2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Lyapunov-Stable Adaptive Stabilization of Nonlinear Systems with Matched Uncertainty

Jesse B. Hoagg and Dennis S. Bernstein
Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140, jhoagg@umich.edu

1. Introduction

Adaptive stabilization of linear time-invariant plants with full state feedback has been considered in [1–4] using Lyapunov-based gradient update laws. Lyapunov-based adaptive stabilization has several extensions to nonlinear systems. In [5], a variation of the controller presented in [4] is shown to stabilize a class of scalar second-order nonlinear systems with partial-state-dependent uncertainty. In particular, the adaptive controller of [5] can stabilize the scalar nonlinear system $m\ddot{q}(t)+g(q(t))\dot{q}(t)+f(q(t))q(t) = bu(t)$, where the functions $f(\cdot)$ and $g(\cdot)$ are lower bounded but otherwise unknown. In the present paper, a novel full-state-feedback adaptive controller is used to stabilize nth-order nonlinear systems with bounded state-dependent uncertainty. First, we develop the controller for linear systems, then extended the result to nonlinear systems.

2. Adaptive Stabilization for Linear Systems

We consider the single-input linear-time invariant system

$$\dot{x} = Ax + Bu, \tag{2.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$. We make the following assumptions.

(i) The system is in companion form, where

$$A \triangleq \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}, \tag{2.2}$$

$$B \triangleq \begin{bmatrix} b & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}. \tag{2.3}$$

(ii) $b \neq 0$, and $\operatorname{sgn}(b)$ is known.

(iii) The full state x is available for feedback.

We begin this section by providing several useful results regarding a matrix in controllable canonical form. Consider the matrix $\mathcal{A} \in \mathbb{R}^{n\times n}$, which has the characteristic polynomial $d(s) \triangleq s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \cdots + d_1 s + d_0$ and is in the companion form (2.2). The Hurwitz matrix associated with the characteristic polynomial $d(s)$ is

$$H \triangleq \begin{bmatrix} d_{n-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ d_{n-3} & d_{n-2} & d_{n-1} & 1 & \cdots & 0 & 0 \\ d_{n-5} & d_{n-4} & d_{n-3} & d_{n-2} & \cdots & 0 & 0 \\ d_{n-7} & d_{n-6} & d_{n-5} & d_{n-4} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d_1 & d_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & d_0 \end{bmatrix}. \tag{2.4}$$

The following result, given in [6], concerns the solution to the Lyapunov equation for a matrix in controllable canonical form.

Lemma 2.1. *Consider the asymptotically stable controllable canonical form $\mathcal{A}$. Let $P \in \mathbb{R}^{n\times n}$ be the positive-definite solution to the Lyapunov equation $\mathcal{A}^{\mathrm{T}}P + P\mathcal{A} = -Q$, where $Q \in \mathbb{R}^{n\times n}$ is positive definite. Let p_1 denote the first column of P. Then p_1 satisfies $2DHDp_1 = q$, where H is given by (2.4), $D \triangleq \operatorname{diag}(1,-1,1,\ldots)$, and*

$$q \triangleq \begin{bmatrix} \sum_{1\le i,j\le n, i+j=2}(-1)^{i-1}Q_{i,j} \\ \sum_{1\le i,j\le n, i+j=4}(-1)^{i-2}Q_{i,j} \\ \vdots \\ \sum_{1\le i,j\le n, i+j=2n}(-1)^{i-n}Q_{i,j} \end{bmatrix}. \tag{2.5}$$

Lemma 2.2. *Let $g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0$ be a Hurwitz polynomial where $g_{n-1} > 0$ and define*

$$A_{\mathrm{s}}(k) \triangleq \begin{bmatrix} -k\,|b|\,g_{n-1} & -k\,|b|\,g_{n-2} & \cdots & -k\,|b|\,g_0 \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 0 \end{bmatrix}, \tag{2.6}$$

where $k \in \mathbb{R}$. Then, there exists $k_{\mathrm{s}} > 0$ such that, for all $k \ge k_{\mathrm{s}}$, $A_{\mathrm{s}}(k)$ is asymptotically stable and thus, there exists a positive definite $P(k)$ such that

$$A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -e^{-\alpha k}Q, \tag{2.7}$$

where $Q > 0$ and $\alpha \ge 0$. Furthermore, $\lim_{k\to\infty} p_1(k) = 0$ and $\lim_{k\to\infty} ke^{\alpha k}p_1(k)$ exists, where $p_1(k)$ denotes the first column of $P(k)$. If, in addition, $\alpha > 0$, then there exists $k_2 \ge k_{\mathrm{s}}$ such that, for all $k \ge k_2$, $\frac{\partial P(k)}{\partial k}$ is negative definite.

Proof. Let $H(k)$ be the Hurwitz matrix associated with the characteristic polynomial of $A_s(k)$. The Hurwitz stability conditions for the characteristic polynomial of $A_{\mathrm{s}}(k)$ are polynomials in k given by

$$\Lambda_1 \triangleq k\,|b|\,g_{n-1} > 0, \tag{2.8}$$

$$\Lambda_2 \triangleq \begin{vmatrix} k\,|b|\,g_{n-1} & 1 \\ k\,|b|\,g_{n-3} & k\,|b|\,g_{n-2} \end{vmatrix} > 0, \tag{2.9}$$

$$\Lambda_3 \triangleq \begin{vmatrix} k\,|b|\,g_{n-1} & 1 & 0 \\ k\,|b|\,g_{n-3} & k\,|b|\,g_{n-2} & k\,|b|\,g_{n-1} \\ k\,|b|\,g_{n-5} & k\,|b|\,g_{n-4} & k\,|b|\,g_{n-3} \end{vmatrix} > 0, \tag{2.10}$$

$$\vdots$$

$$\Lambda_n \triangleq \begin{vmatrix} \Lambda_3 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ & & k\,|b|\,g_2 \\ 0 \;\cdots & 0 & k\,|b|\,g_0 \end{vmatrix} > 0. \tag{2.11}$$

For sufficiently large k, the Hurwitz conditions are satisfied since $g(s)$ is Hurwitz with positive leading coefficient. Therefore, there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, the matrix $A_{\rm s}(k)$ is asymptotically stable. Then, for all $k \geq k_{\rm s}$, there exists $P(k) > 0$ satisfying (2.7).

