Lyapunov-Stable Adaptive Stabilization of Nonlinear Systems with Matched Uncertainty

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1. INTRODUCTION
Adaptive stabilization of linear time-invariant plants with full state feedback has been considered in [1-4] using Lyapunov-based gradient update laws. Lyapunov-based adaptive stabilization has several extensions to nonlinear systems. In [5], a variation of the controller presented in [4] is shown to stabilize a class of scalar second-order nonlinear systems with partial-state-dependent uncertainty. In particular, the adaptive controller of [5] can stabilize the scalar nonlinear system \( m \dot{q}(t) + q(t)g(t) + f(q(t))s(t) = bu(t) \), where the functions \( f(\cdot) \) and \( g(\cdot) \) are lower bounded but otherwise unknown. In the present paper, a novel full-state-feedback adaptive controller is used to stabilize \( n \)-th order nonlinear systems with bounded state-dependent uncertainty. First, we develop the controller for linear systems, then extended the result to nonlinear systems.

2. ADAPTIVE STABILIZATION FOR LINEAR SYSTEMS
We consider the single-input linear-time invariant system
\[
\dot{x} = Ax + Bu,
\]
where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times 1} \). We make the following assumptions.

(i) The system is in companion form, where
\[
A \triangleq \begin{bmatrix}
-a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 & -a_0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \tag{2.2}
\]
\[
B \triangleq \begin{bmatrix} b & 0 & \cdots & 0 \end{bmatrix}^T. \tag{2.3}
\]

(ii) \( b \neq 0 \), and \( \text{sgn}(b) \) is known.

(iii) The full state \( x \) is available for feedback.

We begin this section by providing several useful results regarding a matrix in controllable canonical form. Consider the matrix \( A \in \mathbb{R}^{n \times n} \), which has the characteristic polynomial \( d(s) \triangleq s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1s + d_0 \) and is in the companion form (2.2). The Hurwitz matrix associated with the characteristic polynomial \( d(s) \) is
\[
H \triangleq \begin{bmatrix}
d_{n-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\
d_{n-3} & d_{n-2} & d_{n-1} & 1 & \cdots & 0 & 0 \\
d_{n-5} & d_{n-4} & d_{n-3} & d_{n-2} & \cdots & 0 & 0 \\
d_{n-7} & d_{n-6} & d_{n-5} & d_{n-4} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & d_1 & d_2 \\
0 & 0 & 0 & 0 & \cdots & 0 & d_0
\end{bmatrix}. \tag{2.4}
\]

The following result, given in [6], concerns the solution to the Lyapunov equation for a matrix in controllable canonical form.

Lemma 2.1. Consider the asymptotically stable controllable canonical form \( A \). Let \( P \in \mathbb{R}^{n \times n} \) be the positive definite solution to the Lyapunov equation \( A^TP + PA = -Q \), where \( Q \in \mathbb{R}^{n \times n} \) is positive definite. Let \( p_1 \) denote the first column of \( P \). Then \( p_1 \) satisfies \( 2DHDP_1 = q \), where \( H \) is given by (2.4), \( D \triangleq \text{diag}(1,-1,1,\ldots) \), and
\[
q = \begin{bmatrix}
\sum_{1 \leq i,j \leq n,i+j=2} (-1)^{i-j}Q_{i,j} \\
\sum_{1 \leq i,j \leq n,i+j=4} (-1)^{i-j}Q_{i,j} \\
\vdots \\
\sum_{1 \leq i,j \leq n,i+j=2n} (-1)^{i-n}Q_{i,j}
\end{bmatrix}. \tag{2.5}
\]

Lemma 2.2. Let \( g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0 \) be a Hurwitz polynomial where \( g_{n-1} > 0 \) and define
\[
A_s(k) \triangleq \begin{bmatrix}
-k|b|g_{n-1} & -k|b|g_{n-2} & \cdots & -k|b|g_0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0
\end{bmatrix}, \tag{2.6}
\]
where \( k \in \mathbb{R} \). Then, there exists \( k_n > 0 \) such that, for all \( k \geq k_n \), \( A_s(k) \) is asymptotically stable and thus, there exists a positive definite \( P(k) \) such that
\[
A_s^T(k)P(k) + P(k)A_s(k) = -e^{-\alpha k}Q, \tag{2.7}
\]
where \( Q > 0 \) and \( \alpha \geq 0 \). Furthermore, \( \lim_{k \to \infty} p_1(k) = 0 \) and \( \lim_{k \to \infty} e^{\alpha k}p_1(k) \) exists, where \( p_1(k) \) denotes the first column of \( P(k) \). If, in addition, \( \alpha > 0 \), then there exists \( k_2 \geq k_n \) such that, for all \( k \geq k_2 \), \( e^{\alpha k}P(k) \) is negative definite.

