$l$-Delay Input Reconstruction for Discrete-Time Linear Systems

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Abstract—As an extension of existing results on input reconstruction, we define $l$-delay state and input reconstruction, and we characterize this property through necessary and sufficient conditions. This property is shown to be a stronger notion of left invertibility, in which the initial state is assumed to be known. We demonstrate $l$-delay state and input reconstruction on several numerical examples, which show how the input reconstruction error depends on the locations of the zeros. Specifically, minimum-phase zeros give rise to decaying input reconstruction error, nonminimum-phase zeros give rise to growing reconstruction error, and zeros on the unit circle give rise to persistent reconstruction error.

I. INTRODUCTION

Input reconstruction, that is, the problem of inferring the inputs to a system from available measurements, impacts numerous applications, ranging from machine tool control to cryptography, filtering, and coding. In particular, when unknown inputs represent disturbances or the effects of system uncertainties, estimates can be used to improve the control system performance. In addition, when unknown inputs represent the effect of system faults, estimates can be used to implement fault-tolerant control, and thus enhance system reliability.

The reconstruction of unknown inputs under the assumption that the initial state of the system is either known or zero is addressed by left inversion, a long-studied problem in the systems literature [1]–[5]. As to structural left inversion, necessary and sufficient conditions were obtained in [1]–[4] for the existence of a linear time-invariant dynamical system that, when cascaded with the original system, produces as its output the input to the original system. These conditions are given in terms of a rank condition on matrices made up of either the system matrices or the system Markov parameters. The issue of internal stability of the resulting cascade system was subsequently addressed in [5], which considers both left inversion and the dual problem of right inversion. From then on, the issue of internal stability and the effect of nonminimum-phase zeros were tackled within the context of right inversion along with the related notions of noncausal inversion, preview, preaction, and steering along zeros [6]–[12]. A unified solution of noncausal right and left inversion is given in [13].

Although the problem of left inversion has been extensively studied, the more realistic problem of determining the unknown inputs from the measurements when the system initial state is nonzero and unknown is still open. In the continuous time case, [14] defines the notion of an unknown-state, unknown-input reconstructable system and characterizes that property through a necessary and sufficient geometric condition, which implies, in particular, that the system has no invariant zeros. Later, [15] and [16] present state and input asymptotic reconstructors. In particular, the reconstructor of [16] does not require differentiators but ensures convergence up to an arbitrary degree of accuracy. More recently, [17] introduces a modified version of the reconstructor of [16], which provides an asymptotic estimate of both a function of the state and the unknown input; in fact, in [16] it is observed that, in general, the entire state is not needed to reconstruct the unknown inputs.

For discrete-time, minimum-phase systems, [18] introduces a constructive algorithm for establishing whether the system under consideration is finite-time observable and left invertible with or without sampling delays, and provides a procedure for obtaining an estimator that allows both the state and the unknown inputs to be retrieved after a finite number of sampling intervals.

In this paper we extend the results of [19] on one-step-delay exact input reconstruction in discrete-time systems by deriving a reconstructor for $l$-step-delay reconstruction, where the increase in the delay takes into account the relative degree of the system. This technique is based on an input-output model that depends on the observability matrix and a block-Toeplitz matrix of Markov parameters. We apply this technique to several numerical examples that show how the input reconstruction error depends on the locations of the zeros. Specifically, we show that minimum-phase zeros give rise to decaying input reconstruction error, nonminimum-phase zeros give rise to growing reconstruction error, and zeros on the unit circle give rise to persistent reconstruction error. The effect of nonminimum-phase zeros shows that this feature increases the delay needed to obtain accurate estimates of the unknown inputs.

In Section 2, we state the objective of the paper and introduce notation. In Section 3, we provide a detailed review of necessary and sufficient conditions for $l$-delay left invertibility. These results unify and extend standard conditions for left invertibility, and thus provide the foundation for the subsequent results on input reconstruction. In Section 4, we focus on the problem of input reconstruction with $l$-step delay when the initial conditions of the system are nonzero and unknown. The basic notion of $l$-delay input

This work was partially sponsored by NASA under IRAC grant NNX08AB92A.

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and state observability is defined and characterized through necessary and sufficient algebraic conditions. Sections 5 and 6 illustrates the effect of invariant zeros on the solvability of the state and input reconstruction problem. Several numerical examples help to clarify the new notions and illustrate the effectiveness of the proposed methods.

