Williamsburg, Virginia
ACTIVE 04
2004 September 20-22

Discrete-time Adaptive Feedback Disturbance Rejection using a Retrospective Performance Measure

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ABSTRACT

A discrete-time adaptive feedback disturbance rejection algorithm is developed for applications where the disturbance spectrum is unknown. Feedforward control methods are frequently used to reject disturbances with a known spectrum, but a feedback controller is required when the disturbance is unknown. Multi-input multi-output (MIMO) disturbance rejection algorithms are developed for controllers represented by both state-space models and time-series models. The controller parameters are updated using a gradient algorithm. Adaptive step size functions are developed, which guarantee that the controller matrices asymptotically approach their optimal value. We define a retrospective performance as the performance of the system at the current time assuming the current controller had been used over a previous window of time. The retrospective performance is minimized as a surrogate measure of the true performance. Using the retrospective performance allows us to remove dependence on unknown plant and disturbance information. The algorithm requires knowledge of some Markov parameters of the system from the control signal inputs to the performance variables. No information is required of the disturbance spectrum. We demonstrate feedback disturbance rejection on constant, tonal, and broadband disturbances.

1 INTRODUCTION

The adaptive control literature is dominated by adaptive stabilization and model reference adaptive control [1–5]. Adaptive stabilization has been approached using parameter-estimation-based adaptive controllers, universal stabilizers, and high-gain adaptive controllers. Model reference adaptive control addresses the adaptive tracking problem by forcing an uncertain system to behave like a reference model. In addition to stabilization and tracking, disturbance rejection is a third common control problem. In fact, disturbance rejection is the objective for many applications in noise control, vibration suppression, and structural control. In the present paper, we consider the disturbance rejection problem for uncertain systems with unknown disturbance spectrum. Our approach to this problem is discrete-time adaptive feedback disturbance rejection.

Adaptive feedforward control is frequently used to reject harmonic disturbances when the disturbance frequencies are known or can be estimated [6–9]. Adaptive feedforward algorithms typically rely on least-mean-square (LMS) or recursive least-mean-square (RLMS) algorithms to update parameters. These methods include the filtered-u LMS and filtered-x LMS algorithms. However, adaptive feedforward algorithms do not account for the transfer function from the control signals to the measurements.

As an alternative, we consider adaptive feedback disturbance rejection [10, 11]. In [10], an adaptive feedback disturbance rejection algorithm is developed based on a retrospective performance measure. The retrospective performance of a system is the performance of the system at the current time assuming that the current controller was used over a past window of time. In [10], the retrospective performance measure is used in connection with time-series modelling of the plant and the controller to develop an adaptive disturbance rejection algorithm that requires only knowledge of the numerator of the transfer function from the control to the performance measurement.

In the present paper, we adopt the notion of a retrospective performance measure and develop adaptive disturbance rejection algorithms based on a state-space model of the plant. By using the retrospective performance, we eliminate the algorithms dependence on unknown plant information and unknown disturbance information. The algorithm requires knowledge of Markov parameters from the control to the performance, rather than the numerator of this transfer function as in [10]. We develop gradient-based update laws and adaptive step size functions that minimize the retrospective performance. We consider two adaptive algorithms that are identical in method but differ in the representation of the controller. In one case, the controller is represented by a state-space model and in the other, it is represented by a time-series model.

In Section 2, we formulate the disturbance rejection problem and rewrite the performance variable in terms of Markov parameters. In Section 3, we develop a state-space adaptive feedback disturbance rejection controller. A time-series adaptive feedback disturbance rejection controller is given in Section 4. We provide numerical examples in Section 5 and conclusions in Section 6.

2 DISTURBANCE REJECTION PROBLEM FORMULATION

Consider the linear shift-invariant discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{2.1}$$

$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k), \tag{2.2}$$

$$y(k) = Cx(k) + Du(k) + D_2 w(k), \tag{2.3}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^{l_u}$, $w(k) \in \mathbb{R}^{l_w}$, $y(k) \in \mathbb{R}^{l_y}$, and $z(k) \in \mathbb{R}^{l_z}$. We assume that A is an asymptotically stable matrix. The standard disturbance rejection problem has two input signals and two output signals. The inputs are the disturbance $w(k)$ and the control $u(k)$. The outputs are the measurement $y(k)$ and the performance $z(k)$. The objective of feedback disturbance rejection is to determine a control $u(k)$ using the measurement $y(k)$ that minimizes the performance $z(k)$ in the presence of the external disturbance $w(k)$. We assume that $w(k)$ is not measured.

In fixed-gain feedforward disturbance rejection, the control $u(k)$ is determined without using the measurement $y(k)$. In adaptive feedforward disturbance rejection, a measurement of the performance $z(k)$ is required for adaptation. We consider adaptive feedback disturbance rejection where $y(k)$ is used to determine $u(k)$, and a measurement of $z(k)$ is used for adaptation.

Propagating the state backwards for q times steps, equations (2.1) and (2.2) can be combined to yield

$$z(k) = E_1 A^q x(k-q) + \begin{bmatrix} \hat{H}_0 & \hat{H}_1 & \cdots & \hat{H}_q \end{bmatrix} \begin{bmatrix} w(k) \\ w(k-1) \\ \vdots \\ w(k-q) \end{bmatrix} + \begin{bmatrix} H_0 & H_1 & \cdots & H_q \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-q) \end{bmatrix}, \tag{2.4}$$

where the Markov parameters from the disturbance $w(k)$ to the performance $z(k)$ are given by $\hat{H}_0 \triangleq E_0$ and, for $i = 1, \ldots, q$, $\hat{H}_i \triangleq E_1 A^{i-1} D_1$. The Markov parameters from the control $u(k)$ to the performance $z(k)$ are given by $H_0 \triangleq E_2$ and, for $i = 1, \ldots, q$, $H_i \triangleq E_1 A^{i-1} B$.