Proof. Let \( H(k) \) be the Hurwitz matrix associated with the characteristic polynomial of \( A_s(k) \). The Hurwitz stability conditions for the characteristic polynomial of \( A_s(k) \) are polynomials in \( k \) given by
\[
\Lambda_1 \triangleq k|b|g_{n-1} > 0, \tag{2.8}
\]
\[
\Lambda_2 \triangleq \begin{bmatrix}
k|b|g_{n-1} & 1 \\
k|b|g_{n-3} & k|b|g_{n-2} \\
\end{bmatrix} > 0, \tag{2.9}
\]
\[
\Lambda_3 \triangleq \begin{bmatrix}
k|b|g_{n-1} & 1 & 0 \\
k|b|g_{n-3} & k|b|g_{n-2} & k|b|g_{n-1} \\
\end{bmatrix} > 0, \tag{2.10}
\]
\[
\vdots \\
\Lambda_n \triangleq \begin{bmatrix} L_3 & \cdots & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & k|b|g_0 \end{bmatrix} > 0. \tag{2.11}
\]

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For sufficiently large \( k \), the Hurwitz conditions are satisfied since \( g(s) \) is Hurwitz with positive leading coefficient. Therefore, there exists \( k_0 > 0 \) such that, for all \( k \geq k_0 \), the matrix \( A_s(k) \) is asymptotically stable. Then, for all \( k \geq k_0 \), there exists \( P(k) > 0 \) satisfying (2.7).

Now, we consider the asymptotic properties of \( p_1(k) \). For all \( k \geq k_0 \), the inverse of the Hurwitz matrix exists and can be expressed as \( H^{-1}(k) = \frac{1}{\det(H(k))} \tilde{H}(k) \), where

\[
H(k) \triangleq \begin{bmatrix}
[H(k)_{1,1}] & -[H(k)_{1,2}] & \cdots & -(1)^{n+1} [H(k)_{1,n}] \\
-[H(k)_{2,1}] & [H(k)_{2,2}] & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
-(1)^{n+1} [H(k)_{n,1}] & \cdots & \cdots & [H(k)_{n,n}]
\end{bmatrix}
\]

(2.12)

where \([H(k)]_{i,j}\) is the \((i,j)\)th minor of \( H(k) \). The determinant of \( H(k) \) is a degree \( n \) polynomial in \( k \), while \([H(k)]_{i,j}\) is a polynomial in \( k \) of degree not exceeding \( n - 1 \). Therefore, \( \lim_{k \to \infty} H^{-1}(k) = 0 \), and \( \lim_{k \to \infty} kH^{-1}(k) \) exists. Using Lemma 2.1 we obtain \( \lim_{k \to \infty} p_1(k) = \lim_{k \to \infty} \frac{1}{2}D^{-1}H^{-1}(k)D^{-1} - \alpha \delta kQ = 0 \) and \( \lim_{k \to \infty} k^{\alpha} p_1(k) = \lim_{k \to \infty} \frac{1}{2}D^{-1}kH^{-1}(k)D^{-1}k = 0 \), exists, where \( Q \) is determined from \( Q \) using (2.5).

Next, we show that for \( k \) sufficiently large, \( \frac{\partial P}{\partial k} < 0 \). Taking the partial derivative of (2.7) with respect to \( k \) yields

\[
-\bar{A}_s(k) \frac{\partial P}{\partial k} = \bar{Q}(k)
\]

(2.13)

where

\[
\bar{Q}(k) \triangleq -a \alpha \alpha k + \delta e_1^T P(k) + P(k)e_1 \delta^T,
\]

\( e_1 = [1 \ 0 \ \cdots \ 0]^T \in \mathbb{R}^n \), and \( \delta = -[b \ [g_{n-1} \ \cdots \ g_0]^T \in \mathbb{R}^n \). Define \( \Omega(k) \triangleq \sqrt{\frac{2}{\alpha} e^{\alpha k} P(k) e_1 \delta^T} \), and since \( 0 \leq \Omega^T(k)Q(k) \), it follows that \( \delta e_1^T P(k) + P(k)e_1 \delta^T \leq \frac{\alpha}{2} e^{\alpha k} Q + \frac{\alpha}{2} \delta e_1^T P(k)Q^{-1}P(k)e_1 \delta^T \). Combining this with (2.13) yields

\[
\bar{Q}(k) \leq -e^{-\alpha k} \left[ \frac{\alpha}{2} Q + \frac{\alpha}{2} \delta e_1^T P(k)Q^{-1}P(k)e_1 \delta^T \right].
\]

(2.14)

Since \( \lim_{k \to \infty} k^{\alpha} p_1(k) \) exists, it follows that \( \lim_{k \to \infty} e^{\alpha k} p_1(k) = 0 \), and thus \( \lim_{k \to \infty} \frac{1}{2} e^{2\alpha k} \delta e_1^T P(k)Q^{-1}P(k)e_1 \delta^T \leq \frac{1}{2} Q \). Therefore, \( \bar{Q}(k) \) is asymptotically stable, \( \frac{\partial P}{\partial k} < 0 \). Now, we present a Lyapunov proof of a high-gain stabilizing controller for the linear system (2.1)-(2.3).