II. PROBLEM STATEMENT

Consider the linear discrete-time system

\[ x_{k+1} = Ax_k + Bu_k, \quad (1) \]
\[ y_k = Cx_k + Du_k, \quad (2) \]

where \( u_k \in \mathbb{R}^m, x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^p, A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, \) and \( D \in \mathbb{R}^{p \times m}. \) We assume that \((A, B, C)\) is minimal. The \( p \times m \) transfer function of (1), (2) is \( G(z) = C(zI_n - A)^{-1}B + D. \) For each nonnegative integer \( r, \) we define the output sequence \( Y_{[k:k+r]} \in \mathbb{R}^{r+p} \) and the input sequence \( U_{[k:k+r]} \in \mathbb{R}^{r+p} \) by

\[ Y_{[k:k+r]} \triangleq \begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+r} \end{bmatrix}, \quad U_{[k:k+r]} \triangleq \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+r} \end{bmatrix}. \quad (3) \]

The objective is to use measurements of the output vector \( Y_{[0:r_1]} \) to determine the initial state \( x_0 \) and the input vector \( U_{[0:r_2]} \), where \( r_1 \geq r_2 \geq 0. \) The goal is to determine values of \( r_1 \) and \( r_2 \) for which input reconstruction is either exactly or approximately possible.

Note that \( Y_{[0:r]}, U_{[0:r]}, \) and \( x_0 \) are related by

\[ Y_{[0:r]} = \Gamma_r x_0 + M_{r,r} U_{[0:r]} = \Psi_{r,r} \begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix}, \quad (4) \]

where, for \( r_1 \geq r_2 \geq 0, \) \( \Gamma_{r_1} \in \mathbb{R}^{p(r_1+1) \times n}, \) \( M_{r_1,r_2} \in \mathbb{R}^{p(r_1+1) \times m(r_2+1)} \), and \( \Psi_{r_1,r_2} \in \mathbb{R}^{p(r_1+1) \times [n+m(r_2+1)]} \) are defined by

\[ \Gamma_{r_1} \triangleq \begin{bmatrix} C & CA & \cdots & CA^{r_1} \\ \end{bmatrix}, \quad M_{r_1,r_2} \triangleq \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ H_1 & H_0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ H_{r_1} & H_{r_1-1} & \cdots & H_{r_1-r_2} \end{bmatrix}, \quad \]

and

\[ \Psi_{r_1,r_2} \triangleq \begin{bmatrix} \Gamma_{r_1} & M_{r_1,r_2} \end{bmatrix}, \]

and where

\[ H_k \triangleq \begin{cases} D, & k = 0, \\ CA^{k-1}B, & k \geq 1, \end{cases} \]

follows that

\[ G(z) = \sum_{i=0}^{\infty} z^{-i}H_i. \quad (5) \]

Throughout this paper, \( d \) denotes the relative degree of \( G(z), \) that is, the smallest nonnegative integer \( i \) such that \( H_i \neq 0. \)

III. LEFT INVERTIBILITY OF TRANSFER FUNCTIONS

In this section we unify and extend known results on the left invertibility of \( G(z) \). These results set the stage for results on input reconstruction. A transfer function \( F(z) \) is proper if it is either exactly proper \((F(\infty) \neq 0)\) or strictly proper \((F(\infty) = 0).\)

Definition 1. Let \( l \) be a nonnegative integer. Then \( G(z) \) is \( l \)-delay left invertible and the \( m \times p \) proper transfer function \( G_l(z) \) is an \( l \)-delay left inverse of \( G(z) \) if \( G_l(z)G(z) = z^{-l}I_m \) for almost all \( z \in \mathbb{C}. \) In this case, the smallest nonnegative integer \( l_0 \) for which \( G(z) \) is \( l_0 \)-delay left invertible is the minimal delay of \( G. \) Finally, \( G \) is left invertible if there exists a nonnegative integer \( q \) such that \( G \) is \( q \)-delay left invertible.