Next, we define the retrospective performance window $p > 0$. For all $j = 0, \ldots, p-1$, the performance is

$$z(k-j) = E_1 A^q x(k-q-j) + \begin{bmatrix} \hat{H}_0 & \hat{H}_1 & \cdots & \hat{H}_q \end{bmatrix} \begin{bmatrix} w(k-j) \\ w(k-1-j) \\ \vdots \\ w(k-q-j) \end{bmatrix} + \begin{bmatrix} H_0 & H_1 & \cdots & H_q \end{bmatrix} \begin{bmatrix} u(k-j) \\ u(k-1-j) \\ \vdots \\ u(k-q-j) \end{bmatrix}. \tag{2.5}$$

We define the performance block vector $Z(k)$, the disturbance block vector $W(k)$, and the control block vector $U(k)$ by

$$Z(k) \triangleq \begin{bmatrix} z(k) \\ \vdots \\ z(k-p+1) \end{bmatrix}, \quad W(k) \triangleq \begin{bmatrix} w(k) \\ \vdots \\ w(k-p-q+1) \end{bmatrix}, \quad U(k) \triangleq \begin{bmatrix} u(k) \\ \vdots \\ u(k-p-q+1) \end{bmatrix}. \tag{2.6}$$

Equation (2.5) implies

$$Z(k) = H_{zu} U(k) + H_{zw} W(k) + \begin{bmatrix} E_1 A^q x(k-q) \\ \vdots \\ E_1 A^q x(k-q-p+1) \end{bmatrix}, \tag{2.7}$$

where

$$H_{zw} \triangleq \begin{bmatrix} \hat{H}_0 & \cdots & \hat{H}_q & 0_{l_z\times l_w} & \cdots & 0_{l_z\times l_w} \\ 0_{l_z\times l_w} & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & 0_{l_z\times l_w} \\ 0_{l_z\times l_w} & \cdots & 0_{l_z\times l_w} & \hat{H}_0 & \cdots & \hat{H}_q \end{bmatrix} \in \mathbb{R}^{pl_z\times(p+q)l_w}, \tag{2.8}$$

$$H_{zu} \triangleq \begin{bmatrix} H_0 & \cdots & H_q & 0_{l_z\times l_u} & \cdots & 0_{l_z\times l_u} \\ 0_{l_z\times l_u} & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & & \ddots & 0_{l_z\times l_u} \\ 0_{l_z\times l_u} & \cdots & 0_{l_z\times l_u} & H_0 & \cdots & H_q \end{bmatrix} \in \mathbb{R}^{pl_z\times(p+q)l_u}. \tag{2.9}$$

3 ADAPTIVE FEEDBACK DISTURBANCE REJECTION WITH A STATE-SPACE CONTROLLER

In this section, we consider adaptive feedback disturbance rejection using a state-space model of the controller. Consider the controller

$$x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k), \tag{3.1}$$

$$u(k) = C_c(k)x_c(k), \tag{3.2}$$

where for $x_c(k) \in \mathbb{R}^{n_c}$. By combining (3.1) and (3.2), the control can be written in terms of all three controller matrices

$$u(k) = C_c(k)A_c(k-1)x_c(k-1) + C_c(k)B_c(k-1)y(k-1). \tag{3.3}$$

Now, we define the measurement block vector $Y(k)$ and the controller state block vector $X_c(k)$ by

$$Y(k) \triangleq \begin{bmatrix} y(k-1) \\ \vdots \\ y(k-q-p) \end{bmatrix}, \quad X_c(k) \triangleq \begin{bmatrix} x_c(k-1) \\ \vdots \\ x_c(k-q-p) \end{bmatrix}, \tag{3.4}$$

so that the control can be expressed as

$$u(k) = C_c(k)A_c(k-1)S_1X_c(k) + C_c(k)B_c(k-1)T_1Y(k), \tag{3.5}$$

where, for $i = 1, \ldots, q+p$,

$$S_i \triangleq \begin{bmatrix} 0_{n_c\times(i-1)n_c} & I_{n_c} & 0_{n_c\times n_c(p+q-i)} \end{bmatrix} \in \mathbb{R}^{n_c\times n_c(p+q)}, \tag{3.6}$$

$$T_i \triangleq \begin{bmatrix} 0_{l_y\times(i-1)l_y} & I_{l_y} & 0_{l_y\times l_y(p+q-i)} \end{bmatrix} \in \mathbb{R}^{l_y\times l_y(p+q)}. \tag{3.7}$$

Similarly, the control block vector can be expressed as

$$U(k) = \sum_{i=1}^{p+q} L_i\left[C_c(k-i+1)A_c(k-i)S_iX_c(k) + C_c(k-i+1)B_c(k-i)T_iY(k)\right]. \tag{3.8}$$

Therefore, the performance block vector (2.7) can be written as

$$\begin{aligned} Z(k) =& H_{zu}\sum_{i=1}^{p+q} L_i\left[C_c(k-i+1)A_c(k-i)S_iX_c(k) + C_c(k-i+1)B_c(k-i)T_iY(k)\right] \\ &+ H_{zw}W(k) + \begin{bmatrix} E_1A^qx(k-q) \\ \vdots \\ E_1A^qx(k-q-p+1) \end{bmatrix}. \end{aligned} \tag{3.9}$$

The retrospective performance is the performance of the system at time k assuming that the controller given at time k was used during p previous time steps. The retrospective performance serves as a surrogate measure of the true performance. For the state-space controller, the retrospective performance block vector is defined as

$$\hat{Z}_{ss}(k) \triangleq H_{zu} \sum_{i=1}^{p+q} L_i \left[C_c(k) A_c(k) S_i X_c(k) + C_c(k) B_c(k) T_i Y(k) \right] + H_{zw} W(k) + \begin{bmatrix} E_1 A^q x(k-q) \\ \vdots \\ E_1 A^q x(k-q-p+1) \end{bmatrix}. \tag{3.10}$$

Combining (2.7) and (3.10) yields

$$\hat{Z}_{ss}(k) = Z(k) - H_{zu} U(k) + H_{zu} \sum_{i=1}^{p+q} L_i \left[C_c(k) A_c(k) S_i X_c(k) + C_c(k) B_c(k) T_i Y(k) \right]. \tag{3.11}$$

Note that equation (3.11) does not explicitly contain the disturbance signal. Furthermore, computing the retrospective performance block vector requires knowledge of only q Markov parameters from the control to the performance.

