53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Faux-Riccati Synthesis of Nonlinear Observer-Based Compensators for Discrete-Time Nonlinear Systems

Anna Prach[1], Ozan Tekinalp [1] and Dennis S. Bernstein[2]

***Abstract*— Pseudo-linear models of nonlinear systems use either a state-dependent coefficient or the Jacobian of the vector field to facilitate the use of Riccati techniques. In this paper we use the state-dependent Riccati equation (SDRE) and the forward propagating Riccati equation (FPRE) with pseudo-linear models to construct nonlinear observer-based compensators for output-feedback control of nonlinear discrete-time systems. While attractive due to their simplicity and potentially wide applicability, these techniques remain largely heuristic. The goal of this paper is thus to present numerical experiments to assess the performance of these "faux" Riccati techniques on representative nonlinear systems. The goal is to compare the performance of SDRE and FPRE when used with either a state-dependent coefficient or the Jacobian of the vector field. Stabilization and performance are considered, along with integral control for step command following.**

I. INTRODUCTION

Nonlinear control has seen extensive progress during the last several decades through the development of a wide range of techniques, such as HJB methods, backstepping, nested saturations, and feedback linearization. While these techniques are generally confined to full-state feedback, under some conditions, such as passivity, output feedback control of nonlinear systems is feasible. In many applications, however, control of nonlinear systems without benefit of the full state remains a serious challenge. In particular, difficulties arise in constructing nonlinear observers that can be used in conjunction with a nonlinear separation principle.

In the present paper we consider nonlinear output-feedback compensation of nonlinear systems by taking advantage of the confluence of several ideas and techniques, all of which are, to varying degrees, heuristic. The first idea is to focus on nonlinear systems with state-dependent coefficients (SDC), which have the pseudo-linear form $\dot{x} = A(x)x + B(x)u$. These systems have been widely studied using the state-dependent Riccati equation formulation (SDRE), where an algebraic Riccati equation (ARE) is solved at each time instant [1], [2], [3]. Implementation of SDRE requires that stabilizability and detectability conditions be satisfied at each time instant, and global guarantees of stability and performance are not available. The dual case of estimation for nonlinear systems can also be addressed [4], [5]. If A and B are also time varying, that is, $A(x,t)$ and $B(x,t)$, then ARE can also be solved at each time step, leading to a frozen-time Riccati equation (FTRE) formulation [6].

An alternative approach to SDRE is to solve a forward propagating Riccati equation (FPRE). This approach is antithetical to the classical optimality conditions, where the differential Riccati equation (DRE) is solved backward in time [7]. However, as shown in [8], [9], solving DRE forward in time is often stabilizing and close to optimal. In addition, unlike SDRE, stabilizability and detectability conditions need not be satisfied at each instant of time. FPRE is a natural dual to the Kalman filter error-covariance update, which also propagates forward in time.

While state-dependent coefficients provide a heuristic technique that can be used to apply linear control techniques to nonlinear control problems, there is another approach that is more established, at least within the context of estimation. Here we are referring to the extended Kalman filter (EKF), which uses the Jacobian (linearization along the trajectory) of the vector field for the error covariance update [10], [11]. While the Jacobian is routinely used for the EKF, it apparently has not been used for control, although there is nothing that prevents its use within the context of either SDRE or FPRE. By the same token, although SDC has been used for SDRE-based estimation, SDC does not appear to have been studied within the context of the Kalman filter with differential error-covariance update. Of course, the Jacobian cannot be used if the vector field is not differentiable, just as the SDC cannot be used if the vector field cannot be factored. Together, SDRE and FPRE with either SDC or Jacobian pseudo-linear models constitute "faux Riccati" techniques.

Whether the SDC or Jacobian is used for control and estimation within either the FPRE or SDRE, the resulting regulator and estimator can be combined to form an observer-based compensator. This "forced separation" is, of course, ad hoc, and there is no guarantee that the resulting closed-loop system is asymptotically stable, either locally or globally. Note that, within the context of output feedback, the SDC and Jacobian must be evaluated at the state estimate, which introduces additional error.