Note that if \( G(z) \) is \( l \)-delay left invertible, then \( G(z) \) is \( r \)-delay left invertible for all \( r \geq l. \)

Theorem 1. The following statements are equivalent:

(i) \( G(z) \) is left invertible.
(ii) The normal rank of \( G(z) \) is \( m. \)
(iii) The normal rank of \( [zI-A \quad -B \\ C \quad D] \) is \( n+m. \)

Theorem 1 shows that if \( G(z) \) is \( l \)-delay left invertible, then \( G_l(z)G(z) = z^{-l}I_m \) holds for all \( z \in \mathbb{C} \) except the poles of \( G(z) \) and \( G_l(z) \) and, if \( l \geq 1, z = 0. \)

Now, for \( r \geq l \geq 0, \) partition \( M_{r,r} \) as

\[ M_{r,r} = [M_{r,l} \quad \tilde{M}_{r,r} \quad \tilde{M}_{r,l}], \quad (6) \]

where \( \tilde{M}_{r,l} \in \mathbb{R}^{p(r+1) \times ml}. \) Noting that

\[ \Psi_{r,r} = \begin{bmatrix} \Gamma_r & M_{r,r} \\ \end{bmatrix} = \begin{bmatrix} \Gamma_r & M_{r,l} \quad \tilde{M}_{r,l} \end{bmatrix}, \quad (7) \]

it follows that (4) can be written as

\[ Y_{[0:r]} = \Gamma_r x_0 + M_{r,l} U_{[0:r-l]} + \tilde{M}_{r,r} U_{[r-l+1:r]} = \Psi_{r,r-l} \begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} + \tilde{M}_{r,r} U_{[r-l+1:r]}. \quad (9) \]

Note that \( \tilde{M}_{r,0} \in \mathbb{R}^{p(r+1) \times 0} \) is an empty matrix whose range is \( \{0\}. \) Furthermore,

\[ \tilde{M}_{r,l} = \begin{bmatrix} 0_{p(r+1) \times 0,} \\ 0_{p(r-l+1) \times ml} \\ M_{l-l-1,l-1} \end{bmatrix}, \quad l \geq 1, \quad (11) \]

and thus, for all \( l \geq 1, \)

\[ \text{rank}(\tilde{M}_{r,l}) = \text{rank}(M_{l-l-1,l-1}). \quad (12) \]

Let \( \mathcal{R} \) denote range.

Theorem 2. Let \( l \) be a nonnegative integer. Then the following statements are equivalent:

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(i) $G(z)$ is $l$-delay left invertible.
(ii) There exists a matrix $K \in \mathbb{R}^{m \times p(l+1)}$ such that
$$KM_{l,l} = \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix},$$
(iii) $u_0$ is uniquely determined by $Y_{[0:l]}$ and $x_0$.
(iv) $\mathcal{N}(M_{l,l}) \subseteq \mathbb{R} \left( \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} I_{ml} \right)$.
(v) $M_{l,0}$ has full column rank, and
$$\mathcal{R}(M_{l,0}) \cap \mathcal{R}(M_{l,l}) = \{0\}.$$ (14)
(vi) $m = \left\{ \begin{array}{ll} \text{rank}(M_{0,0}), & l = 0, \\
\text{rank}(M_{l,l}) - \text{rank}(M_{l-1,l-1}), & l \geq 1. \end{array} \right.$

Theorem 2 implies that $G(z)$ is $l$-delay left invertible if and only if each of the first $m$ columns of $M_{l,l}$ is linearly independent of the remaining columns of $M_{l,l}$.

Note that if $G(z)$ is SISO, then $z^{-d}G^{-1}(z)$ is exactly proper, and thus $l_0 = d$. However, the following example shows that $l_0$ and $d$ may be different in the MIMO case.

**Example 1.** Consider the transfer function
$$G(z) = \begin{bmatrix} \frac{z^{-1}}{z^{-2}} & \frac{z^{-2}}{z^{-2}} \end{bmatrix}$$
with the minimal realization
$$A = \begin{bmatrix} 0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\
0 \\
0 \\
0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}.$$

Note that $H_1 = \begin{bmatrix} 1 & 0 \\
0 & 0 \end{bmatrix}$, $H_2 = \begin{bmatrix} 0 & 1 \\
1 & 1 \end{bmatrix}$, $H_i = 0$ for all $i > 2$, and
$$G^{-1}(z) = \frac{1}{z^{-1}} \begin{bmatrix} z^2 & -z^2 \\
-z^2 & z^3 \end{bmatrix}.$$ 

Next, define $G_l(z) = z^{-l}G^{-1}(z)$. Then $G_l(z)$ is proper if and only if $l \geq 2$. Hence $l_0 = 2$. Since $d = 1$, it follows that $l_0 > d$.