**Lemma 2.3.** Consider the linear system (2.1)-(2.3). Let \( g(s) \) be the Hurwitz polynomial

\[
g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0,
\]

(2.15)

where \( g_{n-1} > 0 \). Define \( G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix} \), and consider the feedback

\[
u(t) = -\text{sgn}(b) k G x(t),
\]

(2.16)

where \( k \in \mathbb{R} \). Then there exists \( k_0 > 0 \) such that, for all \( k \geq k_0 \), the origin of the closed-loop system is asymptotically stable.

**Proof.** The system (2.1)-(2.3) with the feedback (2.16) can be written as \( x(t) = \bar{A}_s(k) + \Delta \bar{x}(t) \), where \( \bar{A}_s(k) \) is given by (2.6), \( \Delta \triangleq \begin{bmatrix} \delta \ 0_{n \times (n-1)} \end{bmatrix} \in \mathbb{R}^{n \times n} \), and \( \delta \triangleq [-a_{n-1} \ \cdots \ -a_0]^T \in \mathbb{R}^n \). Lemma 2.2 implies that there exists \( k_1 > 0 \) such that, for all \( k \geq k_1 \), \( \bar{A}_s(k) \) is asymptotically stable. For all \( k \geq k_1 \), let \( P(k) > 0 \) be the solution to the Lyapunov equation \( A_s^T(k) P(k) + P(k) \bar{A}_s(k) = - (Q + I) \), where \( Q > 0 \). Furthermore, let \( p_1(k) \) denote the first column of \( P(k) \). Next, consider the Lyapunov candidate \( V(x) = x^T P(k)x \), where \( k \geq k_1 \). Taking the derivative along the closed-loop trajectory yields

\[
\dot{V}(x) = -x^T (Q + I) x + x^T \Delta^T P(k) + P(k) \Delta x.
\]

(2.17)

Since

\[
0 \leq \left( \sqrt{2} P(k) \Delta - \frac{1}{\sqrt{2}} I \right)^T \left( \sqrt{2} P(k) \Delta - \frac{1}{\sqrt{2}} I \right)
\]

it follows that \( \Delta^T P(k) + P(k) \Delta \leq \frac{1}{2} I + 2 \Delta^T P^2(k) \Delta \). Combining this with (2.17) yields

\[
\dot{V}(x) \leq -x^T Q x - \frac{1}{2} x^T x + 2 x^T \left( \delta e_1^T P(k)e_1 \delta^T \right) \leq \frac{1}{2} I. \]

Therefore, for all \( k \geq k_1 \), \( \dot{V}(x) < 0 \) if \( x \neq 0 \), and the origin is asymptotically stable.

Now, we present a Lyapunov-stable adaptive stabilization algorithm for linear systems.

**Theorem 2.1.** Consider the linear system (2.1)-(2.3). Let \( g(s) \) be the Hurwitz polynomial (2.15) where \( g_{n-1} > 0 \). Define \( G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix} \), and consider the adaptive feedback controller

\[
u(t) = -\text{sgn}(b) k G x(t),
\]

(2.18)

\[
k(t) = e^{-\alpha(t)} x^T(t) R x(t),
\]

(2.19)

where \( R \) is positive definite and \( \alpha > 0 \). Then, there exists \( k_0 > 0 \) such that for all \( k \geq k_0 \), the equilibrium solution \((0,0)\) of the closed-loop system (2.1)-(2.3) and (2.18)-(2.19) is Lyapunov stable. Furthermore, for all initial conditions \( x(0) \) and \( k(0) \), \( k_\infty \triangleq \lim_{t \to \infty} k(t) \) exists and \( \lim_{t \to \infty} x(t) = 0 \).

**Proof.** The dynamics (2.1)-(2.3) with the feedback (2.18) can be expressed as

\[
\dot{x}(t) = \bar{A}(k(t)) x(t),
\]

(2.20)

where \( \bar{A}(k(t)) \triangleq A - k(t) \text{sgn}(b) BG \). Lemma 2.3 implies that there exists \( k_0 \) such that for all \( k \geq k_0 \), \( \bar{A}(k) \) is asymptotically stable. Let \( k_0 \geq k_0 \), define \( A_e \triangleq \bar{A}(k_e) \), and
define $\hat{k}(t) \triangleq k_e - k(t)$ so that (2.19)-(2.20) can be written as
\[ \dot{x} = A_e x + sgn(b)k_B G x. \]
Since $A_e$ is asymptotically stable, there exists $P_e > 0$ such that $A_e^T P_e + P_e A_e = -Q$, where $Q > 0$. Next, consider the Lyapunov candidate

\[ V(x, \hat{k}) = x^T \frac{\partial P(k)}{\partial k} x + \frac{1}{2} x^T P_e x + k^2, \]
where $V : \mathbb{R}^n \times \mathbb{D} \to [0, \infty)$ and the domain $\mathcal{D}$ will be determined. The derivative of $V(x, \hat{k})$ along a closed-loop trajectory is

\[ \dot{V}(x, \hat{k}) = \dot{k} x^T [\text{sgn}(g) G^T B^T P_e + \text{sgn}(g) k_B P_e B G] x + x^T Q x - 2k e^{-\alpha k} x^T R x. \]  \hspace{1cm} (2.21)