Having laid out the various elements of faux Riccati control techniques, the goal of this paper is to illustrate several variations of this technique for output feedback compensation and provide numerical experiments that are intended to motivate further investigation of this approach. To do this, we adopt a discrete-time setting in order to avoid clouding the numerics with issues of integration accuracy. This is for convenience only since all of the techniques can be formulated for continuous-time systems. To do this, we

[1]Authors are with the Department of Aerospace Engineering, Middle East Technical University, 06800, Ankara, Turkey. annaprach@metu.edu.tr, tekinalp@metu.edu.tr

[2]Dennis S. Bernstein is with the Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140. dsbaero@umich.edu

discretize continuous-time examples using Euler integration with a fixed step size. Each resulting discrete-time model is adopted as the truth model for the purposes of the subsequent numerical investigation. The accuracy of the discrete-time model relative to the underlying continuous-time system does not concern us here since that aspect is irrelevant to the objective of the investigation.

The examples we consider are the Van der Pol oscillator (VDP) and the rotational-translational actuator (RTAC) [12], [13]. For each system, we first consider full-state feedback in order to compare the performance of SDC and Jacobian pseudo-linear models. We then consider output feedback, and, finally, we use the same techniques to design controllers with integrators in order to follow step commands and reject constant disturbances. Note that effectiveness of an integrator in following step commands and rejecting constant disturbances is not assured due to the fact that the plants are nonlinear.

One of the basic questions that these numerical experiments are aimed at concerns the relative accuracy of SDRE and FPRE. In addition, we are interested in comparing the accuracy of the SDC and Jacobian for both SDRE and FPRE. These findings are discussed in the Conclusions section.

II. Pseudo-Linear Models

Consider the discrete-time nonlinear system

$$x_{k+1} = f(x_k) + B(x_k)u_k,\ x(0) = x_0, \tag{1}$$

$$y_k = C(x_k)x_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^p$ is the output, and, for all $x_k \in \mathbb{R}^n$, $f(x_k) \in \mathbb{R}^n$, $B(x_k) \in \mathbb{R}^{n\times m}$, and $C(x_k) \in \mathbb{R}^{p\times n}$.

We consider two pseudo-linear versions of (1). The state-dependent coefficient (SDC) form is given by an exact factorization of $f(x_k)$ of the form

$$f(x_k) = A(x_k)x_k. \tag{3}$$

Note that the SDC matrix $A(x_k)$ satisfying (3) is not unique. For example, if $N(x_k) \in \mathbb{R}^{n\times n}$ satisfies $N(x_k)x_k = 0$, then $A(x_k)$ in (1) can replaced by $A(x_k) + N(x_k)$.

Motivated by the extended Kalman filter, we consider the approximation

$$f(x_k) \approx A_{\rm J}(x_k)x_k,\ \ A_{\rm J}(x_k) \stackrel{\triangle}{=} f'(x_k). \tag{4}$$

Note that the Jacobian $A_{\rm J}(x_k)$ of $f(x_k)$ is not defined at an equilibrium, but rather is updated along the state trajectory.

For full-state feedback with perfect measurements of the state, both $A(x_k)$ and $A_{\rm J}(x_k)$ can be used to update the feedback gain. If the state is not measured, then $A(x_k)$ and $A_{\rm J}(x_k)$ must be evaluated at an estimate $\hat{x}_k$ of x_k.

We define the pseudo-linear dynamics

$$x_{k+1} = A_k x_k + B_k u_k, \tag{5}$$

$$y_k = C_k x_k, \tag{6}$$

where $B_k \stackrel{\triangle}{=} B(x_k)$, $C_k \stackrel{\triangle}{=} C(x_k)$ and A_k represents either the SDC $A(x_k)$ or the Jacobian $A_{\rm J}(x_k)$.

III. Full State Feedback

Full-state feedback concerns the case where $C(x_k) = I_n$. In this case we consider the full-state-feedback control law

$$u_k = K_k x_k, \tag{7}$$

where K_k is given by

$$K_k = -(B_k^{\rm T} P_k B_k + R_2)^{-1} B_k^{\rm T} P_k A_k \tag{8}$$

and $P_k \in \mathbb{R}^n$ is the solution of either an algebraic Riccati equation (ARE) in the case of SDRE or a difference Riccati equation (DRE) in the case of FPRE. In particular, let $R_1 \in \mathbb{R}^{n\times n}$ be positive semidefinite and let $R_2 \in \mathbb{R}^{m\times m}$ be positive definite. For SDRE, the ARE has the form

$$P_k = A_k^{\rm T} P_k A_k - A_k^{\rm T} P_k B_k (B_k^{\rm T} P_k B_k + R_2)^{-1} B_k^{\rm T} P_k A_k + R_1, \tag{9}$$

whereas, for FPRE, the DRE has the form

$$P_{k+1} = A_k^{\rm T} P_k A_k - A_k^{\rm T} P_k B_k (B_k^{\rm T} P_k B_k + R_2)^{-1} B_k^{\rm T} P_k A_k + R_1. \tag{10}$$