The following result gives a necessary and sufficient condition for $d = l_0$.

**Proposition 1.** Assume that $G(z)$ is $l$-delay left invertible, and let $l_0$ denote the minimal delay of $G(z)$. Then,
$$\text{rank} \begin{bmatrix} H_0 & \cdots & H_l \end{bmatrix} \geq m, \quad (15)$$
$$\text{rank} \begin{bmatrix} H_0 \\
H_1 \\
\vdots \\
H_l \end{bmatrix} = m, \quad (16)$$
and $l_0 \leq n$. Furthermore, $l_0 \geq d$ with equality if and only if $\text{rank}(H_d) = m$.

Proposition 1 shows that if $G(z)$ is $l$-delay left invertible, then both (15) and (16) are satisfied. However, the converse does not hold. For the following example, both (15) and (16) hold for $l = 1$, whereas $l_0 = 2$.

**Example 2.** Consider the transfer function
$$G(z) = \begin{bmatrix} z^{-1} & 1 \\
-1 & z^{-1} \\
2 & z^{-2} \\
2 & z^{-2} \end{bmatrix},$$
for which $n = 5$ and $d = 0$. Note that $H_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $H_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 \end{bmatrix}$, and $H_i = 0$ for all $i > 2$. Then $\text{rank}(M_{0,0}) = 1$, $\text{rank}(M_{1,1}) = 3$, and $\text{rank}(M_{2,2}) = 6$. Since $\text{rank}(M_{1,1}) - \text{rank}(M_{0,0}) < 3$, and $\text{rank}(M_{2,2}) - \text{rank}(M_{1,1}) = 3$, it follows from condition (vi) of Theorem 2, that $l_0 = 2$.

However, (15) holds for $l = 1$, that is, $\text{rank} \begin{bmatrix} H_0 & H_1 \end{bmatrix} = 3$, and (16) holds for $l = 1$, that is, $\text{rank} \begin{bmatrix} H_0 \\
H_1 \end{bmatrix} = 3$. Note that $l_0 \leq n$, as implied by Proposition 1.

Theorems 1 and 2 provide necessary and sufficient conditions under which the identity transfer function can be attained by a left inverse after $l$ steps. However, this non-causal left inverse has limited value for input reconstruction for two reasons. First, the transfer function formulation of a dynamical system does not account for the initial condition of the state of a corresponding state space realization. Therefore, we must consider a state space formulation in order to have a more complete picture of the free response of the system, which is present in practice. Furthermore, within a state space formulation, if $G(z)$ has a transmission zero, then $G(z)$ can have a nonzero input such that, for some nonzero initial condition of a corresponding state space model, the output is identically zero. We explore these issues in the following sections.

**IV. l-Delay Input Reconstruction with Known Initial State**

In this section we show that if $G(z)$ is $l$-delay left invertible then we can achieve input reconstruction with an $l$-step delay from known initial condition $x_0$ and output measurements $Y_{[0:r]}$.

**Lemma 1.** Assume that $G(z)$ is $l$-delay left invertible. Then, for all $r \geq l$,
$$\mathcal{N}(M_{r,r}) \subseteq \mathbb{R} \left( \begin{bmatrix} 0 \cdots 0 \end{bmatrix} I_{ml} \right). \quad (17)$$

**Theorem 3.** Assume that $G(z)$ is $l$-delay left invertible. Then for all $r \geq l$, $U_{[0:r-l]}$ is uniquely determined by $x_0$ and $Y_{[0:r]}$. Furthermore, the unique solution with reconstruction delay $l$ is
$$U_{[0:r-l]} = \text{row}_{[1:(r-l+1)m]} \left[ M_r^{\dagger} (Y_{[0:r]} - \Gamma_r x_0) \right]. \quad (18)$$

**V. l-Delay Input and Initial State Reconstruction with Unknown Initial State**

In this section we take into account the unknown and possibly nonzero initial condition of (1), (2) in order to achieve input and state reconstruction.