Now, we consider the asymptotic properties of $p_1(k)$. For all $k \geq k_{\rm s}$, the inverse of the Hurwitz matrix exists and can be expressed as $H^{-1}(k) = \frac{1}{\det(H(k))}\hat{H}(k)$, where

$$\hat{H}(k) \triangleq \begin{bmatrix} [H(k)]_{1,1} & -[H(k)]_{2,1} & \cdots (-1)^{n+1}[H(k)]_{n,1} \\ -[H(k)]_{1,2} & [H(k)]_{2,2} & \\ \vdots & \ddots & \vdots \\ (-1)^{n+1}[H(k)]_{1,n} & \cdots & [H(k)]_{n,n} \end{bmatrix}. \tag{2.12}$$

where $[H(k)]_{i,j}$ is the $(i,j)^{\rm th}$ minor of $H(k)$. The determinant of $H(k)$ is a degree n polynomial in k, while $[H(k)]_{i,j}$ is a polynomial in k of degree not exceeding $n-1$. Therefore, $\lim_{k\to\infty} H^{-1}(k) = 0$, and $\lim_{k\to\infty} kH^{-1}(k)$ exists. Using Lemma 2.1 we obtain $\lim_{k\to\infty} p_1(k) = \lim_{k\to\infty} \frac{1}{2}D^{-1}H^{-1}(k)D^{-1}e^{-\alpha k}q = 0$ and $\lim_{k\to\infty} ke^{\alpha k}p_1(k) = \lim_{k\to\infty} \frac{1}{2}D^{-1}kH^{-1}(k)D^{-1}q$ exists, where q is determined from Q using (2.5).

Next, we show that for k sufficiently large, $\frac{\partial P(k)}{\partial k} < 0$. Taking the partial derivative of (2.7) with respect to k yields $-A_{\rm s}^{\rm T}(k)\frac{\partial P(k)}{\partial k} - \frac{\partial P(k)}{\partial k}A_{\rm s}(k) = \hat{Q}(k)$ where

$$\hat{Q}(k) \triangleq -\alpha e^{-\alpha k}Q + \delta e_1^{\rm T}P(k) + P(k)e_1\delta^{\rm T}, \tag{2.13}$$

$e_1 \triangleq \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\rm T} \in \mathbb{R}^n$, and $\delta \triangleq -|b|\begin{bmatrix} g_{n-1} & \cdots & g_0 \end{bmatrix}^{\rm T} \in \mathbb{R}^n$. Define $\Omega(k) \triangleq \sqrt{\frac{\alpha}{2}}e^{-\frac{\alpha}{2}k}Q^{\frac{1}{2}} - \sqrt{\frac{2}{\alpha}}e^{\frac{\alpha}{2}k}Q^{-\frac{1}{2}}P(k)e_1\delta^{\rm T}$, and since $0 \leq \Omega^{\rm T}(k)\Omega(k)$, it follows that $\delta e_1^{\rm T}P(k) + P(k)e_1\delta^{\rm T} \leq \frac{\alpha}{2}e^{-\alpha k}Q + \frac{2}{\alpha}e^{\alpha k}\delta e_1^{\rm T}P(k)Q^{-1}P(k)e_1\delta^{\rm T}$. Combining this with (2.13) yields

$$\hat{Q}(k) \leq -e^{-\alpha k}\left[\frac{\alpha}{2}Q - \frac{2}{\alpha}e^{2\alpha k}\delta p_1^{\rm T}(k)Q^{-1}p_1(k)\delta^{\rm T}\right]. \tag{2.14}$$

Since $\lim_{k\to\infty} ke^{\alpha k}p_1(k)$ exists, it follows that $\lim_{k\to\infty} e^{\alpha k}p_1(k) = 0$, and thus $\lim_{k\to\infty} \frac{2}{\alpha}e^{2\alpha k}\delta p_1^{\rm T}(k)Q^{-1}p_1(k)\delta^{\rm T} = 0$. Let $k_2 \geq k_{\rm s}$ be such that, for all $k \geq k_2$, $\frac{2}{\alpha}e^{2\alpha k}\delta p_1^{\rm T}(k)Q^{-1}p_1(k)\delta^{\rm T} < \frac{\alpha}{2}Q$, and thus $\hat{Q}(k) < 0$. Then it follows that, for all $k \geq k_2$, $-A_{\rm s}^{\rm T}(k)\frac{\partial P(k)}{\partial k} - \frac{\partial P(k)}{\partial k}A_{\rm s}(k) < 0$. Since $A_{\rm s}(k)$ is asymptotically stable, $\frac{\partial P(k)}{\partial k} < 0$. □

Now, we present a Lyapunov proof of a high-gain stabilizing controller for the linear system (2.1)-(2.3).

Lemma 2.3. *Consider the linear system (2.1)-(2.3). Let $g(s)$ be the Hurwitz polynomial*

$$g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0, \tag{2.15}$$

where $g_{n-1} > 0$. Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the feedback

$$u(t) = -{\rm sgn}(b)kGx(t), \tag{2.16}$$

where $k \in \mathbb{R}$. Then there exists $k_{\rm s} > 0$ such that, for all $k \geq k_{\rm s}$, the origin of the closed-loop system is asymptotically stable.

Proof. The system (2.1)-(2.3) with the feedback (2.16) can be written as $\dot{x}(t) = [A_{\rm s}(k) + \Delta]\,x(t)$, where $A_{\rm s}(k)$ is given by (2.6), $\Delta \triangleq \begin{bmatrix} \delta & 0_{n\times(n-1)} \end{bmatrix}^{\rm T} \in \mathbb{R}^{n\times n}$, and $\delta \triangleq \begin{bmatrix} -a_{n-1} & \cdots & -a_0 \end{bmatrix}^{\rm T} \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_1 > 0$ such that, for all $k \geq k_1$, $A_{\rm s}(k)$ is asymptotically stable. For all $k \geq k_1$, let $P(k) > 0$ be the solution to the Lyapunov equation $A_{\rm s}^{\rm T}(k)P(k) + P(k)A_{\rm s}(k) = -(Q+I)$, where $Q > 0$. Furthermore, let $p_1(k)$ denote the first column of $P(k)$. Next, consider the Lyapunov candidate $V(x) = x^{\rm T}P(k)x$, where $k \geq k_1$. Taking the derivative along the closed-loop trajectory yields

$$\begin{aligned}\dot{V}(x) = & -x^{\rm T}(Q+I)\,x \\ & + x^{\rm T}\left[\Delta^{\rm T}P(k) + P(k)\Delta\right]x. \end{aligned} \tag{2.17}$$

Since $0 \leq \left(\sqrt{2}P(k)\Delta - \frac{1}{\sqrt{2}}I\right)^{\rm T}\left(\sqrt{2}P(k)\Delta - \frac{1}{\sqrt{2}}I\right)$, it follows that $\Delta^{\rm T}P(k) + P(k)\Delta \leq \frac{1}{2}I + 2\Delta^{\rm T}P^2(k)\Delta$. Combining this with (2.17) yields $\dot{V}(x) \leq -x^{\rm T}Qx - \frac{1}{2}x^{\rm T}x + 2x^{\rm T}\left[\delta p_1^{\rm T}(k)p_1(k)\delta^{\rm T}\right]x$. Since Lemma 2.2 implies $p_1(k) \to 0$ as $k \to \infty$, let $k_{\rm s} \geq k_1 > 0$ be such that, for all $k \geq k_{\rm s}$, $\delta p_1^{\rm T}(k)p_1(k)\delta^{\rm T} \leq \frac{1}{4}I$. Therefore, for all $k \geq k_{\rm s}$, $\dot{V}(x) \leq -x^{\rm T}Qx < 0$ for $x \neq 0$, and the origin is asymptotically stable. □

Now, we present a Lyapunov-stable adaptive stabilization algorithm for linear systems.