First, consider the case $\dot{k} \geq 0$, then

\[ \dot{V}(x, \hat{k}) \leq -x^T Q x + k \sigma_1 x^T x, \]
where $\sigma_1 \triangleq \lambda_{\max} (\text{sgn}(g) G^T B^T P_e + \text{sgn}(g) k_B P_e B G)$. If $\sigma_1 \leq 0$, then

\[ \dot{V}(x, \hat{k}) \leq -x^T Q x. \]
If $\sigma_1 > 0$, then let $0 < \varepsilon < \lambda_{\min}(Q)$. Thus, for all $\hat{k}$ such that $0 \leq \hat{k} \leq \frac{\lambda_{\min}(Q)}{\sigma_1}$,

\[ \dot{V}(x, \hat{k}) \leq -x^T x. \]
Now, consider the case $\dot{k} \leq 0$, then

\[ \dot{V}(x, \hat{k}) \leq -x^T Q x + k \sigma_2 x^T x, \]
where $\sigma_2 \triangleq \lambda_{\min} (\text{sgn}(g) G^T B^T P_e + \text{sgn}(g) k_B P_e B G - 2R)$. If $\sigma_2 \geq 0$, then

\[ \dot{V}(x, \hat{k}) \leq -x^T Q x. \]
If $\sigma_2 < 0$, then, for all $\hat{k}$ such that $0 \leq \hat{k} \leq \frac{\lambda_{\min}(Q) - \varepsilon}{\sigma_2}$,

\[ \dot{V}(x, \hat{k}) \leq -x^T x. \]
Thus, for all $x \in \mathbb{R}^n$ and all $\hat{k} \in \mathcal{D}$, $\dot{V}(x, \hat{k}) \leq -x^T x$ and the solution $(0, k_e)$ is Lyapunov stable.

Next, we show that, $k(t)$ converges. The dynamics (2.1)- (2.3) with the feedback (2.18) be expressed as

\[ \dot{x}(t) = [A_e(k) + \Delta] x(t), \]  \hspace{1cm} (2.23)
where $A_e(k)$ is given by (2.6), $\Delta \triangleq \left[ \delta \ 0_{n \times (n-1)} \right] \in \mathbb{R}^{n \times n}$, and $\delta \triangleq [-a_{n-1} \ldots -a_0] \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_0 > 0$ such that, for all constant $k \geq k_0$, $A_e(k)$ is asymptotically stable. For $k \geq k_0$, define $V_0(x, k) \triangleq x^T P(k)x$, where $P(k) > 0$ satisfies the Lyapunov equation $A_e^k(k) P(k) + P(k) A_e(k) = -e^{-\alpha k} R$, and $\alpha > 0$. Taking the derivative of $V_0(x, k)$ along a trajectory of (2.19) and (2.23) yields

\[ \dot{V_0}(x, k) = -e^{-\alpha k} x^T R x + k x^T \frac{\partial P(k)}{\partial k} x + x^T [A_e^T(k) P(k) + P(k) \Delta] x \]  \hspace{1cm} (2.24)
Define $\Omega(k) \triangleq \frac{1}{2} e^{-\frac{3}{2} k} R^2 - \sqrt{2} e^{-\frac{3}{2} k} R^{-\frac{1}{2}} P(k) \Delta$, and since $0 \leq \Omega^T(k) \Omega(k)$, it follows that $\Delta^T P(k) + P(k) \Delta \leq \frac{1}{2} e^{-\alpha k} R + 2 e^{\alpha k} \Delta^T P(k) R^{-1} P(k) \Delta$. Combining this with (2.24) yields

\[ \dot{V_0}(x, k) \leq -e^{-\alpha k} x^T \left[ \frac{1}{2} R - 2 e^{2 \alpha k} \delta P_1(k) R^{-1} P_1(k) \delta \right] x + k x^T \frac{\partial P(k)}{\partial k} x. \]  \hspace{1cm} (2.25)

Lemma 2.2 implies that $\lim_{k \to \infty} e^{\alpha k} p_1(k)$ exists, and thus $\lim_{k \to \infty} e^{2 \alpha k} \delta P_1(k) R^{-1} P_1(k) \delta = 0$, and there exists $k_1 \geq k_0$ such that, for all $k \geq k_1$,

\[ 2 e^{2 \alpha k} \delta P_1(k) R^{-1} p_1(k) \delta \leq \frac{1}{4} \Delta. \]  \hspace{1cm} (2.26)