Next, define the state-space controller retrospective performance cost function

$$J_{ss}(k) \triangleq \frac{1}{2} \hat{Z}_{ss}^{\mathrm{T}}(k) \hat{Z}_{ss}(k). \tag{3.12}$$

The gradients of $J_{ss}(k)$ with respect to the controller matrices are given by

$$\frac{\partial J_{ss}(k)}{\partial A_c(k)} = \sum_{i=1}^{p+q} C_c^{\mathrm{T}}(k) L_i^{\mathrm{T}} H_{zu}^{\mathrm{T}} \hat{Z}_{ss}(k) X_c^{\mathrm{T}}(k) S_i^{\mathrm{T}}, \tag{3.13}$$

$$\frac{\partial J_{ss}(k)}{\partial B_c(k)} = \sum_{i=1}^{p+q} C_c^{\mathrm{T}}(k) L_i^{\mathrm{T}} H_{zu}^{\mathrm{T}} \hat{Z}_{ss}(k) Y^{\mathrm{T}}(k) T_i^{\mathrm{T}}, \tag{3.14}$$

$$\frac{\partial J_{ss}(k)}{\partial C_c(k)} = \sum_{i=1}^{p+q} \left[L_i^{\mathrm{T}} H_{zu}^{\mathrm{T}} \hat{Z}_{ss}(k) X_c^{\mathrm{T}}(k) S_i^{\mathrm{T}} A_c^{\mathrm{T}}(k) + L_i^{\mathrm{T}} H_{zu}^{\mathrm{T}} \hat{Z}_{ss}(k) Y^{\mathrm{T}}(k) S_i^{\mathrm{T}} B_c^{\mathrm{T}}(k) \right]. \tag{3.15}$$

The controller parameter updates are given by

$$A_c(k+1) = A_c(k) - \eta_{ss}(k) \frac{\partial J_{ss}(k)}{\partial A_c(k)}, \tag{3.16}$$

$$B_c(k+1) = B_c(k) - \eta_{ss}(k) \frac{\partial J_{ss}(k)}{\partial B_c(k)}, \tag{3.17}$$

$$C_c(k+1) = C_c(k) - \eta_{ss}(k) \frac{\partial J_{ss}(k)}{\partial C_c(k)}, \tag{3.18}$$

where $\eta_{ss}(k)$ is the adaptive step size. Now, we present a result that provides formulas for optimal step size functions and inequalities that guarantee improvement in the estimates of the controller matrices.

Proposition 3.1. *Assume that there exists A_c^*, B_c^*, and C_c^* such that $J_{ss}(k)$ is minimized for all k. Define the controller parameter errors*

$$E_{A_c}(k) \triangleq A_c^* - A_c(k), \qquad E_{B_c}(k) \triangleq B_c^* - B_c(k), \qquad E_{C_c}(k) \triangleq C_c^* - C_c(k), \tag{3.19}$$

the performance errors

$$\varepsilon_{A_c}(k) \triangleq \hat{Z}_{ss}(k)|_{A_c(k)=A_c^*} - \hat{Z}_{ss}(k), \tag{3.20}$$

$$\varepsilon_{B_c}(k) \triangleq \hat{Z}_{ss}(k)|_{B_c(k)=B_c^*} - \hat{Z}_{ss}(k), \tag{3.21}$$

$$\varepsilon_{C_c}(k) \triangleq \hat{Z}_{ss}(k)|_{C_c(k)=C_c^*} - \hat{Z}_{ss}(k), \tag{3.22}$$

and the performance error cost functions

$$\mathcal{J}_{A_c}(k,\eta_{ss}(k)) \triangleq ||E_{A_c}(k+1)||_F^2 - ||E_{A_c}(k)||_F^2\,, \tag{3.23}$$

$$\mathcal{J}_{B_c}(k,\eta_{ss}(k)) \triangleq ||E_{B_c}(k+1)||_F^2 - ||E_{B_c}(k)||_F^2\,, \tag{3.24}$$

$$\mathcal{J}_{C_c}(k,\eta_{ss}(k)) \triangleq ||E_{C_c}(k+1)||_F^2 - ||E_{C_c}(k)||_F^2\,. \tag{3.25}$$

Let $k \geq 0$ and assume that $\frac{\partial J_{ss}(k)}{\partial A_c(k)} \neq 0$, $\frac{\partial J_{ss}(k)}{\partial B_c(k)} \neq 0$, and $\frac{\partial J_{ss}(k)}{\partial C_c(k)} \neq 0$. Then

$$\mathcal{J}_{A_c}(k,\eta_{ss}(k)) < 0, \qquad \mathcal{J}_{B_c}(k,\eta_{ss}(k)) < 0, \qquad \mathcal{J}_{C_c}(k,\eta_{ss}(k)) < 0, \tag{3.26}$$

if and only if

$$0 < \eta_{ss}(k) < 2\eta^*_{A_c}(k), \qquad 0 < \eta_{ss}(k) < 2\eta^*_{B_c}(k), \qquad 0 < \eta_{ss}(k) < 2\eta^*_{C_c}(k), \tag{3.27}$$

where the optimal step size functions are given by

$$\eta^*_{A_c}(k) \triangleq \frac{||\varepsilon_{A_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F^2}, \qquad \eta^*_{B_c} \triangleq \frac{||\varepsilon_{B_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial B_c(k)}\right|\right|_F^2}, \qquad \eta^*_{C_c}(k) \triangleq \frac{||\varepsilon_{C_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial C_c(k)}\right|\right|_F^2}. \tag{3.28}$$