Theorem 2.1. *Consider the linear system (2.1)-(2.3). Let $g(s)$ be the Hurwitz polynomial (2.15) where $g_{n-1} > 0$. Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the adaptive feedback controller*

$$u(t) = -{\rm sgn}(b)k(t)Gx(t), \tag{2.18}$$

$$\dot{k}(t) = e^{-\alpha k(t)}x^{\rm T}(t)Rx(t), \tag{2.19}$$

where R is positive definite and $\alpha > 0$. Then, there exists $k_{\rm s} > 0$, such that for all $k_e \geq k_{\rm s}$, the equilibrium solution $(0, k_e)$ of the closed-loop system (2.1)-(2.3) and (2.18)-(2.19) is Lyapunov stable. Furthermore, for all initial conditions $x(0)$ and $k(0)$, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists and $\lim_{t\to\infty} x(t) = 0$.

Proof. The dynamics (2.1)-(2.3) with the feedback (2.18) can be expressed as

$$\dot{x}(t) = \tilde{A}(k(t))x(t), \tag{2.20}$$

where $\tilde{A}(k(t)) \triangleq A - k(t){\rm sgn}(b)BG$. Lemma 2.3 implies that there exists $k_{\rm s}$ such that for all $k \geq k_{\rm s}$, $\tilde{A}(k)$ is asymptotically stable. Let $k_e \geq k_{\rm s}$, define $A_e \triangleq \tilde{A}(k_e)$, and define $\tilde{k}(t) \triangleq k_e - k(t)$ so that (2.19)-(2.20) can be written as $\dot{x} = A_e x + \mathrm{sgn}(b)\tilde{k}BGx$. Since A_e is asymptotically stable, there exists $P_e > 0$ such that $A_e^{\mathrm{T}} P_e + P_e A_e = -Q$, where $Q > 0$. Next, consider the Lyapunov candidate $V(x, \tilde{k}) \triangleq x^{\mathrm{T}} P_e x + \tilde{k}^2$, where $V : \mathbb{R}^n \times \mathcal{D} \to [0, \infty)$ and the domain $\mathcal{D}$ will be determined. The derivative of $V(x, \tilde{k})$ along a closed-loop trajectory is

$$\begin{aligned}\dot{V}(x, \tilde{k}) = \tilde{k}x^{\mathrm{T}} \left[\mathrm{sgn}(b)G^{\mathrm{T}} B^{\mathrm{T}} P_e + \mathrm{sgn}(b)P_e BG\right] x \\ - x^{\mathrm{T}} Qx - 2\tilde{k}e^{-\alpha k} x^{\mathrm{T}} Rx. \qquad (2.21)\end{aligned}$$

First, consider the case $\tilde{k} \geq 0$, then $\dot{V}(x, \tilde{k}) \leq -x^{\mathrm{T}} Qx + \tilde{k}\sigma_1 x^{\mathrm{T}} x$, where $\sigma_1 \triangleq \lambda_{\max}\left(\mathrm{sgn}(b)G^{\mathrm{T}} B^{\mathrm{T}} P_e + \mathrm{sgn}(b)P_e BG\right)$. If $\sigma_1 \leq 0$, then $\dot{V}(x, \tilde{k}) \leq -x^{\mathrm{T}} Qx$. If $\sigma_1 > 0$, then let $0 < \varepsilon < \lambda_{\min}(Q)$. Thus, for all $\tilde{k}$ such that $0 \leq \tilde{k} \leq \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}$, $\dot{V}(x, \tilde{k}) \leq -\varepsilon x^{\mathrm{T}} x$.

Now, consider the case $\tilde{k} \leq 0$, then $\dot{V}(x, \tilde{k}) \leq -x^{\mathrm{T}} Qx + \tilde{k}\sigma_2 x^{\mathrm{T}} x$, where $\sigma_2 \triangleq \lambda_{\min}\left(\mathrm{sgn}(b)G^{\mathrm{T}} B^{\mathrm{T}} P_e + \mathrm{sgn}(b)P_e BG - 2R\right)$. If $\sigma_2 \geq 0$, then $\dot{V}(x, \tilde{k}) \leq -x^{\mathrm{T}} Qx$. If $\sigma_2 < 0$, then, for all $\tilde{k}$ such that $\frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \leq \tilde{k} \leq 0$, $\dot{V}(x, \tilde{k}) \leq -\varepsilon x^{\mathrm{T}} x$. If $\sigma_1 \leq 0$, then $\sigma_2 < 0$, so define the domain

$$\mathcal{D} \triangleq \begin{cases} \{\tilde{k} \in \mathbb{R} : -\infty < \tilde{k} \leq \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}\}, & \sigma_1 > 0, \sigma_2 \geq 0 \\ \{\tilde{k} \in \mathbb{R} : \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \leq \tilde{k} \leq \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}\}, & \sigma_1 > 0, \sigma_2 < 0 \\ \{\tilde{k} \in \mathbb{R} : \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \leq \tilde{k} < \infty\}, & \sigma_1 \leq 0, \sigma_2 < 0 \end{cases} \qquad (2.22)$$

Thus, for all $x \in \mathbb{R}^n$ and all $\tilde{k} \in \mathcal{D}$, $\dot{V}(x, \tilde{k}) \leq -\varepsilon x^{\mathrm{T}} x$ and the solution $(0, k_e)$ is Lyapunov stable.

Next, we show that, $k(t)$ converges. The dynamics (2.1)-(2.3) with the feedback (2.18) can be expressed as

$$\dot{x}(t) = [A_{\mathrm{s}}(k) + \Delta]\, x(t), \qquad (2.23)$$

where $A_{\mathrm{s}}(k)$ is given by (2.6), $\Delta \triangleq \left[\begin{array}{cc} \delta & 0_{n\times(n-1)} \end{array}\right]^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and $\delta \triangleq \left[\begin{array}{ccc} -a_{n-1} & \cdots & -a_0 \end{array}\right]^{\mathrm{T}} \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_{\mathrm{s}} > 0$ such that, for all constant $k \geq k_{\mathrm{s}}$, $A_{\mathrm{s}}(k)$ is asymptotically stable. For $k \geq k_{\mathrm{s}}$, define $V_0(x, k) \triangleq x^{\mathrm{T}} P(k)x$, where $P(k) > 0$ satisfies the Lyapunov equation $A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -e^{\alpha k}R$, and $\alpha > 0$. Taking the derivative of $V_0(x, k)$ along a trajectory of (2.19) and (2.23) yields

$$\begin{aligned}\dot{V}_0(x, k) = &- e^{-\alpha k} x^{\mathrm{T}} Rx + \dot{k}x^{\mathrm{T}} \frac{\partial P(k)}{\partial k} x \\ &+ x^{\mathrm{T}} \left[\Delta^{\mathrm{T}} P(k) + P(k)\Delta\right] x \qquad (2.24)\end{aligned}$$

