first cell is given by

$$u_{\mathrm{BC},k} = \begin{bmatrix} \varrho_{1,k} \\ m_{1,k} \\ \mathcal{E}_{1,k} \end{bmatrix} = \begin{bmatrix} 1 \\ 12 + \sin(20k) \\ 87 + 0.5\sin^2(20k) + 12\sin(20k) \end{bmatrix}. \tag{7.5}$$

The boundary condition at the first cell in (7.5) is chosen such that the flow is supersonic at all the cells with Mach number between 4 and 5. Assume that the unmodeled driver w_{BC} is zero-mean noise with unit covariance. Let $y_k \in \mathbb{R}^6$ be the measurements of density, momentum and energy at the 5th and 10th cell. The objective is to estimate the density, momentum and energy at the cells where measurements of flow variables are unavailable.

Note that the SDRE observer discussed in the previous sections was based on a deterministic setting but the one-dimensional flow example (7.1) and (7.4) involve nonlinear dynamics in a noisy-environment. Hence, Theorem 4.1 cannot be applied to this example. However, we

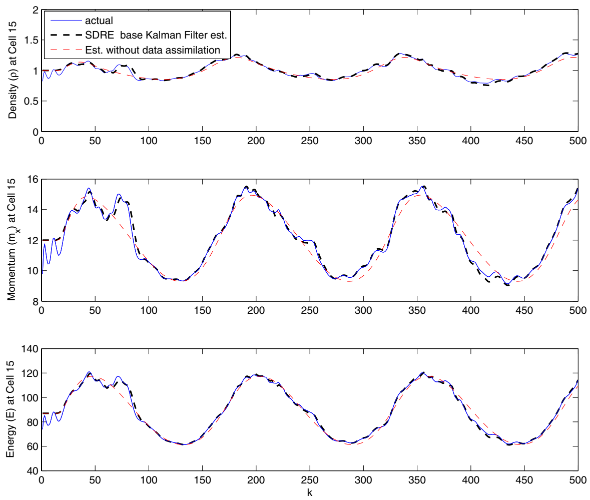

Fig. 7. The actual value of density, momentum and energy at Cell 15 is shown along with the estmates obtained from the SDRE based Kalman filter. The estimates obtained if no data assimilation was performed is shown as dashed lines.

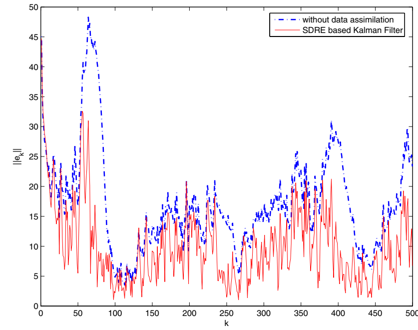

Fig. 8. The error in the estimates when data assimilation is performed using the SDRE based Kalman filter and when no data assimilation is performed is shown in this figure. Even though the performance of the SDRE based observer was analyzed in a deterministic setting, the SDRE based Kalman filter also yields good estimates of the states in a noisy environment.

use the SDRE based observer discussed in Section 2 as an estimator to obtain an estimate $\hat{x}_k$ of the state x_k. The estimator dynamics are given by

$$\begin{aligned} \hat{x}_{k+1} &= f(\hat{x}_k^+, u_{\mathrm{BC},k}), \\ \hat{x}_k^+ &= \hat{x}_k + F_k\left(y_k - \hat{y}_k\right), \end{aligned} \tag{7.6}$$

where F_k is obtained using (2.6) and (2.7). We let $A_k = A(\hat{x}_k^+)$ in (2.7), where $A(x)$ is a parametrization such that for all $x \in \mathbb{R}^n$

$$A(x)x + B(x, u_{\mathrm{BC}} + w_{\mathrm{BC}}) = f(x, u_{\mathrm{BC}} + w_{\mathrm{BC}}). \tag{7.7}$$

Furthermore, since the unmodeled driver $w_{\mathrm{BC},k}$ enters the system through $B(x_k, u_{\mathrm{BC},k} + w_{\mathrm{BC},k})$ in (7.3), we replace V_1 in (2.7) by $V_{1,k} = B(\hat{x}_k^+, u_{\mathrm{BC},k})B(\hat{x}_k^+, u_{\mathrm{BC},k})^{\mathrm{T}}$, and choose D_2 in (7.4) such that $V_2 = D_2 D_2^{\mathrm{T}}$ is positive definite.