To prove that $\lim_{k \to \infty} k(t)$ exists, suppose that $k(t)$ diverges to infinity in either finite or infinite time. Then there exists $t_3 > 0$ such that $k(t_3) = k_3$. Since $k(t)$ does not escape at $t_3$, it follows from (2.19) that $x(t)$ does not escape at time $t_3$. Let $t > t_3$ be such that $k(t)$ exists on $[t_3, t)$. Integrating (2.26) from $t_3$ to $t$ and from $k_3$ to $k(t)$ and solving for $k(t)$ yields $k(t) \leq k(t_3) + 4 V_0(x(t_3), k_3) - 4 V_0(x(t), k(t)) \leq k(t_3) + 4 V_0(x(t_3), k_3)$. Hence $k(t)$ is bounded on $[0, \infty)$, and thus $k(t)$ does not diverge to infinity. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t \to \infty} k(t)$ exists.

Next, we show that $x(t)$ is bounded. Define the function $V_1(x) = x^T x$. Taking the derivative of $V_1(x)$ along a trajectory of (2.19)-(2.20) yields $V_1(x, k) = x^T \left[ A_e^k(k) + \bar{A}(k) \right] x$. Since $k(t)$ converges, there exist $\eta > 0$ such that $V_1(x, k) \leq \eta x^T x = \eta e^{-\alpha k} k$, which implies $\dot{V}_1(x, k) \leq \eta e^{-\alpha k} k$. Integrating from 0 to $t$ and from $k(0)$ to $k(t)$ and solving for $V_1(x(t))$ yields $V_1(x(t)) = \frac{1}{\alpha} e^{-\alpha k(t)} + V_1(x(0)) - \frac{1}{\alpha} e^{-\alpha k(0)}$. Since $k(t)$ is bounded, we conclude that $\dot{V}_1(x(t))$ is bounded. Thus, $x(t)$ is bounded.

Now, we show that $\lim_{t \to \infty} x(t) = 0$. Since $\lim_{t \to \infty} k(t)$ exists, $A(k(t))$ is bounded. The dynamics (2.20) implies $|\dot{x}(t)| \leq |\dot{A}(k(t))| |x(t)|$ and since $\dot{A}(k(t))$ and $x(t)$ are bounded, it follows that $\dot{x}(t)$ is bounded. Thus, $\frac{1}{\alpha} |e^{-\alpha k(t)} x^T(t) R x(t)| = e^{-\alpha k(t)} \left(-\dot{k}(t) x^T(t) R x(t) + 2 x^T(t) R \dot{x}(t) \right)$ is bounded. Since the derivative of $\dot{k}(t) = e^{-\alpha k(t)} x^T(t) R x(t)$ is bounded, $\dot{k}(t)$ is uniformly continuous. Since $k(t)$ is uniformly continuous and $\lim_{t \to \infty} k(t)$ exists, Barabat’s lemma implies that $\lim_{t \to \infty} e^{-\alpha k(t)} x^T(t) R x(t) = 0$. Thus, $\lim_{t \to \infty} x(t) = 0$.

3. Adaptive Stabilization for Nonlinear Systems

In this section, we consider adaptive stabilization for the $n$th order nonlinear system

\[ q^{(n)}(t) + m_n q^{(n-1)}(t) + \ldots + q^{(1)}(t) = bu(t), \]  \hspace{1cm} (3.1)
where, for $i = 0, \ldots, n - 1$, $m_i : \mathbb{R}^n \to \mathbb{R}$ and $b \in \mathbb{R}$. We make the following assumptions.

(i) The functions $m_0, \ldots, m_{n-1}$ are locally Lipschitz.

(ii) The functions $m_0, \ldots, m_{n-1}$ are bounded. That is, for $i = 0, \ldots, n - 1$, there exists $\mu > 0$ such that, for all $q^{(n-1)}, \ldots, q, q \in \mathbb{R}$, $[m_i(q^{(n-1)}, \ldots, q, q)] \leq \mu$.

The bound $\mu$ is unknown.

(iii) $b \neq 0$, and $\text{sgn}(b)$ is known.

(iv) The full state $q, \dot{q}, \ldots, q^{(n-1)}$ is available for feedback.

The system (3.1) can be written in the state-dependent controllable canonical form
\[
\dot{x}(t) = A(x(t))x(t) + Bu(t),
\]
where
\[
A(x) \triangleq \begin{bmatrix}
-m_{n-1}(x) & -m_{n-2}(x) & \cdots & -m_0(x) \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix},
\]
\[
B^T \triangleq \begin{bmatrix}
b & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0
\end{bmatrix},
\]
\[
x^T \triangleq \begin{bmatrix}
q^{(n-1)} & q^{(n-2)} & \cdots & \dot{q} & q
\end{bmatrix}.
\]

We now present a nonlinear extension of Lemma 2.3.