**Definition 2.** Let $l$ be a nonnegative integer. The system (1), (2) is $l$-delay input and state observable if there exists
\[ r \geq l \text{ such that} \]
\[ \mathcal{N}(\Psi_{r,r}) \subseteq \mathbb{R}^{n+m(r-l+1) \times lm}. \tag{19} \]

In this case, the smallest nonnegative integer \( l' \) for which (1), (2) is \( l' \)-delay input and state observable is the minimal ISO delay. Furthermore, (1), (2) is input and state observable if there exists a nonnegative integer \( q \) such that (1), (2) is \( q \)-delay input and state observable.

In Definition 2, \( r + 1 \) is the length of the output data window used to reconstruct the initial state and input. Specifically, if \( Y_{[0:r]} = 0 \) in (4), that is, \[ \begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix} \in \mathcal{N}(\Psi_{r,r}), \]
condition (19) implies that \[ \begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} = 0, \]
and thus \( x_0 \) and \( U_{[0:r-l]} \) can be uniquely reconstructed from \( Y_{[0:r]} \).

The following result provides necessary and sufficient conditions for \( l \)-delay input and state observability.

**Proposition 2.** Let \( l \) be a nonnegative integer. The system (1), (2) is \( l \)-delay input and state observable if and only if there exists \( r \geq l \) such that both of the following statements hold:

(i) \( \text{rank} (\Psi_{r,r}) = \text{rank} (\Psi_{r-r,l}) + \text{rank} (M_{r,l}). \)

(ii) \( \Psi_{r-r,l} \) has full column rank.

**Proposition 3.** Assume that (1), (2) is \( l \)-delay input and state observable. Then \( G(z) \) is \( l \)-delay left invertible. Furthermore, \[ l_0 \leq l' \]. \tag{20} \]

**Lemma 2.** Let \( \zeta \in \mathbb{C} \). Then \( \tilde{x}_0 \in \mathbb{C}^n \) and \( \tilde{u}_0 \in \mathbb{C}^m \) satisfy
\[ \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{x}_0 \\ \tilde{u}_0 \end{bmatrix} = 0. \tag{21} \]
if and only if, for all \( r \geq 0 \),
\[ \Psi_{r,r} \begin{bmatrix} \tilde{x}_0 \\ \tilde{u}_{[0:r]} \end{bmatrix} = 0, \tag{22} \]
where
\[ \tilde{u}_{[0:r]} \triangleq \begin{bmatrix} \tilde{u}_0^T \\ \zeta \tilde{u}_0^T \\ \cdots \\ \zeta^r \tilde{u}_0^T \end{bmatrix}^T \in \mathbb{C}^{(r+1)m}. \tag{23} \]

Recall that \( \zeta \in \mathbb{C} \) is an invariant zero of (1), (2) if rank \[ \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \] is less than the normal rank of \[ \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix}. \]

**Proposition 4.** Assume that (1), (2) is \( l \)-delay input and state observable. Then \( G(z) \) is \( l \)-delay left invertible and (1), (2) has no invariant zeros.

**Proposition 5.** Assume that \( \text{rank} (H_d) = m \) and let \( r \geq n \). Then, \( \tilde{x}_0 \in \mathbb{C}^n \) and \( \tilde{u}_0, \ldots, \tilde{u}_r \in \mathbb{C}^m \) satisfy
\[ \Psi_{r,r} \begin{bmatrix} \tilde{x}_0 \\ \tilde{u}_0 \\ \vdots \\ \tilde{u}_r \end{bmatrix} = 0. \tag{24} \]
if and only if
\[ \tilde{x}_0 \in \left\{ \begin{array}{l} \bigcap_{l=0}^{n-1} \mathcal{N}([I_p - DD^T]CK_d^l], \quad d = 0, \\
\bigcap_{l=0}^{d-1} \mathcal{N}(CK_d^l) \bigcap \mathcal{N}(CA'K_d^{d-l}], \quad d \geq 1, \end{array} \right. \tag{25} \]
and, for all \( k \leq r - d, \)
\[ \tilde{u}_k = -H_dCA^dK_d^k\tilde{x}_0. \tag{26} \]

where
\[ K_d \triangleq A - B\tilde{H}_dCA^d. \tag{27} \]