Define $\Omega(k) \triangleq \sqrt{\frac{1}{2}} e^{-\frac{\alpha}{2}k} R^{\frac{1}{2}} - \sqrt{2} e^{\frac{\alpha}{2}k} R^{-\frac{1}{2}} P(k)\Delta$, and since $0 \leq \Omega^{\mathrm{T}}(k)\Omega(k)$, it follows that $\Delta^{\mathrm{T}} P(k) + P(k)\Delta \leq \frac{1}{2} e^{-\alpha k} R + 2e^{\alpha k} \Delta^{\mathrm{T}} P(k)R^{-1} P(k)\Delta$. Combining this with (2.24) yields

$$\begin{aligned}\dot{V}_0(x, k) \leq &- e^{-\alpha k} x^{\mathrm{T}} \left[\frac{1}{2} R - 2e^{2\alpha k} \delta p_1^{\mathrm{T}}(k) R^{-1} p_1(k)\delta^{\mathrm{T}}\right] x \\ &+ \dot{k}x^{\mathrm{T}} \frac{\partial P(k)}{\partial k} x. \qquad (2.25)\end{aligned}$$

Lemma 2.2 implies that $\lim_{k\to\infty} ke^{\alpha k} p_1(k)$ exists, and thus $\lim_{k\to\infty} e^{\alpha k} p_1(k) = 0$. It follows that $\lim_{k\to\infty} 2e^{2\alpha k} \delta p_1^{\mathrm{T}}(k) R^{-1} p_1(k)\delta^{\mathrm{T}} = 0$, and there exists $k_1 \geq k_{\mathrm{s}}$ such that, for all $k \geq k_1$, $2e^{2\alpha k} \delta p_1^{\mathrm{T}}(k) R^{-1} p_1(k)\delta^{\mathrm{T}} \leq \frac{1}{4} R$. Then, for all $k \geq k_1$, $\dot{V}_0(x, k) \leq -\frac{1}{4} e^{-\alpha k} x^{\mathrm{T}} Rx + \dot{k}x^{\mathrm{T}} \frac{\partial P(k)}{\partial k} x$. Lemma 2.2 also implies that there exists $k_2 \geq k_{\mathrm{s}}$ such that, for all $k \geq k_2$, $\frac{\partial P(k)}{\partial k} < 0$. Thus, for all $k \geq k_3 \triangleq \max(k_1, k_2)$, $\dot{V}_0(x, k) \leq -\frac{1}{4} e^{-\alpha k} x^{\mathrm{T}} Rx = -\frac{1}{4}\dot{k}$, which implies

$$\dot{V}_0(x, k)dt \leq -\frac{1}{4} dk. \qquad (2.26)$$

To prove that $\lim_{t\to\infty} k(t)$ exists, suppose that $k(t)$ diverges to infinity in either finite or infinite time. Then there exists $t_3 > 0$ such that $k(t_3) = k_3$. Since $k(t)$ does not escape at t_3, it follows from (2.19) that $x(t)$ does not escape at time t_3. Let $t > t_3$ be such that $k(\cdot)$ exists on $[t_3, t]$. Integrating (2.26) from t_3 to t and from k_3 to $k(t)$ and solving for $k(t)$ yields $k(t) \leq k(t_3) + 4V_0(x(t_3), k_3) - 4V_0(x(t), k(t)) \leq k(t_3) + 4V_0(x(t_3), k_3)$. Hence $k(t)$ is bounded on $[0, \infty)$, and thus $k(t)$ does not diverge to infinity. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists.

Next, we show that $x(t)$ is bounded. Define the function $V_1(x) = x^{\mathrm{T}} x$. Taking the derivative of $V_1(x)$ along a trajectory of (2.19)-(2.20) yields $\dot{V}_1(x, k) = x^{\mathrm{T}} \left[\tilde{A}^{\mathrm{T}}(k) + \tilde{A}(k)\right] x$. Since $k(t)$ converges, there exist $\eta > 0$ such that $\dot{V}_1(x, k) \leq \eta x^{\mathrm{T}} Rx = \eta e^{\alpha k}\dot{k}$, which implies $\dot{V}_1(x, k)dt \leq \eta e^{\alpha k} dk$. Integrating from 0 to t and from $k(0)$ to $k(t)$ and solving for $V_1(x(t))$ yields $V_1(x(t)) \leq \frac{\eta}{\alpha} e^{\alpha k(t)} + V_1(x(0)) - \frac{\eta}{\alpha} e^{\alpha k(0)}$. Since $k(t)$ is bounded, we conclude that $V_1(x(t))$ is bounded. Thus, $x(t)$ is bounded.

Now, we show that $\lim_{t\to\infty} x(t) = 0$. Since $\lim_{t\to\infty} k(t)$ exists, $\tilde{A}(k(t))$ is bounded. The dynamics (2.20) implies $||\dot{x}(t)|| \leq \left|\left|\tilde{A}(k(t))\right|\right| ||x(t)||$ and since $\tilde{A}(k(t))$ and $x(t)$ are bounded, it follows that $\dot{x}(t)$ is bounded. Thus, $\frac{d}{dt}\left[e^{-\alpha k(t)} x^{\mathrm{T}}(t)Rx(t)\right] = e^{-\alpha k(t)}\left(-\alpha\dot{k}(t)x^{\mathrm{T}}(t)Rx(t) + 2x^{\mathrm{T}}(t)R\dot{x}(t)\right)$ is bounded. Since the derivative of $\dot{k}(t) = e^{-\alpha k(t)} x^{\mathrm{T}}(t)Rx(t)$ is bounded, $\dot{k}(t)$ is uniformly continuous. Since $\dot{k}(t)$ is uniformly continuous and $\lim_{t\to\infty} k(t)$ exists, Barabalat's lemma implies that $\lim_{t\to\infty} e^{-\alpha k(t)} x^{\mathrm{T}}(t)Rx(t) = 0$. Thus, $\lim_{t\to\infty} x(t) = 0$. □

3. Adaptive Stabilization for Nonlinear Systems

In this section, we consider adaptive stabilization for the nth order nonlinear system

$$\begin{aligned}q^{(n)}(t) &+ m_{n-1}(q^{(n-1)}, \ldots, \dot{q}, q)q^{(n-1)}(t) \\ &+ m_{n-2}(q^{(n-1)}, \ldots, \dot{q}, q)q^{(n-2)}(t) \\ &+ \cdots + m_1(q^{(n-1)}, \ldots, \dot{q}, q)\dot{q}(t) \\ &+ m_0(q^{(n-1)}, \ldots, \dot{q}, q)q(t) = bu(t), \qquad (3.1)\end{aligned}$$

where, for $i = 0, \ldots, n-1$, $m_i : \mathbb{R}^n \to \mathbb{R}$ and $b \in \mathbb{R}$. We make the following assumptions.