For both ARE and DRE, the matrix A_k in (8), (9), (10) represents either the SDC or the Jacobian. Note that, since x_k is measured in the case of full-state feedback, both the SDC and the Jacobian are evaluated at the true state x_k. In both cases, the closed-loop system is given by

$$x_{k+1} = (A_k + B_k K_k)x_k. \tag{11}$$

For SDRE, where A_k is not unique, solvability of ARE requires that A_k be chosen such that (A_k, B_k) is stabilizable. This restriction does not apply to FPRE, however.

IV. Output Feedback

For output feedback, the measurement is given by (2). In this case we consider the observer-based compensator

$$x_{{\rm c},k+1} = (A_k + B_k K_k - F_k C_k)x_{{\rm c},k} + F_k y_k, \tag{12}$$

$$u_k = K_k x_{{\rm c},k}. \tag{13}$$

The regulator gain K_k is the full-state-feedback gain (8), and the observer gain F_k is given by

$$F_k = A_k Q_k C_k^{\rm T} (C_k^{\rm T} Q_k C_k + V_2)^{-1}, \tag{14}$$

where Q_k in (14) is a solution to either the SDRE ARE

$$Q_k = A_k Q_k A_k^{\rm T} - A_k Q_k C_k^{\rm T} (C_k Q_k C_k^{\rm T} + V_2)^{-1} C_k Q_k A_k^{\rm T} + V_1, \tag{15}$$

or the FPRE DRE

$$Q_{k+1} = A_k Q_k A_k^{\rm T} - A_k Q_k C_k^{\rm T} (C_k Q_k C_k^{\rm T} + V_2)^{-1} C_k Q_k A_k^{\rm T} + V_1. \tag{16}$$

The matrix A_k in the equations for the regulator and observer optimal gains, (8) and (14), and also in the corresponding equations for the covariances P_k and Q_k, (9), (10) (15), and (16), is given by either SDC form (3) as $A(x_{\rm c,k})$ or as the Jacobian (4) as $A_{\rm J}(x_{\rm c,k})$, which are evaluated at the estimated state.

Note that (15) and (16) are the duals of (9) and (10), respectively. Furthermore, note that (16) is the Kalman filter covariance update Riccati equation applied to a time-varying trajectory.The structure of the observer-based compensator (12), (13) represents a regulator/observer structure. Of course, the use of this "forced separation" structure is heuristic in the sense that stability is not guaranteed.

The closed-loop system with the observer-based dynamic compensator (12) is given by

$$\begin{aligned} x_{k+1} &= f(x_k) + B_k K_k x_{\mathrm{c},k}, \\ x_{\mathrm{c},k+1} &= (A_k + B_k K_k - F_k C_k) x_{\mathrm{c},k} + F_k y_k. \end{aligned} \tag{17}$$

As in the case of full-state feedback, we consider output feedback using SDRE and FPRE.

V. Discrete-time models

The examples we consider are based on continuous-time systems. We apply Euler integration to obtain discrete-time models, which are used as the basis of all numerical examples. These discretized models are viewed as "truth" models, and the performance of the control laws is considered only within the context of the discretized models.

Consider the continuous-time system

$$\dot{x}(t) = f_{\mathrm{cont}}(x(t)) + B_{\mathrm{cont}}(x(t))u(t), \tag{18}$$
$$y(t) = C_{\mathrm{cont}}(x(t))x(t). \tag{19}$$

Using Euler integration, we obtain a discrete-time version of the continuous-time system (18), (19) given by

$$x_{k+1} = x_k + T_s[f_{\mathrm{cont}}(x_k) + B_{\mathrm{cont}}(x_k)u_k], \tag{20}$$
$$y_k = C_{\mathrm{cont}}(x_k)x_k, \tag{21}$$

where T_s is the sampling time, which is chosen to be 0.01 sec for all subsequent examples, and x_k denotes $x(kT_s)$. In terms of the notation of (1), (2), it follows that

$$f(x_k) = x_k + T_s f_{\mathrm{cont}}(x(kT_s)), \tag{22}$$
$$B(x_k) = B_{\mathrm{cont}}(x(kT_s)), \tag{23}$$
$$C(x_k) = C(x(kT_s)). \tag{24}$$