In this case, let \( \tilde{x}_{k+1} \triangleq A\tilde{x}_k + B\tilde{u}_k \) for all \( k \geq 0 \). Then, for all \( k \in \{0, \ldots, r - d + 1\}, \tilde{x}_k = K^k_d\tilde{x}_0. \)

Now, let \( s \triangleq \text{rank} (H_d) \). Then there exist \( \tilde{H}_1 \in \mathbb{R}^{p \times s}, \tilde{H}_2 \in \mathbb{R}^{s \times m} \) such that \( H_d = \tilde{H}_1\tilde{H}_2 \). Note that \( \text{rank} (\tilde{H}_1) = \text{rank} (\tilde{H}_2) = s \). Now, let \( u_k \) be given by
\[ u_k = \tilde{H}_2^T v_k, \tag{28} \]
where \( v_k \in \mathbb{R}^s \). Suppose \( d = 0 \). Then \( D \neq 0 \), and thus
\[ x_{k+1} = A\tilde{x}_k + B'v_k, \tag{29} \]
\[ y_k = C\tilde{x}_k + D'v_k, \tag{30} \]
where \( B' \triangleq B\tilde{H}_2^T \) and \( D' \triangleq D\tilde{H}_2^T = \tilde{H}_1\tilde{H}_2\tilde{H}_2^T \). Note that \( \tilde{H}_2\tilde{H}_2^T \) is positive definite, and thus \( D' \) has full column rank.

Alternatively, suppose \( d > 0 \). Then \( D = 0 \), and thus
\[ x_{k+1} = A\tilde{x}_k + B'v_k, \tag{31} \]
\[ y_k = C\tilde{x}_k, \tag{32} \]
where \( B' \triangleq B\tilde{H}_2^T \). Note that \( CA^{d-1}B' = H_d\tilde{H}_2^T \), which has full column rank. Hence, for all \( d \geq 0 \), the input transformation (28) yields a modified system such that the first nonzero Markov parameter has full column rank. With this transformation, Proposition 5 is applicable to all systems, and by reconstructing \( v_k \), we can compute \( u_k \) using (28).

Now, suppose \( H_d \) has full column rank. Then define the discrete time system
\[ \tilde{x}_{k+1} = K_d\tilde{x}_k, \tag{33} \]
\[ \tilde{y}_k = \left\{ \begin{array}{l} (I_p - DD^T)C\tilde{x}_k, \quad d = 0, \\
C\tilde{x}_k, \quad d \geq 1, \end{array} \right. \tag{34} \]
where \( K_d \) is given by (27).

**Proposition 6.** Assume that (33), (34) is unobservable and \( r \geq n \). Then there exist \( \tilde{x}_0 \in \mathbb{R}^n \) and \( \tilde{u}_0, \ldots, \tilde{u}_r \in \mathbb{R}^m \) such that \( \tilde{x}_0 \neq 0 \) and
\[ \Psi_{r,r} \begin{bmatrix} \tilde{x}_0 \\ \tilde{u}_0 \\ \vdots \\ \tilde{u}_r \end{bmatrix} = 0. \tag{35} \]
Furthermore, let \( \tilde{x}_{k+1} \triangleq A\tilde{x}_k + B\tilde{u}_k \) for all \( k \geq 0 \). Then, for all \( k \in \{0, \ldots, r + 1\}, \tilde{x}_k = K^k_d\tilde{x}_0. \)
Proposition 7. Assume that, for all \( \zeta \in \mathbb{C} \),
\[
\text{rank} \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} = n + m.
\]
Then \( G(z) \) is left invertible, and at least one of the following statements holds:

(i) For all \( r \geq n \), \( \mathcal{R}(\Gamma_r) \cap \mathcal{R}(M_{r,r}) = \{0\} \).

(ii) (33), (34) is observable.

Theorem 4. Assume that \( G(z) \) is \( l \)-delay left invertible,
(1), (2) has no invariant zeros, (33), (34) is unobservable and \( K_r \) is given by (27). Then (1), (2) is \( l \)-delay input and state observable.

Corollary 1. Assume that \( G(z) \) is SISO, left invertible and has no zeros. Then (1), (2) is input and state observable and
\[
l_0 = l'_0 = n.
\]
Furthermore, for all \( r \geq n - 1 \), \( \text{def}(\Psi_{r,r}) = n \).