(i) The functions $m_0, \ldots, m_{n-1}$ are locally Lipschitz.

(ii) The functions $m_0, \ldots, m_{n-1}$ are bounded. That is, for $i = 0, \ldots, n-1$, there exists $\mu > 0$ such that, for all $q^{(n-1)}, \ldots, \dot{q}, q \in \mathbb{R}$, $\left|m_i(q^{(n-1)}, \ldots, \dot{q}, q)\right| \le \mu$. The bound μ is unknown.

(iii) $b \neq 0$, and $\mathrm{sgn}(b)$ is known.

(iv) The full state $q, \dot{q}, \ldots, q^{(n-1)}$ is available for feedback.

The system (3.1) can be written in the state-dependent controllable canonical form

$$\dot{x}(t) = A(x(t))x(t) + Bu(t), \tag{3.2}$$

where

$$A(x) \triangleq \begin{bmatrix} -m_{n-1}(x) & -m_{n-2}(x) & \cdots & -m_0(x) \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 0 \end{bmatrix}, \tag{3.3}$$

$$B^{\mathrm{T}} \triangleq \begin{bmatrix} b & 0 & \cdots & 0 \end{bmatrix}, \tag{3.4}$$

$$x^{\mathrm{T}} \triangleq \begin{bmatrix} q^{(n-1)} & q^{(n-2)} & \cdots & \dot{q} & q \end{bmatrix}. \tag{3.5}$$

We now present a nonlinear extension of Lemma 2.3.

Lemma 3.1. *Consider the system (3.2)-(3.4). Let $g(s)$ be the Hurwitz polynomial*

$$g(s) \triangleq g_{n-1}s^{n-1} + g_{n-2}s^{n-2} + \cdots + g_0, \tag{3.6}$$

where $g_{n-1} > 0$. Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the feedback

$$u(t) = -\mathrm{sgn}(b)kGx(t), \tag{3.7}$$

where $k \in \mathbb{R}$. Then there exists $k_{\mathrm{s}} > 0$ such that, for all $k \ge k_{\mathrm{s}}$, the origin of the closed-loop system is globally asymptotically stable.

Proof. The system (3.2)-(3.4) with the feedback (3.7) can be written as

$$\dot{x}(t) = \left[A_{\mathrm{s}}(k) + \Delta(x(t))\right] x(t), \tag{3.8}$$

where where $A_{\mathrm{s}}(k)$ is given by (2.6), $\Delta(x) \triangleq \begin{bmatrix} \delta(x) & 0_{n\times(n-1)} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and $\delta(x) \triangleq \begin{bmatrix} -m_{n-1}(x) & \cdots & -m_0(x) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. Lemma 2.2 implies that there exists $k_1 > 0$ such that, for all $k \ge k_1$, the matrix $A_{\mathrm{s}}(k)$ is asymptotically stable. For all $k \ge k_1$, let $P(k) > 0$ be the solution to the Lyapunov equation $A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -(Q+I)$ where $Q > 0$. Furthermore, let $p_1(k)$ denote the first column of $P(k)$. Lemma 2.2 also provides the asymptotic property $\lim_{k\to\infty} p_1(k) = 0$. Now, consider the Lyapunov candidate $V(x) \triangleq x^{\mathrm{T}}P(k)x$, where $k \ge k_1$. Taking the derivative along a closed-loop trajectory yields

$$\dot{V}(x) = -x^{\mathrm{T}}\left(Q+I\right)x + x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(x)P(k) + P(k)\Delta(x)\right]x. \tag{3.9}$$

Since $0 \le \Omega(x)^{\mathrm{T}}\Omega(x)$ where $\Omega(x) \triangleq \sqrt{2}P(k)\Delta(x) - \frac{1}{\sqrt{2}}I$, it follows that $\Delta^{\mathrm{T}}(x)P(k) + P(k)\Delta(x) \le \frac{1}{2}I + 2\Delta^{\mathrm{T}}(x)P^2(k)\Delta(x)$. Combining this with (3.9) yields

$$\dot{V}(x) \le -x^{\mathrm{T}}Qx - \frac{1}{2}x^{\mathrm{T}}x + 2x^{\mathrm{T}}\left[\delta(x)p_1^{\mathrm{T}}(k)p_1(k)\delta^{\mathrm{T}}(x)\right]x. \tag{3.10}$$

Since $p_1(k) \to 0$ as $k \to \infty$, let $k_{\mathrm{s}} \ge k_1$ be such that, for all $k \ge k_{\mathrm{s}}$, $p_1^{\mathrm{T}}(k)p_1(k) \le \left(\frac{1}{4\hat{\delta}^{\mathrm{T}}\hat{\delta}}\right)I$, where $\hat{\delta}^{\mathrm{T}} \triangleq \begin{bmatrix} \mu & \cdots & \mu \end{bmatrix}$. Therefore, for all $k \ge k_{\mathrm{s}}$, $\dot{V}(x) \le -x^{\mathrm{T}}Qx$. Hence, for all $k \ge k_{\mathrm{s}}$, the origin is globally asymptotically stable. □

Now we present the main result of this paper, namely, Lyapunov-stable adaptive stabilization of a class of nonlinear systems.

Theorem 3.1. *Consider the nonlinear system (3.2)-(3.4). Let $g(s)$ be the Hurwitz polynomial (3.6), where $g_{n-1} > 0$. Define $G \triangleq \begin{bmatrix} g_{n-1} & g_{n-2} & \cdots & g_0 \end{bmatrix}$, and consider the adaptive feedback controller*

$$u(t) = -\mathrm{sgn}(b)k(t)Gx(t), \tag{3.11}$$

$$\dot{k}(t) = e^{-\alpha k(t)}x^{\mathrm{T}}(t)Rx(t), \tag{3.12}$$

where R is positive definite and $\alpha > 0$. Then, there exists $k_{\mathrm{s}} > 0$, such that for all $k_e \ge k_{\mathrm{s}}$, the equilibrium solution $(0, k_e)$ of the closed-loop system (3.2)-(3.4) and (3.11)-(3.12) is Lyapunov stable. Furthermore, for all initial conditions $x(0)$ and $k(0)$, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists and $\lim_{t\to\infty} x(t) = 0$.