**Lemma 3.1.** Consider the system (3.2)-(3.4). Let $g(s)$ be the Hurwitz polynomial
\[
g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0,
\]
where $g_{n-1} > 0$. Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the feedback
\[
u(t) = -\text{sgn}(b(k))Gx(t),
\]
where $k \in \mathbb{R}$. Then there exists $k_s > 0$ such that, for all $k \geq k_s$, the origin of the closed-loop system is globally asymptotically stable.

**Proof.** The system (3.2)-(3.4) with the feedback (3.7) can be written as
\[
\dot{x}(t) = [A_s(k) + \Delta(x(t))]x(t),
\]
where $A_s(k)$ is given by (2.6), $\Delta(x) \triangleq \begin{bmatrix} \delta(x) & 0_{n \times (n-1)} \end{bmatrix}^T \in \mathbb{R}^{n \times n}$, and $\delta(x) \triangleq \begin{bmatrix} -m_{n-1}(x) & \cdots & -m_0(x) \end{bmatrix}^T \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_1$ such that, for all $k \geq k_1$, the matrix $A_s(k)$ is asymptotically stable. For all $k \geq k_1$, let $P(k)$ be the solution to the Lyapunov equation $A^T_s(k)P(k) + P(k)A_s(k) = -Q + I$, where $Q > 0$. Furthermore, let $p_1(k)$ denote the first column of $P(k)$. Lemma 2.2 also provides the asymptotic property $\lim_{k \to \infty} p_1(k) = 0$. Now, consider the Lyapunov candidate $V(x) \triangleq x^T P(x) x$, where $k \geq k_1$. Taking the derivative along a closed-loop trajectory yields
\[
\dot{V}(x) = -x^T (Q + I) x + x^T [\Delta^T(x) P(k) + P(k) \Delta(x)] x.
\]
Since $0 \leq \Omega(x)^T \Omega(x)$ where $\Omega(x) \triangleq \sqrt{2}P(k)\Delta(x) - \frac{1}{\sqrt{2}}I$, it follows that $\Delta^T(x) P(k) + P(k) \Delta(x) \leq \frac{1}{2} I + 2\Delta^T(x) P^2(k) \Delta(x)$. Combining this with (3.9) yields
\[
\dot{V}(x) \leq -x^T Qx - \frac{1}{2} x^T x + 2x^T [\delta(x)p_1^T(k)p_1(k)\delta(x)] x.
\]
(3.10)

Since $p_1(k) \to 0$ as $k \to \infty$, let $k_a \geq k_1$ be such that, for all $k \geq k_a$, $p_1^T(k)p_1(k) \leq \left( \frac{1}{2\sqrt{2}} \right) I$, where $\delta^T \triangleq \begin{bmatrix} \mu & \cdots & \mu \end{bmatrix}$. Therefore, for all $k \geq k_a$, $\dot{V}(x) \leq -x^T Qx$. Hence, for all $k \geq k_a$, the origin is globally asymptotically stable.

Now we present the main result of this paper, namely, Lyapunov-stable adaptive stabilization of a class of nonlinear systems.

**Theorem 3.1.** Consider the nonlinear system (3.2)-(3.4). Let $g(s)$ be the Hurwitz polynomial (3.6), where $g_{n-1} > 0$.

Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the adaptive feedback controller
\[
u(t) = -\text{sgn}(k)Gx(t),
\]
where $R$ is positive definite and $\alpha > 0$. Then, there exists $k_s > 0$, such that for all $k_c \geq k_s$, the equilibrium solution $(0, k_c)$ of the closed-loop system (3.2)-(3.4) and (3.11)-(3.12) is Lyapunov stable. Furthermore, for all initial conditions $x(0)$ and $k(0)$, $k_c \triangleq \lim_{t \to \infty} k(t)$ exists and $\lim_{t \to \infty} x(t) = 0$.

**Proof.** The dynamics (3.2)-(3.4) with the feedback (3.11) can be written as
\[
\dot{x}(t) = [A_s(k) + \Delta(x(t))]x(t),
\]
where $A_s(k)$ is given by (2.6), $\Delta(x) \triangleq \begin{bmatrix} \delta(x) & 0_{n \times (n-1)} \end{bmatrix}^T \in \mathbb{R}^{n \times n}$, and $\delta(x) \triangleq \begin{bmatrix} -m_{n-1}(x) & \cdots & -m_0(x) \end{bmatrix}^T \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_1$ such that for all $k \geq k_1$, $A_s(k)$ is asymptotically stable. For all $k \geq k_1$, let $P(k)$ be the solution to the Lyapunov equation $A^T_s(k)P(k) + P(k)A_s(k) = -(Q + I)$, where $Q > 0$. Furthermore, let $p_1(k)$ denote the first column of $P(k)$. Lemma 2.2 also provides the asymptotic property $\lim_{k \to \infty} p_1(k) = 0$. Since $p_1(k) \to 0$ as $k \to \infty$, let $k_a \geq k_1$ be such that, for all $k \geq k_a$, $p_1^T(k)p_1(k) \leq \left( \frac{1}{2\sqrt{2}} \right) I$, where $\delta^T \triangleq \begin{bmatrix} \mu & \cdots & \mu \end{bmatrix}$. Let $k_c \geq k_s$, define $A_e \triangleq A_s(k_c)$, and define $\tilde{k}(t) \triangleq k_c - k(t)$ so that (3.12)-(3.13) can be written as
\[
\dot{x} = A_e x + \text{sgn}(b)\tilde{k}BGx + \Delta(x)x.
\]
(3.14)