Proof. Subtracting A_c^* from the update law (3.16) yields

$$E_{A_c}(k+1) = E_{A_c}(k) + \eta_{ss}(k)\frac{\partial J_{ss}(k)}{\partial A_c(k)}. \tag{3.29}$$

Combining (3.23) with (3.29) results in

$$\mathcal{J}_{A_c}(k,\eta_{ss}(k)) = 2\eta_{ss}(k)\mathrm{tr}\left(E_{A_c}(k)\frac{\partial J_{ss}(k)}{\partial A_c(k)}^{\mathrm{T}}\right) + \eta_{ss}^2(k)\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F^2. \tag{3.30}$$

Considering only the trace term

$$\mathrm{tr}\left(E_{A_c}(k)\frac{\partial J_{ss}(k)}{\partial A_c(k)}^{\mathrm{T}}\right) = \mathrm{tr}\left(E_{A_c}(k)\left(\frac{\partial J_{ss}(k)}{\partial A_c(k)} - \left.\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|_{A_c(k)=A_c^*}\right)^{\mathrm{T}}\right) \tag{3.31}$$

$$= \mathrm{tr}\left(E_{A_c}(k)\left(-\sum_{i=1}^{p+q} S_i X_c(k)\varepsilon_{A_c}^{\mathrm{T}}(k) H_{zu} L_i C_c(k)\right)\right) \tag{3.32}$$

$$= -\left(\varepsilon_{A_c}^{\mathrm{T}}(k)\left(\sum_{i=1}^{p+q} H_{zu} L_i C_c(k) E_{A_c}(k) S_i X_c(k)\right)\right) \tag{3.33}$$

$$= -\,||\varepsilon_{A_c}(k)||_2^2\,. \tag{3.34}$$

Therefore, (3.30) is equivalent to

$$\mathcal{J}_{A_c}(k,\eta_{ss}(k)) = -2\eta_{ss}(k)\,||\varepsilon_{A_c}(k)||_2^2 + \eta_{ss}^2(k)\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F^2. \tag{3.35}$$

It follows that $\mathcal{J}_{A_c}(k,\eta_{ss}(k)) < 0$ if and only if

$$0 < \eta_{ss}(k) < 2\frac{||\varepsilon_{A_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F^2} = 2\eta^*_{A_c}(k). \tag{3.36}$$

The results for $\mathcal{J}_{B_c}(k,\eta_{ss}(k))$ and $\mathcal{J}_{C_c}(k,\eta_{ss}(k))$ follow by similar reasoning. □

Unfortunately, the optimal step size functions given by (3.28) are not implementable since measurements of $\varepsilon_{A_c}(k)$, $\varepsilon_{B_c}(k)$, and $\varepsilon_{C_c}(k)$ are not available.. Therefore, we desire a step size function that satisfies the inequalities (3.27). Consider the implementable step size function

$$\hat{\eta}_{ss}(k) = \min(\hat{\eta}_{A_c}, \hat{\eta}_{B_c}, \hat{\eta}_{C_c}), \tag{3.37}$$

where

$$\hat{\eta}_{A_c} \triangleq \frac{1}{(p+n)\,||H_{zu}||_F^2\,||C_c(k)||_F^2\,||X_c(k)||_2^2}, \tag{3.38}$$

$$\hat{\eta}_{B_c} \triangleq \frac{1}{(p+n)\,||H_{zu}||_F^2\,||C_c(k)||_F^2\,||Y(k)||_2^2}, \tag{3.39}$$

$$\hat{\eta}_{C_c} \triangleq \frac{1}{(p+n)\,||H_{zu}||_F^2\,[||A_c(k)||_F\,||X_c(k)||_2 + ||B_c(k)||_F\,||Y(k)||_2]^2}. \tag{3.40}$$

Proposition 3.2. *The implementable step size function given by (3.37)-(3.40) satisfies the inequalities*

$$0 < \hat{\eta}_{ss}(k) \le \eta^*_{A_c}(k), \qquad 0 < \hat{\eta}_{ss}(k) \le \eta^*_{B_c}(k), \qquad 0 < \hat{\eta}_{ss}(k) \le \eta^*_{C_c}(k). \tag{3.41}$$

Proof. Consider

$$\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F = \left|\left|-\sum_{i=1}^{p+q} C_c^{\mathrm{T}}(k)L_i^{\mathrm{T}}H_{zu}^{\mathrm{T}}\varepsilon_{A_c}(k)X_c^{\mathrm{T}}(k)S_i^{\mathrm{T}}\right|\right|_F, \tag{3.42}$$

$$\le \sum_{i=1}^{p+q}\left|\left|C_c^{\mathrm{T}}(k)L_i^{\mathrm{T}}H_{zu}^{\mathrm{T}}\varepsilon_{A_c}(k)X_c^{\mathrm{T}}(k)S_i^{\mathrm{T}}\right|\right|_F \tag{3.43}$$

$$\le ||\varepsilon_{A_c}(k)||_2\,||C_c(k)||_F\sum_{i=1}^{p+q}\sigma_{\max}(H_{zu}L_i)\,||S_iX_c(k)||_2 \tag{3.44}$$

$$\le ||\varepsilon_{A_c}(k)||_2\,||C_c(k)||_F\,||X_c(k)||_2\left[\sum_{i=1}^{p+q}\sigma_{\max}(H_{zu}L_i)\right]. \tag{3.45}$$