Proof. The dynamics (3.2)-(3.4) with the feedback (3.11) can be written as

$$\dot{x}(t) = \left[A_{\mathrm{s}}(k) + \Delta(x(t))\right] x(t), \tag{3.13}$$

where $A_{\mathrm{s}}(k)$ is given by (2.6), $\Delta(x) \triangleq \begin{bmatrix} \delta(x) & 0_{n\times(n-1)} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, and $\delta(x) \triangleq \begin{bmatrix} -m_{n-1}(x) & \cdots & -m_0(x) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^n$. Lemma 2.2 implies that there exists k_1 such that for all $k \ge k_1$, $A_{\mathrm{s}}(k)$ is asymptotically stable. For all $k \ge k_1$, let $P(k) > 0$ be the solution to the Lyapunov equation $A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -(Q+I)$, where $Q > 0$. Furthermore, let $p_1(k)$ denote the first column of $P(k)$. Lemma 2.2 also provides the asymptotic property $\lim_{k\to\infty} p_1(k) = 0$. Since $p_1(k) \to 0$ as $k \to \infty$, let $k_{\mathrm{s}} \ge k_1$ be such that, for all $k \ge k_{\mathrm{s}}$, $p_1^{\mathrm{T}}(k)p_1(k) \le \left(\frac{1}{4\hat{\delta}^{\mathrm{T}}\hat{\delta}}\right)I$, where $\hat{\delta}^{\mathrm{T}} \triangleq \begin{bmatrix} \mu & \cdots & \mu \end{bmatrix}$. Let $k_e \ge k_{\mathrm{s}}$, define $A_e \triangleq A_{\mathrm{s}}(k_e)$, and define $\tilde{k}(t) \triangleq k_e - k(t)$ so that (3.12)-(3.13) can be written as

$$\dot{x} = A_e x + \mathrm{sgn}(b)\tilde{k}BGx + \Delta(x)x. \tag{3.14}$$

Since A_e is asymptotically stable, then there exists $P_e > 0$ such that $A_e^{\mathrm{T}}P_e + P_eA_e = -(Q+I)$, where $Q > 0$. Let p_e denote the first column of P_e. Next, consider the Lyapunov candidate $V(x,\tilde{k}) \triangleq x^{\mathrm{T}}P_e x + \tilde{k}^2$, where $V : \mathbb{R}^n \times \mathcal{D} \to [0,\infty)$ and the domain $\mathcal{D}$ will be determined. The derivative of $V(x,\tilde{k})$ along a closed-loop trajectory is

$$\begin{aligned}\dot{V}(x,\tilde{k}) &= x^{\mathrm{T}}\left[A_e^{\mathrm{T}}P_e + P_eA_e\right]x - 2\tilde{k}\dot{k}\\ &\quad + \mathrm{sgn}(b)\tilde{k}x^{\mathrm{T}}G^{\mathrm{T}}B^{\mathrm{T}}P_e x + \mathrm{sgn}(b)\tilde{k}x^{\mathrm{T}}P_eBGx\\ &\quad + x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(x)P_e + P_e\Delta(x)\right]\\ &\le -x^{\mathrm{T}}(Q+I)x - 2\tilde{k}e^{-\alpha k}x^{\mathrm{T}}Rx\\ &\quad + \tilde{k}x^{\mathrm{T}}\left[\mathrm{sgn}(b)G^{\mathrm{T}}B^{\mathrm{T}}P_e + \mathrm{sgn}(b)P_eBG\right]x\\ &\quad + x^{\mathrm{T}}\left[\frac{1}{2}I + 2\Delta^{\mathrm{T}}(x)P_e^2\Delta(x)\right]x\\ &\le -x^{\mathrm{T}}(Q+I)x - 2\tilde{k}e^{-\alpha k}x^{\mathrm{T}}Rx\\ &\quad + \tilde{k}x^{\mathrm{T}}\left[\mathrm{sgn}(b)G^{\mathrm{T}}B^{\mathrm{T}}P_e + \mathrm{sgn}(b)P_eBG\right]x\\ &\quad + \frac{1}{2}x^{\mathrm{T}}x + 2x^{\mathrm{T}}\left[\delta(x)p_e^{\mathrm{T}}p_e\delta^{\mathrm{T}}(x)\right]x\\ &\le \tilde{k}x^{\mathrm{T}}\left[\mathrm{sgn}(b)G^{\mathrm{T}}B^{\mathrm{T}}P_e + \mathrm{sgn}(b)P_eBG\right]x\\ &\quad - x^{\mathrm{T}}Qx - 2\tilde{k}e^{-\alpha k}x^{\mathrm{T}}Rx. \end{aligned} \tag{3.15}$$

First, consider the case $\tilde{k} \ge 0$, then $\dot{V}(x,\tilde{k}) \le -x^{\mathrm{T}}Qx + \tilde{k}\sigma_1 x^{\mathrm{T}}x$, where $\sigma_1 \triangleq \lambda_{\max}\left(\mathrm{sgn}(b)G^{\mathrm{T}}B^{\mathrm{T}}P_e + \mathrm{sgn}(b)P_eBG\right)$. If $\sigma_1 \le 0$, then $\dot{V}(x,\tilde{k}) \le -x^{\mathrm{T}}Qx$. If $\sigma_1 > 0$, then let $0 < \varepsilon < \lambda_{\min}(Q)$. Thus, for all $\tilde{k}$ such that $0 \le \tilde{k} \le \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}$, $\dot{V}(x,\tilde{k}) \le -\varepsilon x^{\mathrm{T}}x$.

Now, consider the case $\tilde{k} \le 0$, then $\dot{V}(x,\tilde{k}) \le -x^{\mathrm{T}}Qx + \tilde{k}\sigma_2 x^{\mathrm{T}}x$, where $\sigma_2 \triangleq \lambda_{\min}\left(\mathrm{sgn}(b)G^{\mathrm{T}}B^{\mathrm{T}}P_e + \mathrm{sgn}(b)P_eBG - 2R\right)$. If $\sigma_2 \ge 0$, then $\dot{V}(x,\tilde{k}) \le -x^{\mathrm{T}}Qx$. If $\sigma_2 < 0$, then, for all $\tilde{k}$ such that $\frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \le \tilde{k} \le 0$, $\dot{V}(x,\tilde{k}) \le -\varepsilon x^{\mathrm{T}}x$. If $\sigma_1 \le 0$, then $\sigma_2 < 0$, so define the domain

$$\mathcal{D} \triangleq \begin{cases} \{\tilde{k} \in \mathbb{R} : -\infty < \tilde{k} \le \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}\}, & \sigma_1 > 0, \sigma_2 \ge 0\\ \{\tilde{k} \in \mathbb{R} : \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \le \tilde{k} \le \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_1}\}, & \sigma_1 > 0, \sigma_2 < 0\\ \{\tilde{k} \in \mathbb{R} : \frac{\lambda_{\min}(Q)-\varepsilon}{\sigma_2} \le \tilde{k} < \infty\}, & \sigma_1 \le 0, \sigma_2 < 0 \end{cases} \tag{3.16}$$

Thus, for all $x \in \mathbb{R}^n$ and all $\tilde{k} \in \mathcal{D}$, $\dot{V}(x,\tilde{k}) \le -\varepsilon x^{\mathrm{T}}x$ and the solution $(0, k_e)$ is Lyapunov stable.