VI. Numerical example: Van der Pol Oscillator

Consider the Van der Pol oscillator

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + \mu(1 - x_1^2)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \tag{25}$$

with the SDC parametrization

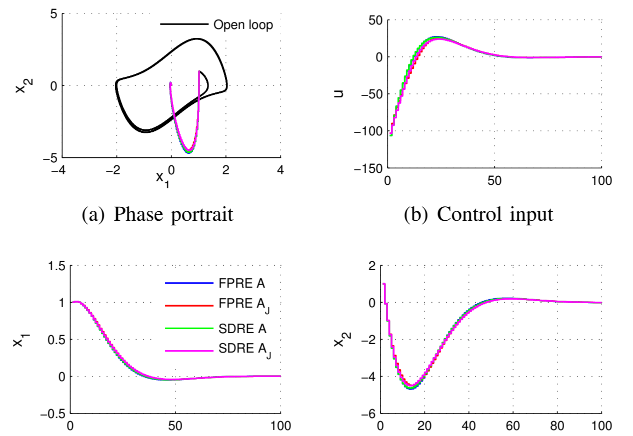
$$A(x) = \begin{bmatrix} 0 & 1 \\ -1 & \mu(1 - x_1^2) \end{bmatrix}, \tag{26}$$

and the Jacobian

$$A_{\mathrm{J}}(x) = \begin{bmatrix} 0 & 1 \\ -(1 + 2\mu x_1 x_2) & \mu(1 - x_1^2) \end{bmatrix}. \tag{27}$$

The discrete-time model of VDP is obtained according to (20), (21).

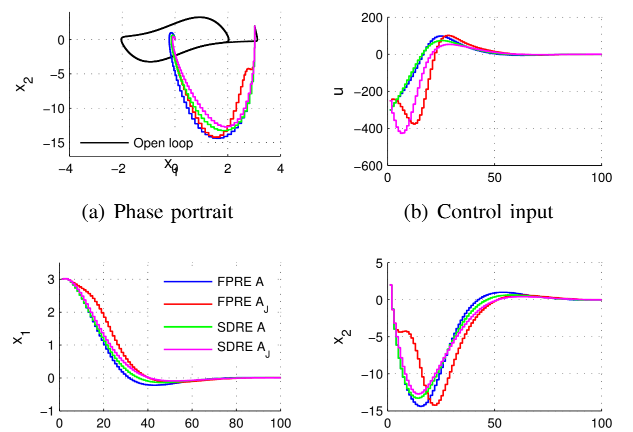
Let $\mu = 1.5$, $R_1 = V_1 = [10 \quad 0]^{\mathrm{T}}[10 \quad 0]$, and $R_2 = V_2 = 0.01$. For full-state and output-feedback control we test two sets of initial conditions for the state vector. The initial condition $[1\, 1]^{\mathrm{T}}$ is inside the open-loop limit cycle, whereas the initial condition $[3 \;\, 2]^{\mathrm{T}}$ is outside the limit cycle. The numerical simulation results for the SDRE and FPRE full-state feedback controllers are shown in Figs. 1 and 2.



(a) Phase portrait (b) Control input

(c) State trajectory x_1 (d) State trajectory x_2

Fig. 1. State trajectories and control input for the full-state SDRE and FPRE feedback compensators for VDP with the initial condition $[1 \;\, 1]^{\mathrm{T}}$. The phase portrait of the uncontrolled system is plotted along with the phase portraits of the closed-loop system.



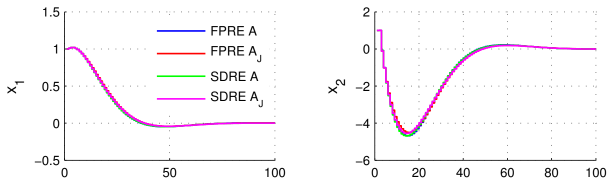
(a) Phase portrait (b) Control input

(c) State trajectory x_1 (d) State trajectory x_2

Fig. 2. State trajectories and control input for the full-state SDRE and FPRE feedback compensators for VDP with the initial condition $[3\;2]^{\mathrm{T}}$. (d) shows that the controllers with the Jacobian $A_{\mathrm{J}}(x)$ require larger control input than the controllers with the SDC $A(x)$.