Similarly,

$$\left|\left|\frac{\partial J_{ss}(k)}{\partial B_c(k)}\right|\right|_F \le ||\varepsilon_{B_c}(k)||_2\,||C_c(k)||_F\,||Y(k)||_2\left[\sum_{i=1}^{p+q}\sigma_{\max}(H_{zu}L_i)\right], \tag{3.46}$$

$$\left|\left|\frac{\partial J(k)}{\partial C_c(k)}\right|\right|_F \le ||\varepsilon_{C_c}(k)||_2\,[||A_c(k)||_F\,||X_c(k)||_2 + ||B_c(k)||_F\,||Y(k)||_2]\left[\sum_{i=1}^{p+q}\sigma_{\max}(H_{zu}L_i)\right]. \tag{3.47}$$

Inequality (3.45) implies

$$\frac{||\varepsilon_{A_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial A_c(k)}\right|\right|_F^2} \ge \frac{||\varepsilon_{A_c}(k)||_2^2}{||\varepsilon_{A_c}(k)||_2^2\,||C_c(k)||_F^2\,||X_c(k)||_2^2\left[\sum_{i=1}^{p+q}\sigma_{\max}(H_{zu}L_i)\right]^2} \tag{3.48}$$

$$\ge \frac{1}{(p+q)\,||H_{zu}||_F^2\,||C_c(k)||_F^2\,||X_c(k)||_2^2} = \hat{\eta}_{A_c}(k). \tag{3.49}$$

Similarly, inequality (3.46) implies

$$\frac{||\varepsilon_{B_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial B_c(k)}\right|\right|_F^2} \ge \frac{1}{(p+q)\,||H_{zu}||_F^2\,||C_c(k)||_F^2\,||Y(k)||_2^2} = \hat{\eta}_{B_c}(k), \tag{3.50}$$

and inequality (3.47) implies

$$\frac{||\varepsilon_{C_c}(k)||_2^2}{\left|\left|\frac{\partial J_{ss}(k)}{\partial C_c(k)}\right|\right|_F^2} \ge \frac{1}{(p+q)\,||H_{zu}||_F^2\left[||A_c(k)||_F^2\,||X_c(k)||_2^2 + ||B_c(k)||_F^2\,||Y(k)||_2^2\right]} = \hat{\eta}_{C_c}(k). \tag{3.51}$$

It follows from inequalities (3.49)-(3.51) that $\hat{\eta}_{ss}(k)$ satisfies the inequalities (3.41). □

To summarize, the state-space based adaptive feedback disturbance rejection algorithm is given by the controller (3.1)-(3.2), and the controller update laws (3.16)-(3.18) with the step size function $\eta_{ss}(k) = \hat{\eta}_{ss}(k)$.

4 ADAPTIVE FEEDBACK DISTURBANCE REJECTION WITH A TIME-SERIES CONTROLLER

In this section, we consider adaptive feedback disturbance rejection using a time-series model of the controller. Using a time-series model of the controller offers several advantages over the state-space model. First, for the same controller order, a time-series controller requires fewer discrete states to update the controller parameters. This can lessen the computational burden of the algorithm. Furthermore, we do not need to propagate the controller states when we use a time-series model. Finally, the time-series controller performs better than the state-space controller in simulation. Consider the controller

$$u(k) = \sum_{i=1}^{n_c} -a_{c_i}(k)u(k-i) + \sum_{i=1}^{n_c} b_{c_i}(k)y(k-i), \tag{4.1}$$

where for $i = 1, \ldots, n_c$, $a_{c_i}(k) \in \mathbb{R}^{l_u \times l_u}$ and $b_{c_i}(k) \in \mathbb{R}^{l_u \times l_y}$. We define the controller parameter matrix

$$\theta(k) \triangleq \begin{bmatrix} -a_{c_1}(k) & \cdots & -a_{c_{n_c}}(k) & b_{c_1}(k) & \cdots & b_{c_{n_c}}(k) \end{bmatrix}, \tag{4.2}$$

and the regressor

$$\Phi_{uy}(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_c-q-p+1) \\ y(k-1) \\ \vdots \\ y(k-n_c-q-p+1) \end{bmatrix}, \tag{4.3}$$

where $\theta(k) \in \mathbb{R}^{l_u \times n_c(l_u+l_y)}$ and $\Phi_{uy}(k) \in \mathbb{R}^{(l_u+l_y)(n_c+q+p-1)}$. The controller (4.1) can be written in terms of (4.2) and (4.3) as

$$u(k) = \theta(k)R_1\Phi_{uy}(k), \tag{4.4}$$

where, for $i = 1, \ldots, q+p$, $R_i \in \mathbb{R}^{n_c(l_u+l_y)\times(p+q)(l_u+l_y)}$ is given by

$$R_i \triangleq \begin{bmatrix} 0_{n_cl_u\times(i-1)l_u} & I_{n_cl_u} & 0_{n_cl_u\times((p-i)l_u+(i-1)l_y)} & 0_{n_cl_u\times n_cl_y} & 0_{n_cl_u\times(p-i)l_y} \\ 0_{n_cl_y\times(i-1)l_u} & 0_{n_cl_y\times n_cl_u} & 0_{n_cl_y\times((p-i)l_u+(i-1)l_y)} & I_{n_cl_y} & 0_{n_cl_y\times(p-i)l_y} \end{bmatrix}. \tag{4.5}$$

Similarly, the control block vector can be expressed as

$$U(k) = \sum_{i=1}^{p+q} L_i\theta(k-i+1)R_i\Phi_{uy}(k), \tag{4.6}$$

where

$$L_i \triangleq \begin{bmatrix} 0_{(i-1)l_u\times l_u} \\ I_{l_u} \\ 0_{(p+q-i)l_u\times l_u} \end{bmatrix} \in \mathbb{R}^{(p+q)l_u\times l_u}. \tag{4.7}$$