Proposition 8. Assume that \( G(z) \) is left invertible and, for all \( r \geq n \), \( \mathcal{R}(\Gamma_r) \cap \mathcal{R}(M_{r,r}) = \{0\} \). Then for all \( \zeta \in \mathbb{C} \),
\[
\text{rank} \begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} = n + m.
\]
The following result provides necessary conditions for \( l \)-delay input and state observability.

Proposition 9. Let \( l \) be a nonnegative integer, and assume that (1), (2) is \( l \)-delay input and state observable. Then the following statements hold:

(i) (1), (2) is \( k \)-delay input and state observable for all \( k \geq l \).

(ii) \( \text{rank}(\Psi_{r,r-l}) = \text{rank}(\Psi_{r,r-l-1}) + m \) for all \( r \geq n \).

(iii) \( m \leq p \).

(iv) If \( m = p \), then \( n \leq ml \).

(v) \( \text{rank}(\Gamma_{n-1}) = n \).

The following result provides an explicit solution of the \( l \)-delay state and input reconstruction problem.

Proposition 10. Assume that (1), (2) is \( l \)-delay input and state observable and (19) holds for some \( q \geq l \). Then, for all \( r \geq q \), \( x_0 \) and \( U_{[0:r-l]} \) are uniquely determined by \( Y_{[0:r]} \). Furthermore, the unique solution of (9) with reconstruction delay \( l \) is given by
\[
\begin{bmatrix} x_0 \\ U_{[0:r-l]} \end{bmatrix} = (Q^T\Psi_{r,r-l})^TQ^TY_{[0:r]},
\]
where \( \dagger \) represents the Moore-Penrose generalized inverse, \( k \overset{\Delta}{=} \text{rank}(\tilde{M}_{r,l}) \), and the columns of \( Q \in \mathbb{R}^{(r+1) \times [p+(r+1)-k]} \) are an orthonormal basis of \( \mathcal{N}(\tilde{M}_{r,l}) \).

The special case \( l = 1, d = 1 \) is given in [19]. Proposition 2 shows that for this special case (1), (2) is 1-delay input and state observable if and only if there exists some \( r \geq 1 \), such that \( \Psi_{r,r-1} \) has full column rank. Furthermore, since \( \tilde{M}_{r,1} = 0_{(r+1)p \times x_m} \), it follows from Proposition 5 that \( Q = I_{(r+1)p} \) and it follows from (37) that the unique solution of (9) for each \( r \geq 1 \) is given by
\[
\begin{bmatrix} x_0 \\ U_{[0:r-1]} \end{bmatrix} = \Psi_{r,r-1}^T Y_{[0:r]}.
\]

The following example demonstrates the case where \( l_0 = l'_0 \) and Figure 1 shows 1-delay and 2-delay input and state reconstruction for a 2-delay input and state observable system with \( d = 1 \). Note that, here 1-delay input and state reconstruction estimate is given by
\[
\begin{bmatrix} \hat{x}_0 \\ \hat{U}_{[0:r-1]} \end{bmatrix} = \Psi_{r,r-1}^T Y_{[0:r]},
\]

Example 4. Consider the transfer function
\[
G(z) = \frac{1}{(z - 0.5)(z - 0.6)}
\]
with the minimal realization
\[
A = \begin{bmatrix} 1.1 & -0.6 \\ 0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix},
\]
\[
C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \end{bmatrix}.
\]
Since \( \Psi_{2,0} \) has full column rank, \( \Psi_{2,1} \) does not have full column rank, and \( \text{rank}(\Psi_{2,0}) + \text{rank}(\tilde{M}_{2,2}) \), it follows from Proposition 2 that \( G(z) \) is 2-delay input and state observable but not 1-delay input and state observable. Therefore, \( l'_0 = 2 \). Since \( G(z) \) is SISO with \( d = 2 \), it follows that \( l_0 = d \) and thus \( l_0 = l'_0 = 2 \).

For both cases we compute the least-squares expression in (37), and Figure 1 compares 1-delay and 2-delay input and state reconstruction. As expected, 2-delay input and state reconstruction described by (37) correctly estimates the initial state and input, whereas, for \( l = 1 \), the solution given by (37) fails.