Next, we show that, $k(t)$ converges. Lemma 2.2 implies that there exists $k_{\mathrm{s}} > 0$ such that, for all constant $k \ge k_{\mathrm{s}}$, $A_{\mathrm{s}}(k)$ is asymptotically stable. For $k \ge k_{\mathrm{s}}$, define $V_0(x,k) \triangleq x^{\mathrm{T}}P(k)x$, where $P(k) > 0$ satisfies the Lyapunov equation $A_{\mathrm{s}}^{\mathrm{T}}(k)P(k) + P(k)A_{\mathrm{s}}(k) = -e^{\alpha k}R$ and $\alpha > 0$. Taking the derivative of $V_0(x,k)$ along a trajectory of (3.12) and (3.13) yields

$$\begin{aligned}\dot{V}_0(x,k) = &-e^{-\alpha k}x^{\mathrm{T}}Rx + \dot{k}x^{\mathrm{T}}\frac{\partial P(k)}{\partial k}x\\ &+ x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(x)P(k) + P(k)\Delta(x)\right]x \end{aligned} \tag{3.17}$$

Define $\Omega(x,k) \triangleq \sqrt{\frac{1}{2}}e^{-\frac{\alpha}{2}k}R^{\frac{1}{2}} - \sqrt{2}e^{\frac{\alpha}{2}k}R^{-\frac{1}{2}}P(k)\Delta(x)$, and since $0 \le \Omega^{\mathrm{T}}(k)\Omega(k)$, it follows that $\Delta^{\mathrm{T}}(x)P(k) + P(k)\Delta(x) \le \frac{1}{2}e^{-\alpha k}R + 2e^{\alpha k}\Delta^{\mathrm{T}}(x)P(k)R^{-1}P(k)\Delta(x)$. Combining this with (3.17) yields

$$\begin{aligned}\dot{V}_0(x,k) \le &-\frac{1}{2}e^{-\alpha k}x^{\mathrm{T}}Rx + \dot{k}x^{\mathrm{T}}\frac{\partial P(k)}{\partial k}x\\ &+ 2e^{\alpha k}x^{\mathrm{T}}\Delta^{\mathrm{T}}(x)P(k)R^{-1}P(k)\Delta(x)x. \end{aligned} \tag{3.18}$$

Lemma 2.2 implies that $\lim_{k\to\infty} ke^{\alpha k}p_1(k)$ exists, and thus $\lim_{k\to\infty} e^{\alpha k}p_1(k) = 0$. Since $\delta(x)$ is bounded for all $x \in \mathbb{R}$, it follows that $\lim_{k\to\infty} 2e^{2\alpha k}\delta(x)p_1^{\mathrm{T}}(k)R^{-1}p_1(k)\delta^{\mathrm{T}}(x) = 0$, and there exists $k_1 \ge k_{\mathrm{s}}$ such that, for all $k \ge k_1$, $2e^{2\alpha k}\delta(x)p_1^{\mathrm{T}}(k)R^{-1}p_1(k)\delta^{\mathrm{T}}(x) \le \frac{1}{4}R$. Then, for all $k \ge k_1$, $\dot{V}_0(x,k) \le -\frac{1}{4}e^{-\alpha k}x^{\mathrm{T}}Rx + \dot{k}x^{\mathrm{T}}\frac{\partial P(k)}{\partial k}x$. Lemma 2.2 also implies that there exists $k_2 \ge k_{\mathrm{s}}$ such that, for all $k \ge k_2$, $\frac{\partial P(k)}{\partial k} < 0$. Thus, for all $k \ge k_3 \triangleq \max(k_1, k_2)$, $\dot{V}_0(x,k) \le -\frac{1}{4}e^{-\alpha k}x^{\mathrm{T}}Rx = -\frac{1}{4}\dot{k}$, which implies

$$\dot{V}_0(x,k)dt \le -\frac{1}{4}dk. \tag{3.19}$$

To prove that $\lim_{t\to\infty} k(t)$ exists, suppose that $k(t)$ diverges to infinity in either finite or infinite time. Then there exists $t_3 > 0$ such that $k(t_3) = k_3$. Since $k(t)$ does not escape at t_3, it follows from (3.12) that $x(t)$ does not escape at time t_3. Let $t > t_3$ be such that $k(\cdot)$ exists on $[t_3, t]$. Integrating (3.19) from t_3 to t and from k_3 to $k(t)$ and solving for $k(t)$ yields $k(t) \le k(t_3) + 4V_0(x(t_3), k_3) - 4V_0(x(t), k(t)) \le k(t_3) + 4V_0(x(t_3), k_3)$. Hence $k(t)$ is bounded on $[0,\infty)$, and thus $k(t)$ does not diverge to infinity. Since $k(t)$ is non-decreasing, $k_\infty \triangleq \lim_{t\to\infty} k(t)$ exists.

Next, we show that $x(t)$ is bounded. Define the function $V_1(x) = x^{\mathrm{T}}x$. Taking the derivative of $V_1(x)$ along a trajectory of (3.12)-(3.13) yields $\dot{V}_1(x,k) = x^{\mathrm{T}}\left[A_{\mathrm{s}}^{\mathrm{T}}(k) + A_{\mathrm{s}}(k)\right]x + x^{\mathrm{T}}\left[\Delta^{\mathrm{T}}(x) + \Delta(x)\right]x$. Since $k(t)$ converges and $\Delta(x)$ is bounded, there exist $\eta > 0$ such that $\dot{V}_1(x,k) \le \eta x^{\mathrm{T}}Rx = \eta e^{\alpha k}\dot{k}$, which implies $\dot{V}_1(x,k)dt \le \eta e^{\alpha k}dk$. Integrating from 0 to t and from $k(0)$ to $k(t)$ and solving for $V_1(x(t))$ yields $V_1(x(t)) \le \frac{\eta}{\alpha}e^{\alpha k(t)} + V_1(x(0)) - \frac{\eta}{\alpha}e^{\alpha k(0)}$. Since $k(t)$ is bounded, we conclude that $V_1(x(t))$ is bounded. Thus, $x(t)$ is bounded.

Next, we show that $\lim_{t\to\infty} x(t) = 0$. The dynamics (3.13) implies

$$||\dot{x}(t)|| \le (||A_{\mathrm{s}}(k)|| + ||\Delta(x(t))||)\,||x(t)||. \tag{3.20}$$

Since $\lim_{t\to\infty} k(t)$ exists, $A_{\mathrm{s}}(k)$ is bounded. Furthermore, for all $x \in \mathbb{R}^n$, $\Delta(x(t))$ is bounded. Since $\tilde{A}(k(t))$, $\Delta(x(t))$, and $x(t)$ are bounded, it follows from (3.20) that $\dot{x}(t)$ is bounded. Thus, $\frac{d}{dt}\left[e^{-\alpha k(t)}x^{\mathrm{T}}(t)Rx(t)\right] = e^{-\alpha k(t)}\left(-\alpha\dot{k}(t)x^{\mathrm{T}}(t)Rx(t) + 2x^{\mathrm{T}}(t)R\dot{x}(t)\right)$ is bounded. Since the derivative of $\dot{k}(t) = e^{-\alpha k(t)}x^{\mathrm{T}}(t)Rx(t)$ is bounded, $\dot{k}(t)$ is uniformly continuous. Since $\dot{k}(t)$ is uniformly continuous and $\lim_{t\to\infty} k(t)$ exists, Barabalat's lemma implies that $\lim_{t\to\infty} e^{-\alpha k(t)}x^{\mathrm{T}}(t)Rx(t) = 0$. Thus, $\lim_{t\to\infty} x(t) = 0$. $\square$