Therefore, the performance block vector (2.7) can be written as

$$Z(k) = H_{zu}\sum_{i=1}^{p+q} L_i\theta(k-i+1)R_i\Phi_{uy}(k) + H_{zw}W(k) + \begin{bmatrix} E_1A^qx(k-q) \\ \vdots \\ E_1A^qx(k-q-p+1) \end{bmatrix}. \tag{4.8}$$

The retrospective performance block vector for a time-series controller is defined as

$$\hat{Z}_{ts}(k) \triangleq H_{zu}\sum_{i=1}^{p+q} L_i\theta(k)R_i\Phi_{uy}(k) + H_{zw}W(k) + \begin{bmatrix} E_1A^qx(k-q) \\ \vdots \\ E_1A^qx(k-q-p+1) \end{bmatrix}. \tag{4.9}$$

By combining (2.7) and (4.9), we obtain

$$\hat{Z}_{ts}(k) = Z(k) - H_{zu}U(k) + H_{zu}\sum_{i=1}^{p+q} L_i\theta(k)R_i\Phi_{uy}(k). \tag{4.10}$$

Define the retrospective performance cost function

$$J_{ts}(k) \triangleq \frac{1}{2}\hat{Z}_{ts}^{\mathrm{T}}(k)\hat{Z}_{ts}(k). \tag{4.11}$$

The gradient with respect to the controller parameters is given by

$$\frac{\partial J_{ts}(k)}{\partial \theta(k)} = \sum_{i=1}^{q+p} L_i^{\mathrm{T}} H_{zu}^{\mathrm{T}} \hat{Z}_{ts}(k)\Phi_{uy}^{\mathrm{T}}(k)R_i^{\mathrm{T}}. \tag{4.12}$$

Therefore, the controller gradient update law is

$$\theta(k+1) = \theta(k) - \eta_{ts}(k)\frac{\partial J_{ts}(k)}{\partial \theta(k)}, \tag{4.13}$$

where $\eta_{ts}(k)$ is the adaptive step size. The following result provides a formula for the step size function that causes the greatest improvement in the controller parameter matrix at each time step. This is the optimal step size function. In addition, the result provides a bound on the step size function, which guarantees the controller parameter matrix asymptotically approaches its optimal value.

Proposition 4.1. *Assume that there exists θ^* such that $J_{ts}(k)$ is minimized for all k. Define the parameter error*

$$E(k) \triangleq \theta^* - \theta(k), \tag{4.14}$$

the performance error

$$\varepsilon(k) \triangleq \hat{Z}_{ts}|_{\theta(k)=\theta^*} - \hat{Z}_{ts}(k), \tag{4.15}$$

and the performance error cost function

$$\mathcal{J}(k, \eta_{ts}(k)) \triangleq ||E(k+1)||_F^2 - ||E(k)||_F^2. \tag{4.16}$$

Let $k \geq 0$ and assume that $\frac{\partial J_{ts}(k)}{\partial \theta(k)} \neq 0$. Then

$$\mathcal{J}(k, \eta_{ts}(k)) < 0 \tag{4.17}$$

if and only if

$$0 < \eta_{ts}(k) < 2\eta_{ts}^*(k), \tag{4.18}$$

where the optimal step size is given by

$$\eta_{ts}^*(k) \triangleq \frac{||\varepsilon(k)||_2^2}{\left|\left|\frac{\partial J_{ts}(k)}{\partial \theta(k)}\right|\right|_F^2}. \tag{4.19}$$

Proof. The proof follows by substituting H_{zu} for B_{zu} in Appendix B of [10]. □

Unfortunately, the optimal step size given by (4.19) is not implementable since a measurement of $\varepsilon(k)$ is not available. Therefore, we need to find an implementable step size function that satisfies (4.18). Consider the step size function

$$\hat{\eta}_{ts}(k) \triangleq \frac{1}{(p+q)\,||H_{zu}||_F^2\,||\Phi_{uy}(k)||_2^2}. \tag{4.20}$$

By an argument similar to the one presented in Appendix C of [10], it can be shown that the candidate step size function $\hat{\eta}_{ts}(k)$ satisfies

$$0 < \hat{\eta}_{ts}(k) \le \eta^*_{ts}(k). \tag{4.21}$$

The adaptive feedback disturbance rejection algorithm is given by the time-series controller (4.4), and the controller parameter matrix update law (4.13) with the step size function $\eta_{ts}(k) = \hat{\eta}_{ts}(k)$.

5 ACOUSTIC DUCT EXAMPLE

Consider the equations of motion for an acoustic duct, given in [12, 13],

$$\frac{1}{c^2}p_{tt}(\chi,t) = p_{\chi\chi}(\chi,t) + \rho_0\dot{v}_u(t)\delta(\chi-\chi_u) + \rho_0\dot{v}_w(t)\delta(\chi-\chi_w), \tag{5.1}$$

$$z(t) = p(\chi_z,t), \tag{5.2}$$

$$y(t) = p(\chi_y,t), \tag{5.3}$$

where $p(\chi,t)$ is the acoustic pressure, $\rho_0 = 1.21$ kg/m^3 is the equilibrium density of air, $c = 343$ m/s is the acoustic wave speed in air at room conditions, $v_u(t)$ and $v_w(t)$ are the speaker cone velocities of the control and disturbance speakers respectively, and χ_u, χ_w, χ_z, and χ_y are the positions of the control, disturbance, performance, and measurement, respectively. The boundary conditions are open at $\chi = 0$ and open at $\chi = L$, where L is the length of the duct. The system (5.1)-(5.3) can be written in the standard state-space form by truncating the system to finite dimensions. Assuming separation of variables and retaining the first r modes, the acoustic pressure is given by