VI. INVARIANT ZEROS AND THE UNOBSERVABLE INPUTS

If the system (1), (2) has invariant zeros, then it follows from Proposition 4 that (1), (2) is not input and state observable. In this case, the initial state and input sequence cannot be exactly reconstructed from output measurements. In this section we relate the error in input reconstruction to the locations of the invariant zeros relative to the unit circle.

Definition 3. Let \( r \geq 0 \). Then the input \( U_{[0:r]} \) is unobservable if there exists a nonzero initial condition \( x_0 \) such that
\[
\begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix} \in \mathcal{N}(\Psi_{r,r}).
\]

Lemma 5 implies that if \( G(z) \) is left invertible and (1), (2) has at least one invariant zero, then there exists an initial condition and an input sequence, not both zero, such that, the
output is identically zero. In fact, the following immediate result shows that such initial conditions and input sequences are nonzero elements in the null space of $\Psi_{r,r}$. These initial conditions and input sequences are thus unobservable for the purpose of initial state and input reconstruction.

**Proposition 11.** Let $\xi \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^{(r+1)m}$ be such that

$$
\Psi_{r,r} \begin{bmatrix} \xi \\ \nu \end{bmatrix} = \Gamma_r \xi + M_{r,r} \nu = 0. 
$$

(40)

Then, with $x_0 = \xi$ and $U_{[0:r]} = \nu$, it follows that $Y_{[0:r]} = 0$.

For SIMO systems the following two results provide explicit expressions for unobservable initial state and input sequences.

**Proposition 12.** Let $m = 1$ and let $\zeta \in \mathbb{C}$ be an invariant zero of (1), (2). Furthermore, let $c \in \mathbb{C}$, define the input sequence

$$
u_k = \text{Re}(c \zeta^k), \quad k = 0, 1, \ldots, 
$$

(41)

and let

$$
x_0 = -\Gamma_{n-1}^l \text{Re}c \begin{bmatrix} H_0 \\
H_0 \zeta + H_1 \\
\vdots \\
\vdots \\
\sum_{j=0}^i H_j \zeta^{i-j} \\
\vdots \\
\sum_{j=0}^{n-1} H_j \zeta^{n-1-j} \end{bmatrix} . 
$$

(42)

Then, for all $r \geq 0$, $Y_{[0:r]} = 0$, that is,

$$
\begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix} \in \mathcal{N}(\Psi_{r,r}). 
$$

**Proposition 13.** Let $m = 1$ and let $\zeta \in \mathbb{C}$ be an invariant zero of (1), (2) such that $|\zeta| > \rho(A)$. Furthermore, let $c \in \mathbb{C}$, define the input sequence

$$
u_k = \text{Re}(c \zeta^k), \quad k = 0, 1, \ldots, 
$$

(43)

and let

$$
x_0 = \text{Re}c \sum_{i=1}^\infty \zeta^{-i} A^{i-1} B .
$$

(44)

Then, for all $r \geq 0$, $Y_{[0:r]} = 0$, that is,

$$
\begin{bmatrix} x_0 \\ U_{[0:r]} \end{bmatrix} \in \mathcal{N}(\Psi_{r,r}). 
$$

(45)

Note that, if $\zeta$ is a zero of (1), (2), then rank

$$
\begin{bmatrix} \zeta I - A & -B \\
C & D \end{bmatrix} < n + m .
$$

Furthermore, the values of $x_0$ given by (44) and (42) satisfy

$$
\begin{bmatrix} \zeta I - A & -B \\
C & D \end{bmatrix} \begin{bmatrix} x_0 \\ c \end{bmatrix} = 0 .
$$

(46)

Note that if $\zeta$ is an element of the open unit disk, then (43) decays, whereas (43) grows if $\zeta$ is outside the unit disk. If however, $\zeta$ is on the unit disk, then (43) neither grows nor decays.

**VII. CONCLUSIONS**

In this paper defined $l$-delay state and input reconstruction, and we characterized this property through necessary and sufficient conditions. We demonstrated this technique on several numerical examples, which shows how the input reconstruction error depends on the locations of the zeros. Specifically, minimum-phase zeros give rise to decaying input reconstruction error, nonminimum-phase zeros give rise to growing reconstruction error, and zeros on the unit circle give rise to persistent reconstruction error. Future research will focus on the effects of modeling errors and noisy measurements.

**REFERENCES**