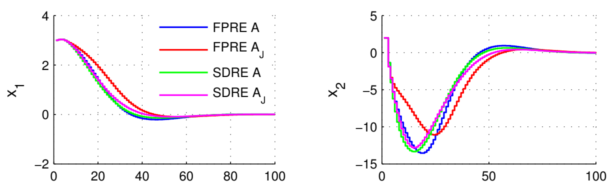
For output feedback, we assume that $y = x_1$, and thus $C = [1 \quad 0]$. For the observer-based controller, A and A_{J} must be evaluated at the estimate x_{c} of x. The weights R_1, R_2, V_1, V_2 are chosen as in the full-state feedback case. We use zero initial conditions for the state estimates x_{c}. Responses for the output-feedback case are shown in Figs. 3 and 4.

Additionally, within the framework of output feedback, we want the output to follow a step command. This is achieved by aserting an integrator into the loop and thus, we obtain a tracking output-feedback controller. The responses of the tracking output-feedback controller for VDP are shown in Figs. 5 and 6.



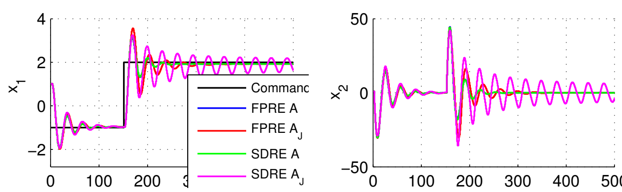
(a) State trajectory x_1 (b) State trajectory x_2

Fig. 3. State trajectories for the output-feedback SDRE and FPRE compensators for VDP with the initial condition $[1\ 1]^{\mathrm{T}}$.

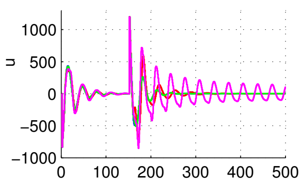


(a) State trajectory x_1 (b) State trajectory x_2

Fig. 4. State trajectories for the output feedback SDRE and FPRE compensators for VDP with the initial condition $[3\ 2]^{\mathrm{T}}$.



(a) State trajectory x_1 (b) State trajectory x_2



(c) Control input

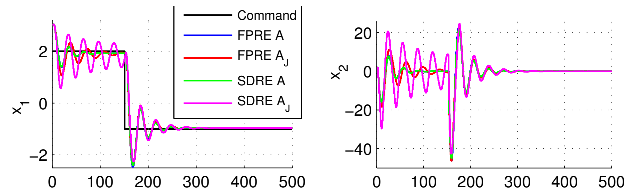
Fig. 5. State trajectories and control input for the tracking output feedback SDRE and FPRE compensators for VDP with the initial condition $[1\ 1]^{\mathrm{T}}$. This figure shows that SDRE with Jacobian yields poor performance, whereas, SDRE with SDC and FPRE controllers are able to follow the given command.

VII. Numerical example: RTAC

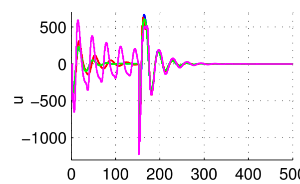
We consider the translational oscillator with rotational proof-mass actuator [12], [13]. The equations of motion of RTAC are given by

$$(M+m)\ddot{q}+kq=-me(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+F,$$
$$(I+me^2)\ddot{\theta}=-me\ddot{q}\cos\theta+N, \qquad (28)$$

where q and $\dot{q}$ are the translational position and velocity of the cart, θ and $\dot{\theta}$ are the angular position and velocity of the rotational proof-mass, respectively. M is the mass of the cart, k is the spring stiffness, m is the mass of the proof-mass actuator, I is the moment of inertia, e is the distance from the point about which the proof mass rotates to the center of mass, N is the control torque applied to the proof mass,



(a) State trajectory x_1 (b) State trajectory x_2



(c) Control input

Fig. 6. State trajectories and control input for the tracking output feedback SDRE and FPRE compensators for the VDP with the initial condition $[3\ 2]^{\mathrm{T}}$. This figure shows that all tracking controllers ensure that the output successfully attains the given command.

and F is the disturbance force on the cart. The normalized equations of motion of RTAC are

$$\ddot{\xi}+\xi=\varepsilon(\dot{\theta}^2\sin\theta-\ddot{\theta}\cos\theta)+w,$$
$$\ddot{\theta}=-\varepsilon\ddot{\xi}\cos\theta+u, \qquad (29)$$

where ξ is the normalized cart position, w is the non-dimensionalized disturbance, u is the non-dimensionalized control torque, and ε is the coupling parameter.