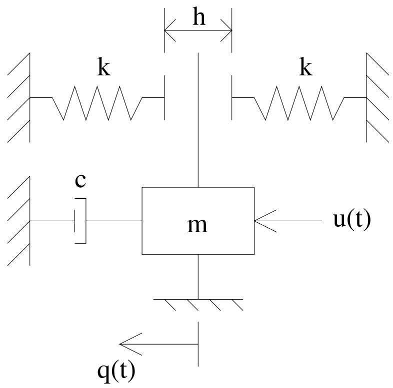

Fig. 1. Nonlinear spring-mass-damper with Coulomb friction and a dead zone in the spring stiffness.

4. NONLINEAR SPRING-MASS-DAMPER EXAMPLE

In this section, we consider Lyapunov-stable adaptive stabilization of the nonlinear spring-mass-damper

$$m\ddot{q}(t) + \hat{c}(\dot{q}(t)) + \hat{k}(q(t)) = u(t), \tag{4.1}$$

where

$$\hat{c}(\dot{q}(t)) \triangleq \begin{cases} \left(c + \frac{d}{\delta}\right)\dot{q}(t), & |\dot{q}(t)| < \delta, \\ c\dot{q}(t) + \operatorname{sgn}(\dot{q}(t))d, & |\dot{q}(t)| \geq \delta, \end{cases} \tag{4.2}$$

$$\hat{k}(q(t)) \triangleq \begin{cases} k_0(q(t) + \frac{h}{2}), & q(t) \leq \frac{-h}{2}, \\ 0, & |q(t)| < \frac{h}{2}, \\ k_0(q(t) - \frac{h}{2}), & q(t) \geq \frac{h}{2}, \end{cases} \tag{4.3}$$

and $\delta, c, d, h, k_0 > 0$. The function $\hat{c}(\cdot)$ is a continuous approximation of Coulomb friction and satisfies assumption *(i)*. The function $\hat{k}(\cdot)$ is a linear spring with a deadzone. This nonlinear system is shown in Figure 1. Note that the uncontrolled system has a continuum of equilibria, and the origin of the system is semistable, but not asymptotically stable. (For the definition of semistability, see [7]). For this example, the mass $m = 3$ kg, the viscous friction $c = 2$ kg/s, the Coulomb friction $d = 20$ N, the spring stiffness $k_0 = 2$ kg/s^2, the deadzone gap $h = 10$ m, and $\delta = 0.1$ m/s.

Note that the system (4.1) satisfies assumptions *(i)*-*(iv)* and the adaptive controller presented in Theorem 3.1 can be used to stabilize the origin. This controller is given by

$$u(t) = -k(t) \begin{bmatrix} g_1 & g_0 \end{bmatrix} \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}, \tag{4.4}$$

$$k(t) = e^{-\alpha k(t)} \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}^{\mathrm{T}} R \begin{bmatrix} \dot{q}(t) \\ q(t) \end{bmatrix}, \tag{4.5}$$

where $g_0 > 0$, $g_1 > 0$, $k(0) \geq 0$, $\alpha > 0$, and R is positive definite. We choose the controller parameters $g_0 = 11$, $g_1 = 7$, $\alpha = 0.1$, and $R = I$. The system (4.1) with the adaptive controller (4.4)-(4.5) is simulated with the initial conditions $k(0) = 0$, $q(0) = -25$ m, and $\dot{q}(0) = 10$ m/s. The time histories of the position $q(t)$ and velocity $\dot{q}(t)$ for the open-loop and closed-loop systems are shown in Figure 2. The the equilibrium of the open-loop system is semistable, and, while the velocity converges to zero, the position converges to approximately -15.6 m. The adaptive controller stabilizes the equilibrium so that both the velocity and position converge to zero. The adaptive parameter k converges to approximately 42.3.

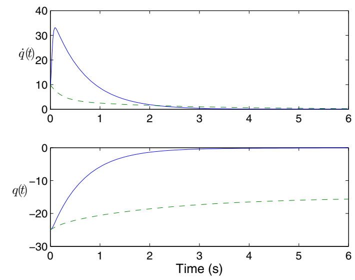

Fig. 2. The open-loop (dashed) and closed-loop (solid) time histories for the position and velocity of the mass. The open-loop system is semistable, and the adaptive controller stabilizes the origin.

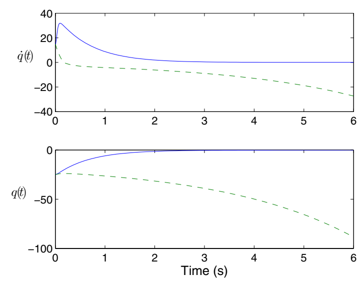

Fig. 3. The open-loop (dashed) and closed-loop (solid) time histories for the position and velocity of the mass. The open-loop system is unstable, and the adaptive controller stabilizes the origin.

Next we apply Theorem 3.1 to the unstable system $m\ddot{q}(t)+\hat{c}(\dot{q}(t))-\hat{k}(q(t)) = u(t)$, which is a modification of (4.1) in which the sign of the stiffness term is negative. This system is simulated with the adaptive controller (4.4)-(4.5) connected in feedback. The initial conditions are $k(0) = 0$, $q(0) = -25$ m, and $\dot{q}(0) = 15$ m/s. Figure 3 shows the time histories of the position and velocity for the open-loop and closed-loop system. The open-loop system is unstable, and the adaptive controller stabilizes the origin.

REFERENCES

[1] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. New Jersey: Prentice Hall, 1989.
[2] K. J. Åström and B. Wittenmark, *Adaptive Control*, 2nd ed. Reading, MA: Addison-Wesley, 1995.
[3] P. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice Hall, 1996.
[4] J. Hong and D. S. Bernstein, "Adaptive stabilization of nonlinear oscillators using direct adaptive control," *Int. J. Contr.*, vol. 74, pp. 432–444, 2001.
[5] A. V. Roup and D. S. Bernstein, "Adaptive stabilization of a class of nonlinear systems with nonparametric uncertainty," *IEEE Trans. Autom. Contr.*, vol. 46, pp. 1821–1825, 2001.
[6] A. Betser, N. Cohen, and E. Zeheb, "On solving the Lyapunov and Stein equations for a companion matrix," *Sys. Contr. Lett.*, vol. 25, pp. 211–218, 1995.
[7] S. P. Bhat and D. S. Bernstein, "Nontangency-based lyapunov tests for convergence and stability in systems having a continuum of equilibria," *SIAM J. Contr. Optim.*, vol. 42, pp. 1745–1775, 2003.