Since $A_e$ is asymptotically stable, then there exists $P_e > 0$ such that $A^T_e P_e + P_e A_e = -(Q + I)$, where $Q > 0$. Let $p_e$ denote the first column of $P_e$. Next, consider the Lyapunov...
candidate $V(x, \tilde{k}) \triangleq x^T P x + \tilde{k}^2$, where $V : \mathbb{R}^n \times \mathbb{D} \to [0, \infty)$ and the domain $\mathbb{D}$ will be determined. The derivative of $V(x, \tilde{k})$ along a closed-loop trajectory is

$$\dot{V}(x, \tilde{k}) = x^T \left[ A^T P_e + P_e A_e \right] x - 2\tilde{k} \frac{\partial P(k)}{\partial k} x + \sigma^2 + \frac{1}{2} \frac{T}{\eta} \frac{T}{\Delta^T(x)} P(k) R^{-1} P(k) \Delta(x) x.$$  

Combining this with (3.17) yields

$$\dot{V}(x, k) \leq -\frac{1}{2} e^{-\alpha k} x^T R x + \frac{1}{2} \frac{T}{\eta} \frac{T}{\Delta^T(x)} P(k) R^{-1} P(k) \Delta(x) x.$$  

(3.18)

Lemma 2.2 implies that $\lim_{t \to \infty} e^{\alpha k} p_1(k)$ exists, and thus $\lim_{t \to \infty} e^{\alpha k} p_1(k) = 0$. Since $\delta(x)$ is bounded for all $x \in \mathbb{R}$, it follows that $\lim_{t \to \infty} 2 e^{\alpha k} \delta(x) p_1(k) R^{-1} p_1(k) \Delta(x) x = 0$, and there exists $k_1 \geq k_2$ such that, for all $k \geq k_1$, $2 e^{\alpha k} \delta(x) p_1(k) R^{-1} p_1(k) \delta(x) x \leq \frac{1}{2} R$. Then, for all $k \leq k_1$, $V_0(x, k) \leq -\frac{1}{2} e^{-\alpha k} x^T R x + \frac{1}{2} \frac{T}{\eta} \frac{T}{\Delta^T(x)} P(k) R^{-1} P(k) \Delta(x) x$. Lemma 2.2 also implies that there exists $k_2 \geq k_1$, for all $k \geq k_2$, $\eta e^{\alpha k} x^T R x \leq \frac{1}{4} k$, which implies

$$\dot{V}(x, k) \leq -\frac{1}{4} k,$$  

(3.19)

To prove that $\lim_{t \to \infty} k(t)$ exists, suppose that $k(t)$ diverges to infinity in either finite or infinite time. Then there exists $t_0 > 0$ such that $k(t_0) = k_0$. Since $k(t)$ does not escape at $t_0$, it follows from (3.12) that $x(t)$ does not escape at time $t_0$. Let $t > t_0$ be such that $k(t)$ exists on $[t_0, t]$. Integrating (3.19) from $t_0$ to $t$ and from $k_0$ to $k(t)$ and solving for $k(t)$ yields $k(t) \leq k(t_0) \leq 4 V_0(x(t_0), k_0) - 4 V_0(x(t), k_0) \leq k(t_0) + 4 V_0(x(t_0), k_0)$. Hence $k(t)$ is bounded on $[0, \infty)$, and thus $k(t)$ does not diverge to infinity. Since $k(t)$ is non-decreasing, $\lim_{t \to \infty} k(t)$ exists.

Next, we show that $x(t)$ is bounded. Define the function $V_1(x) = x^T x$. Taking the derivative of $V_1(x)$ along a trajectory of (3.12)-(3.13) yields $V_1(x, k) = x^T \left[ A^T(k) + A_s(k) \right] x + x^T \left[ A^T(x) + A(x) \right] x$. Since $k(t)$ converges and $\Delta(x)$ is bounded, there exist $\eta > 0$ such that $V_1(x, k) \leq \eta x^T R x = \eta e^{\alpha k} x^T R x$, which implies $V_1(x, k) \leq \eta e^{\alpha k} x^T R x$. Integrating from $0$ to $t$ and from $k_0$ to $k(t)$ and solving for $V_1(x(t))$ yields $V_1(x(t)) \leq e^{\alpha k} x^T R x + \eta e^{\alpha k} x^T R x$. Since $k(t)$ is bounded, we conclude that $V_1(x(t))$ is bounded. Thus, $x(t)$ is bounded.