For the state vector $x=[x_1\ x_2\ x_3\ x_4]^{\mathrm{T}}=[\xi\ \dot{\xi}\ \theta\ \dot{\theta}]^{\mathrm{T}}$, the non-dimensional equations of motion in first-order form are

$$\dot{x}=f(x)+g(x)u+d(x)w \qquad (30)$$

where

$$f(x)=\begin{bmatrix} x_2 \\ (-x_1+\varepsilon x_4^2\sin x_3)/\Delta \\ x_4 \\ \varepsilon\cos x_3(x_1-\varepsilon x_4^2\sin x_3)/\Delta \end{bmatrix},$$
$$g(x)=\begin{bmatrix} 0 \\ -\varepsilon\cos x_3/\Delta \\ 0 \\ 1/\Delta \end{bmatrix}, d(x)=\begin{bmatrix} 0 \\ 1/\Delta \\ 0 \\ -\varepsilon\cos x_3/\Delta \end{bmatrix},$$

where $\Delta \triangleq 1-\varepsilon^2\cos^2 x_3$. SDC matrices for RTAC are

$$A(x)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/\Delta & 0 & 0 & \varepsilon x_4\sin x_3/\Delta \\ 0 & 0 & 0 & 1 \\ \varepsilon\cos x_3/\Delta & 0 & 0 & -\varepsilon^2 x_4\cos x_3\sin x_3/\Delta \end{bmatrix},$$

and $B(x)=g(x)$. The expression for $A_{\mathrm{J}}(x)$ is lengthy, and thus is not shown. A discrete-time model is obtained according to (20), (21). We adopt the RTAC parameters given in [12]. For the numerical simulations we consider the initial conditions $[0.5\ 0.1\ 0.5\ 0.1]^{\mathrm{T}}$ and $[1\ 0.5\ 0.3\ 0.3]^{\mathrm{T}}$. The weights R_1 and R_2 are selected for SDRE and FPRE separately for each controller and may not be equal. Figures

7 and 8 show the simulation results for the full-state feedback controllers.

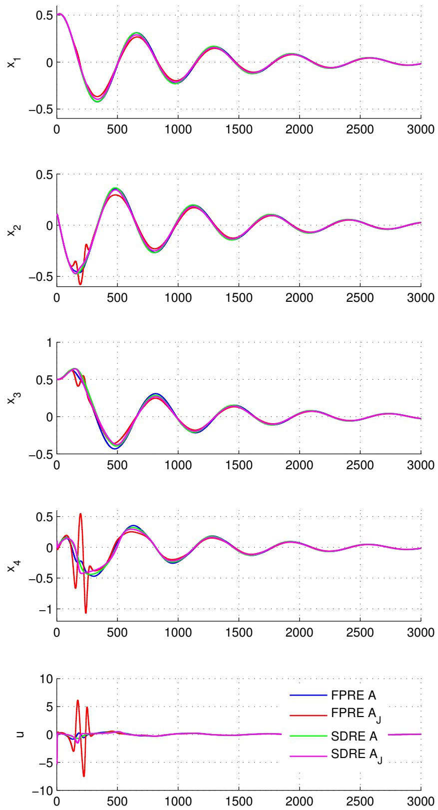


Fig. 7. State trajectories and control input for the full-state SDRE and FPRE feedback compensators for the RTAC with the initial condition $[0.5\ 0.1\ 0.5\ 0.1]^{\mathrm{T}}$. This figure shows that all four control methods provide stabilizing performance. FPRE with the Jacobian yields poor performance and requires larger control input than the other controllers.

For output feedback we assume that the measurement of the angular position θ of the rotational proof-mass be available, thus $C = [0 \quad 0 \quad 1 \quad 0]$. Let $V_2 = 0.01$ and $V_1 = \alpha I + D(x_k)D(x_k)^{\mathrm{T}}$, where $D(x_k) = d(x(kT_s))$. The state trajectories and control inputs for the output-feedback are given in Figs. 9 and 10. In the tracking output-feedback compensator, the command is given to the angular position of the rotational proof-mass, θ. The state trajectories and the control input of the tracking output-feedback controller for RTAC are given in Fig. 11.