Figure 7 shows the actual density, momentum and energy at the 15th cell along with the estimates obtained using the SDRE based Kalman filter. The estimate when no data assimilation is performed, that is, $F_k \equiv 0$ for all $k \geqslant 0$, is also shown in Figure 1. The norm of the error in the estimates $\|x_k - \hat{x}_k\|$ for the two cases, the SDRE based Kalman filter and when no data assimilation is performed is shown in Figure 8.

VIII. Conclusion

In this paper, we develop an observer for discrete-time nonlinear systems. The nonlinear dynamic equation is viewed as a frozen-in-time linear equation and the state dependent Riccati update equation is used to determine the Kalman filter gain at every time step. An advantage of using the SDRE-based observer over the extended Kalman filter is that knowledge of the Jacobian of the nonlinearity in the dynamics, which may be difficult to evaluate for large scale systems, is not necessary. We analyze the performance of the SDRE based Kalman filter when used as an observer in a deterministic setting. We provide sufficient conditions that guarantee asymptotic convergence of the state estimates to the actual state. Furthermore, we show by an example that the conditions that guarantee asymptotic convergence are only sufficient and not necessary. The SDRE-based Kalman filter is then used as an estimator in a one-dimensional hydrodynamic flow example.

References

[1] D. G. Luenberger, "Observers for Multivariable Systems," *IEEE Trans. Auto. Contr.,* vol. AC-11, 1966, pp. 563-603.

[2] R. E. Kalman, "A New Approach to Linear Filter and Prediction Theory," *J. Basic. Engr.*, vol. 82D, pp. 35-45, 1960.

[3] F. E. Daum, "Exact Finite-Dimensional Nonlinear Filters," *IEEE Trans. Auto. Contr.,* vol. AC-31, 1986, pp. 616-622.

[4] M. Athans, R. P. Wishner and A. Bertolini, "Suboptimal State Estimation for Continous-Time Nonlinear Systems from Discrete Noisy Measurements," *IEEE Trans. Auto. Contr.,* vol. AC-13, 1968, pp. 504-514.

[5] A. Gelb, *Applied Optimal Estimation,* The M.I.T Press, 1974.

[6] B. D. O. Anderson and J. B. Moore, *Optimal filtering,* Englewood Cliffs, NJ, 1979.

[7] J. W. Grizzle and Y. Song, "The Extended Kalman Filter as a Local Asymptotic Observer for Nonlinear Discrete-Time Systems," *Journal of Mathematical Systems, Estimation and Control,* vol. 5, no. 1, 1995, pp. 59-78.

[8] K. Reif and R. Unbehauen, "The Extended Kalman Filter as an Exponential Observer for Nonlinear System," *IEEE Trans. Sig. Proces.,* vol. 47, no. 8, 1999, pp. 2324-2328.

[9] G. Evensen, "The ensemble Kalman filter : theoretical formulation and practical implementation," *Ocean Dynamics,* vol. 53, 2003.

[10] E. A. Wan and Rudolph van er Merwe, "The Unscented Kalman Filter for Nonlinear Estimation," *Adap. Sys. Sig. Proces., Comm., and Contr. Symp.,* 2000, pp. 153 - 158.

[11] C. P. Mracek, J. R. Cloutier, and C. A. D'Souza, "A New Technique for Nonlinear Estimation," *Proc. IEEE Conf. Contr. App.,* 1996, pp. 338-343.

[12] J. J. Deyst, Jr., and C. F. Price, "Conditions for Asymptotic Stability of the Discrete Minimum-Variance Linear Estimator," *IEEE Trans. Auto. Cont.,* vol. 13, no. 6, 1968, pp. 702-705.

[13] C. Hirsch, *Numerical Computation of Internal and External Flows,* pp. 408-469, John Wiley and Sons, 1994.