Next, we show that $\lim_{t \to \infty} x(t) = 0$. The dynamics (3.13) implies

$$\dot{x}(t) = \left[ -\frac{\partial A_s(k)}{\partial k} + A_s(k) \right] x(t) + \left[ A_s(k) + \frac{1}{2} \frac{T}{\Delta^T(x)} P(k) \Delta(x) R^{-1} P(k) \Delta(x) \right] x(t).$$  

(3.20)

Since $\lim_{t \to \infty} k(t)$ exists, $A_s(k)$ is bounded. Furthermore, for all $x \in \mathbb{R}^n$, $\Delta(x) = \Delta(x(t))$ is bounded. Since $\dot{A}(k(t)) = \Delta(x(t))$, and $x(t)$ is bounded, it follows from (3.20) that $\dot{x}(t)$ is bounded. Then, $\frac{1}{2} \frac{T}{\Delta^T(x)} P(k) \Delta(x) R^{-1} P(k) \Delta(x)$ is bounded. Since the derivative of $k(t)$ is uniformly continuous, $k(t)$ is uniformly continuous and $\lim_{t \to \infty} k(t)$ exists, Barabanov’s lemma implies that $\lim_{t \to \infty} e^{-\alpha k(t)} x^T(t) R x(t) = 0$. Thus, $\lim_{t \to \infty} x(t) = 0$. 

\[ \square \]
4. Nonlinear Spring-Mass-Damper Example

In this section, we consider Lyapunov-stable adaptive stabilization of the nonlinear spring-mass-damper

\[ m\ddot{q}(t) + c\dot{q}(t) + k\dot{q}(t) = u(t), \]  

where

\[ \ddot{q}(t) = \begin{cases} \left( c + \frac{d}{2} \right) \dot{q}(t), & |\dot{q}(t)| < \delta, \\ c\dot{q}(t) + \text{sgn}(\dot{q}(t))d, & |\dot{q}(t)| \geq \delta, \end{cases} \]  

and \( \delta, c, d, h, k_0 > 0 \). The function \( \dot{c}(\cdot) \) is a continuous approximation of Coulomb friction and satisfies assumption (i). The function \( k(\cdot) \) is a linear spring with a deadzone.

This nonlinear system is shown in Figure 1. Note that the uncontrolled system has a continuum of equilibria, and the origin of the system is semistable, but not asymptotically stable. (For the definition of semistability, see [7]). For this example, the mass \( m = 3 \) kg, the viscous friction \( c = 2 \) kg/s, the Coulomb friction \( d = 20 \) N, the spring stiffness \( k_0 = 2 \) kg/s², the deadzone gap \( h = 10 \) m, and \( \delta = 0.1 \) m/s.

Note that the system (4.1) satisfies assumptions (i)-(iv) and the adaptive controller presented in Theorem 3.1 can be used to stabilize the origin. This controller is given by

\[ u(t) = -k(t) \left[ g_1 \right. \left. \begin{array}{c} g_1 \\ g_0 \end{array} \right] \left[ \begin{array}{c} \dot{q}(t) \\ q(t) \end{array} \right], \]  

\[ k(t) = e^{-\alpha k(t)} \left[ \begin{array}{c} \dot{q}(t) \\ q(t) \end{array} \right]^T R \left[ \begin{array}{c} \dot{q}(t) \\ q(t) \end{array} \right], \]

where \( g_0 > 0, g_1 > 0, k(0) \geq 0, \alpha > 0 \), and \( R \) is positive definite. We choose the controller parameters \( g_0 = 11, g_1 = 7, \alpha = 0.1, \text{and } R = I \). The system (4.1) with the adaptive controller (4.4)-(4.5) is simulated with the initial conditions \( k(0) = 0, q(0) = -25 \) m, and \( \dot{q}(0) = 10 \) m/s. The time histories of the position \( q(t) \) and velocity \( \dot{q}(t) \) for the open-loop and closed-loop systems are shown in Figure 2. The equilibrium of the open-loop system is semistable, and, while the velocity converges to zero, the position converges to approximately -15.6 m. The adaptive controller stabilizes the equilibrium so that both the velocity and position converge to zero. The adaptive parameter \( k \) converges to approximately 42.3.

Next we apply Theorem 3.1 to the unstable system \( m\ddot{q}(t) + c\dot{q}(t) - k\dot{q}(t) = u(t) \), which is a modification of (4.1) in which the sign of the stiffness term is negative. This system is simulated with the adaptive controller (4.4)-(4.5) connected in feedback. The initial conditions are \( k(0) = 0, q(0) = -25 \) m, and \( \dot{q}(0) = 15 \) m/s. Figure 3 shows the time histories of the position and velocity for the open-loop and closed-loop system.

The open-loop system is unstable, and the adaptive controller stabilizes the origin.

REFERENCES