VIII. CONCLUSIONS

We compared four heuristic techniques for output-feedback compensation of nonlinear systems. These techniques use either a forward-propagating Riccati equation or an algebraic Riccati equation in conjunction with either a state-dependent coefficient or the Jacobian of the vector field. The nonlinear output-feedback compensator is an observer-based compensator with a separation structure. These methods are not equally applicable for various reasons. For example, for systems with a nondifferentiable vector field, a state-dependent coefficient may exist but the Jacobian may not. Furthermore, while the algebraic Riccati equation

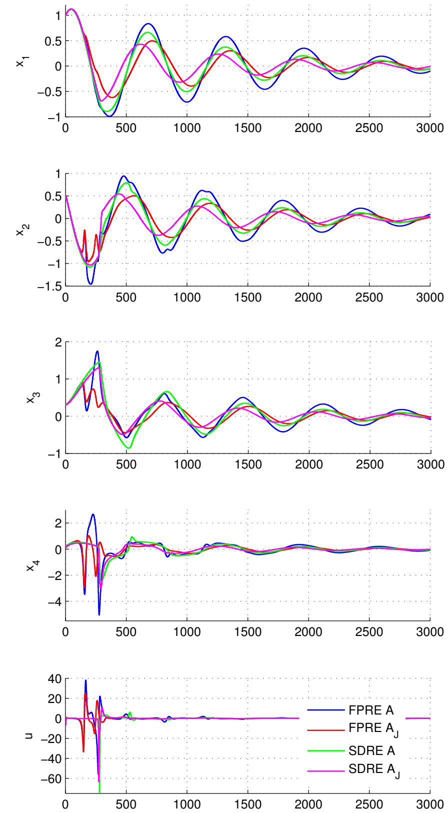


Fig. 8. State trajectories and control input for the full-state SDRE and FPRE feedback compensators for the RTAC with the initial condition $[1\ 0.5\ 0.3\ 0.3]^{\mathrm{T}}$. This figure shows that all compensators are stabilizing when utilizing both SDC and Jacobian as pseudo-linear models.

requires stabilizability at each step, the forward-propagating Riccati equation does not. In this paper we considered nonlinear examples of orders two and four. All four methods successfully controlled both plants with differences in speed of response and control effort.

ACKNOWLEDGMENT

This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) within 2215-Ph.D. Fellowship Program.

REFERENCES

[1] J. R. Cloutier, "State-Dependent Riccati Equation Techniques: an Overview," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, June 1997, pp. 932–936.
[2] T. Cimen, "Systematic and Effective Design of Nonlinear Feedback Controllers via the State-Dependent Riccati Equation (SDRE) method," *Annual Reviews of Control*, vol. 34, pp. 32–51, 2010.
[3] E. B. Erdem and A. Alleyne, "Design of a Class of Nonlinear Controllers via State Dependent Riccati Equations," *IEEE Trans. Contr. Syst. Tech.*, no. 12, pp. 2986–2991, 2004.
[4] C. P. Mracek, J. R. Cloutier, and C. A. D'Souza, "A new technique for nonlinear estimation," in *Proc. Conf. Contr. Appl.*, Dearborn, MI, September 1996, pp. 338–343.
[5] C. Jaganath, A. Ridley, and D. S. Bernstein, "A SDRE-Based Asymptotic Observer for Nonlinear Discrete-Time Systems," in *Proc. Amer. Contr. Conf.*, Portland, OR, June 2005, pp. 3630–3635.
[6] A. Prach, O. Tekinalp, and D. S. Bernstein, "A Numerical Comparison of Frozen-Time and Forward-Propagating Riccati Equations for Stabilization of Periodically Time-Varying Systems," in *Proc. Amer. Contr. Conf.*, Portland, OR, June 2014, pp. 5633–5638.

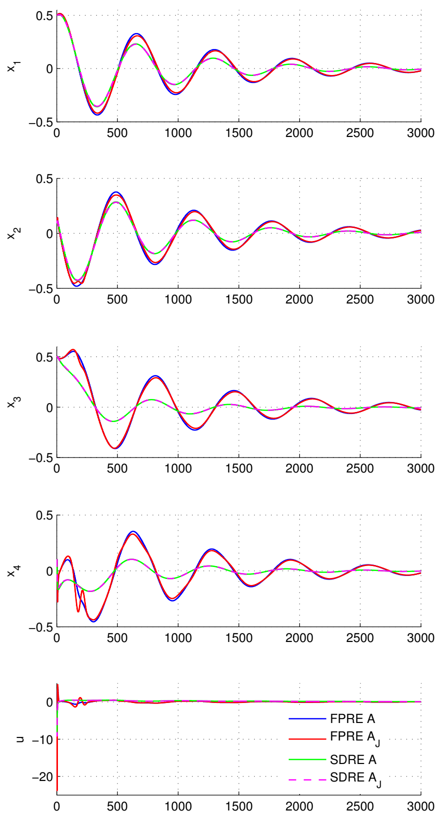


Fig. 9. State trajectories and control input for the output feedback SDRE and FPRE compensators for the RTAC with the initial condition $[0.5\ \ 0.1\ \ 0.5\ \ 0.1]^{\mathrm{T}}$. The responses show that SDRE and FPRE exhibit similar performance.