$$p(\chi,t) = \sum_{i=1}^{r} q_i(t)V_i(\chi). \tag{5.4}$$

Now the system may be written as the $2r$-dimensional state-space system

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t) + \hat{D}_1 w(t), \tag{5.5}$$

$$z(t) = \hat{E}_1 x(t), \tag{5.6}$$

$$y(t) = \hat{C}x(t), \tag{5.7}$$

where $x(t) \triangleq \begin{bmatrix} q_1(t) & \dot{q}_1(t) & \cdots & q_r(t) & \dot{q}_r(t) \end{bmatrix}$, $u(t) \triangleq \dot{v}_u(t)$, $w(t) \triangleq \dot{v}_w(t)$,

$$\hat{A} \triangleq \begin{bmatrix} 0 & 1 & & & \\ -\omega_1^2 & -2\zeta_1\omega_1 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -\omega_r^2 & -2\zeta_1\omega_r \end{bmatrix}, \quad \hat{B} \triangleq \begin{bmatrix} 0 \\ \rho_0 V_1(\chi_u) \\ \vdots \\ 0 \\ \rho_0 V_r(\chi_u) \end{bmatrix}, \quad \hat{D}_1 \triangleq \begin{bmatrix} 0 \\ \rho_0 V_1(\chi_w) \\ \vdots \\ 0 \\ \rho_0 V_r(\chi_w) \end{bmatrix}, \tag{5.8}$$

$$\hat{E}_1 \triangleq \begin{bmatrix} V_1(\chi_z) & 0 & \cdots & V_r(\chi_z) & 0 \end{bmatrix}, \quad \hat{C} \triangleq \begin{bmatrix} V_1(\chi_y) & 0 & \cdots & V_r(\chi_y) & 0 \end{bmatrix}. \tag{5.9}$$

Furthermore, for $i = 1,\dots,r$ the modal frequencies are $\omega_i \triangleq \frac{i\pi}{L}$, the mode shapes are $V_i \triangleq c\sqrt{\frac{2}{L}}\sin\frac{i\pi}{L}\chi$, and the damping ratios, due to the introduction of proportional damping, are ζ_i.

For this numerical example, we let $r = 6$ modes, $L = 2$ m, $\chi_w = 0.1$ m, $\chi_y = 0.5$ m, $\chi_z = 1.0$ m, $\chi_u = 1.5$ m, and for $i = 1,\dots,6$, $\zeta_i = 0.05$. The modal frequencies are 85.75 Hz, 171.5 Hz, 257.25 Hz, 343.0 Hz, 428.75 Hz, and 514.5 Hz. The continuous-time system is sampled with a zero-order-hold at 2 kHz. Now, we implement the time-series adaptive feedback disturbance rejection controller presented in Section 4 to demonstrate disturbance rejection on constant, tonal, and broadband disturbances.

5.1 Acoustic Duct with Constant Disturbance

The 6 mode acoustic duct model described above is excited by the constant disturbance $w(k) = 20$ m/s^2. The initial conditions of the acoustic duct are assumed to be zero. The time-series adaptive feedback disturbance rejection algorithm is implemented with $n_c = 12$, $p = 5$, and $q = 3$. Figure 1 shows the open-loop and closed-loop response to the constant disturbance $w(k)$. For 0.5 seconds, the plant is allowed to run open-loop. The adaptive controller is then turned on and asymptotically rejects the constant disturbance.

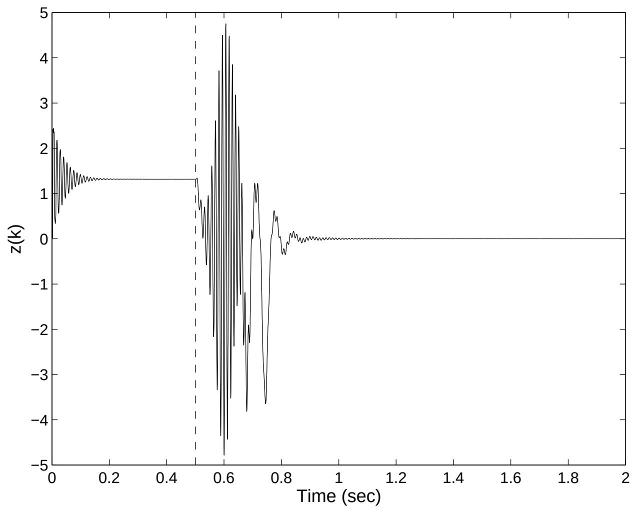

Figure 1: At 0.5 seconds, the adaptive feedback disturbance rejection controller is turned on and rejects the constant disturbance.

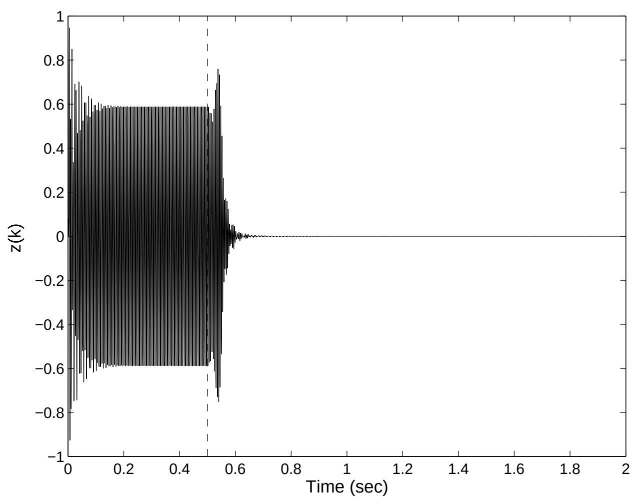

Figure 2: At 0.5 seconds, the adaptive feedback disturbance rejection controller is turned on and rejects the 200 Hz tonal disturbance.