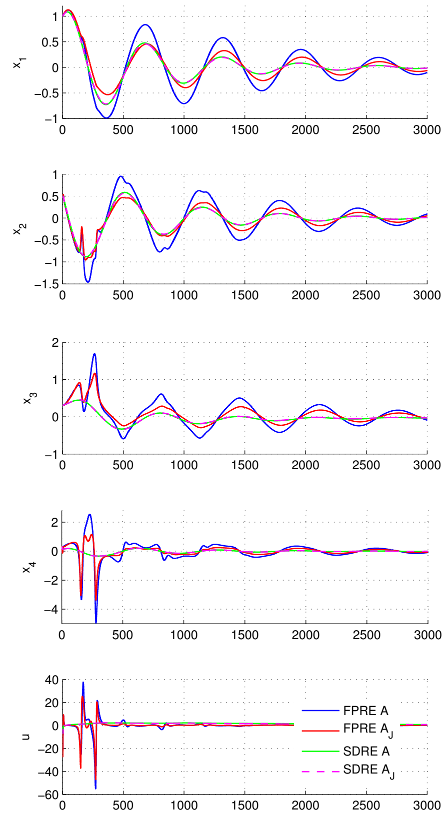


Fig. 10. State trajectories and control input for the output feedback SDRE and FPRE compensators for the RTAC with the initial condition $[1\ \ 0.5\ \ 0.3\ \ 0.3]^{\mathrm{T}}$. This figure shows that SDRE provides slightly faster stabilization with less control effort than FPRE.

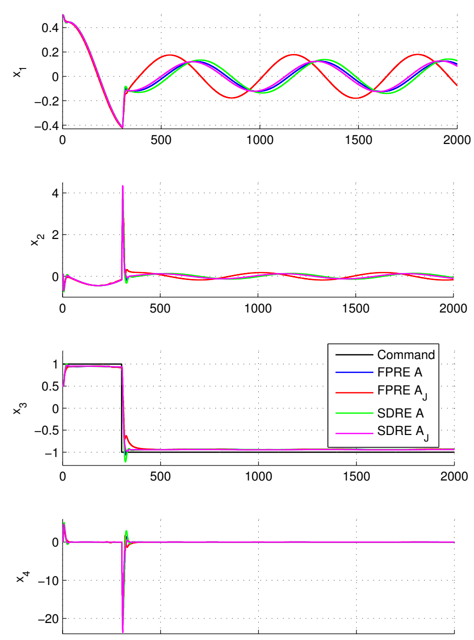


Fig. 11. State trajectories of the tracking output feedback compensator for the RTAC for the initial condition $[0.5\ 0.1\ 0.5\ 0.1]^{\mathrm{T}}$. For this case, SDRE and FPRE with both SDC and Jacobian yield fast stabilization.

[7] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*. New York, N.Y: J. Wiley and Sons, 1972.

[8] M.-S. Chen and C.-Y. Kao, "Control of Linear Time-Varying Systems Using Forward Riccati Equation," *J. Dyn. Syst., Meas. Contr.*, vol. 119, no. 3, pp. 536–540, 1997.

[9] A. Weiss, I. Kolmanovsky, and D. S. Bernstein, "Forward-Integration Riccati-Based Output-Feedback Control of Linear Time-Varying Systems," in *Proc. Amer. Contr. Conf.*, Montreal, Canada, June 2012, pp. 6708–6714.

[10] J. L. Crassidis and J. L. Junkins, *Optimal Estimation of Dynamic Systems*. CRC Press, 2004.

[11] J. W. Grizzle and Y. Song, "The Extended Kalman Filter as a Local Asymptotic Observer for Nonlinear Discrete-Time Systems," *Proc. Amer. Contr. Conf.*, pp. 3365–3369, June 1992.

[12] R. T. Bupp, D. S. Bernstein, and V. Coppola, "A Benchmark Problem for Nonlinear Control Design," *Int. J. Robust Nonl. Contr.*, no. 8, pp. 307–310, 1998.

[13] M. Jankovic, D. Fontaine, and P. V. Kokotovic, "TORA Example: Cascade- and Passivity-Based Control Designs," *IEEE Trans. Contr. Syst. Tech.*, vol. 4, no. 3, pp. 292–297, 1996.