5.2 Acoustic Duct with Tonal Disturbance

Now, we demonstrate disturbance rejection with a tonal disturbance. The disturbance $w(k)$ is given by a single-tone at 200 Hz. The time-series adaptive feedback disturbance rejection algorithm is implemented with $n_c = 12$, $p = 5$, and $q = 3$. The acoustic duct is assumed to have zero initial conditions. For 0.5 seconds, the plant operates in open-loop. Then the disturbance rejection algorithm is turned on and asymptotically rejects the constant disturbance. Figure 2 shows the disturbance rejection in the time domain. Figure 3 shows the magnitude and phase of the controller transfer function after the controller parameters $\theta(k)$ converge. To reject the disturbance, the adaptive controller converges to a notch near the 200 Hz disturbance frequency.

Next, we consider a dual-tone disturbance with frequencies of 200 Hz and 348 Hz. The time-series adaptive feedback disturbance rejection algorithm is implemented with $n_c = 24$, $p = 5$, and $q = 5$. The initial conditions of the duct and the controller are assumed to be zero. After operating in open-loop for 0.5 seconds, the adaptive controller is turned on. Figure 4 shows that the disturbance is asymptotically rejected.

5.3 Acoustic Duct with Broadband Disturbance

Finally, we demonstrate disturbance attenuation on the acoustic duct with a broadband disturbance. The disturbance $w(k)$ is given by Gaussian white noise. The initial conditions of the acoustic duct and the controller

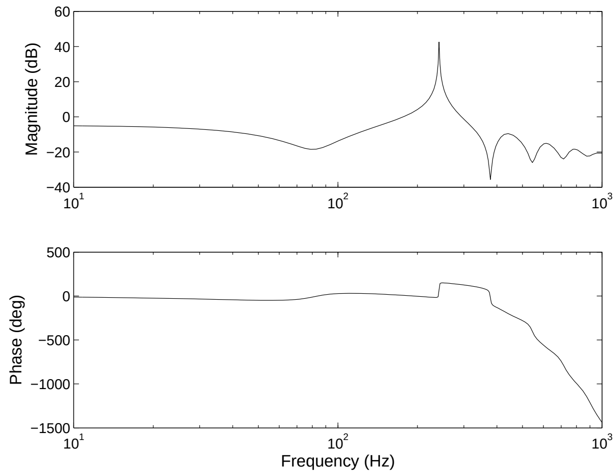

Figure 3: The magnitude and phase of the controller after the controller parameters $\theta(k)$ converge. The controller placed a notch near the 200 Hz disturbance frequency.

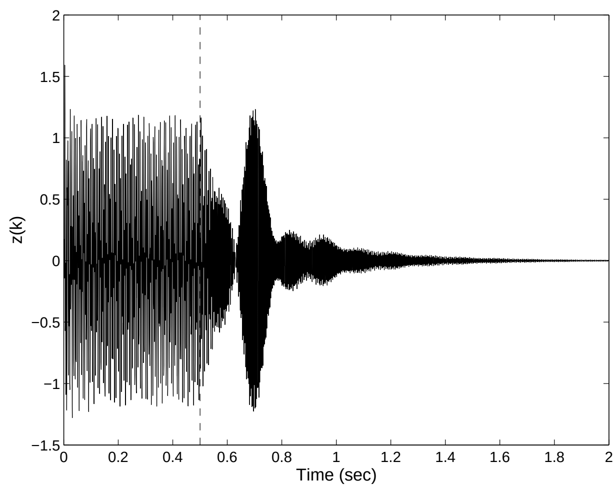

Figure 4: At 0.5 seconds, the adaptive feedback disturbance rejection controller is turned on and rejects the 200 Hz and 348 Hz tonal disturbances.

are assumed to be zero. The time-series adaptive feedback disturbance rejection algorithm is implemented with $n_c = 24$, $p = 10$, and $q = 6$. We allowed the adaptive controller to adapt until $\theta(k)$ converged. Using this fixed controller, we calculated the closed-loop transfer function from $w(k)$ to $z(k)$. Figure 5 compares the open-loop and closed-loop transfer function from the disturbance $w(k)$ to the performance $z(k)$.

6 CONCLUSIONS

In this paper, we developed discrete-time adaptive feedback disturbance rejection algorithms. Algorithms are developed for controllers represented by both a state-space model and a time-series model. The method does not require any information of the disturbance spectrum. A retrospective performance measure was used to develop gradient update laws and adaptive step size functions that do not depend upon the unknown disturbance spectrum or unknown plant information. The novel feature of this paper is that the algorithm requires information of some Markov parameters from the control input to the performance output rather then the numerator of this transfer function. Lastly, we numerically demonstrated the feedback disturbance rejection algorithm on constant, tonal, and broadband disturbances.

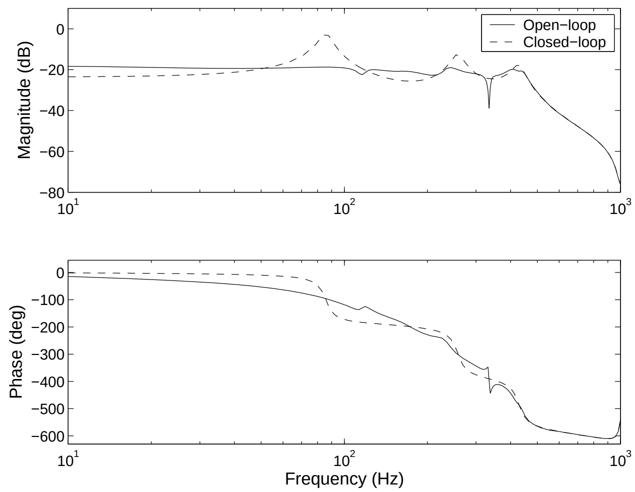

Figure 5: The open-loop and closed-loop transfer functions from $w(k)$ to $z(k)$.

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