2014 American Control Conference (ACC)
June 4-6, 2014. Portland, Oregon, USA

# A Numerical Comparison of Frozen-Time and Forward-Propagating Riccati Equations for Stabilization of Periodically Time-Varying Systems

Anna Prach[1], Ozan Tekinalp[1], and Dennis S. Bernstein[2]

*Abstract*— **Feedback control of linear time-varying systems arises in numerous applications. In this paper we numerically investigate and compare the performance of two heuristic techniques. The first technique is the frozen-time Riccati equation, which is analogous to the state-dependent Riccati equation, where the instantaneous dynamics matrix is used within an algebraic Riccati equation solved at each time step. The second technique is the forward-propagating Riccati equation, which solves the differential algebraic Riccati equation forward in time rather than backward in time as in optimal control. Both techniques are heuristic and suboptimal in the sense that neither stability nor optimal performance is guaranteed. To assess the performance of these methods, we construct Pareto efficiency curves that illustrate the state and control cost tradeoffs. Three examples involving periodically time-varying dynamics are considered, including a second-order exponentially unstable Mathieu equation, a fourth-order rotating disk with rigid body unstable modes, and a 10th-order parametrically forced beam with exponentially unstable dynamics. The first two examples assume full-state feedback, while the last example uses a scalar displacement measurement with state estimation performed by a dual Riccati technique.**

## I. INTRODUCTION

Time-varying systems arise when the dynamics of a system involve parameters or exogenous signals that are prescribed functions of time rather than functions of evolving states as in the case of linear or nonlinear time-invariant systems [1], [2], [3], [4]. These systems arise in diverse applications and for diverse reasons. For example, rotating components in a mechanical system that give rise to forces in the rotating frame are modeled as time-varying dynamics in the body frame, as in the case of a helicopter [5], [6]. More generally, a time-varying exogenous input that multiplies the states of a linear system gives rise to linear time-varying (LTV) dynamics, whose stability is classically analyzed by Floquet theory [7], [8], [10]. As another example, the linearized dynamics of a vehicle following a prescribed trajectory may depend on the location or attitude of the vehicle, thus producing dynamics that are time-varying.

Optimal control of LTV systems is handled by the backward-in-time Riccati equation, where advance knowledge of the dynamics is used to compute the feedback gains for use over the control horizon [11]. The special case of systems with periodic dynamics entails solution of a Riccati equation with periodic dynamics [12], [13]. Receding horizon techniques are also used to stabilize LTV systems [14], [15], [16].

The goal of the present paper is to numerically investigate and compare the ability of two techniques to stabilize LTV systems. The first technique, called *frozen-time Riccati equation* (FTRE), is inspired by the analogous state-dependent Riccati equation (SDRE) method. SDRE is used for nonlinear systems whose dynamics $\dot{x} = f(x, u)$ can be written in the pseudo-linear' form $\dot{x} = A(x, u)x + B(x, u)u$, for which the algebraic Riccati equation is solved at each instant of time $t$, with the dynamics and input matrices chosen to be $A(x(t), u(t))$ and $B(x(t), u(t))$, respectively. This technique is ad hoc in nature, but is easy to implement and often provides surprisingly good performance, at least for some choices of the functions $A(x, u)$ and $B(x, u)$, which are generally nonunique [17], [18], [19], [20].

The second method we consider is the *forward propagating Riccati equation* (FPRE). This approach is analogous to the backward-propagating Riccati equation in optimal control except that, like the Kalman filter, the integration is performed forward in time [21], [22]. Although this approach is suboptimal, and is not stabilizing in all cases, forward integration removes the need to know the dynamics matrix in the future, and thus may be applicable to cases where the optimal controller cannot be used.

In the present paper we compare FTRE and FPRE by considering stabilization of three benchmark LTV systems. The first is the classical Mathieu equation, which is a second-order differential equation with a constant-plus-sinusoidal stiffness term. The second is a two-degree-of-freedom rotating rigid body with force applied to its center of mass in a direction that is fixed in the body frame. The constant rotation of the body gives rise to LTV dynamics with rigid-body instability and with a periodically time-varying input matrix. The last, which is developed in [23], is an elastic beam with periodic loading and transverse control force. The equations of motion given in [23] have the form of a multi-degree-of-freedom Mathieu equation.

Both FTRE and FPRE are heuristic techniques in the sense that neither technique guarantees stabilization or closed-loop performance. The goal of this paper is therefore to investigate their properties as a further motivation and inspiration for future developments.

The contents of the paper are as follows. Section II presents a concept of the FTRE approach, while Section III gives description of the FPRE method. In Section IV we demonstrate a numerical example on application to the FTRE and FPRE controllers for stabilization of Mathieu equation.

[1]Authors are with the Department of Aerospace Engineering, Middle East Technical University, 06800, Ankara, Turkey. annaprach@metu.edu.tr, tekinalp@metu.edu.tr
[2]Dennis S. Bernstein is with the Department of Aerospace Engineering, The University of Michigan, Ann Arbor, MI 48109-2140. dsbaero@umich.edu

In Section V a second numerical example demonstrates the utility of the methods on a system with a periodically time-varying input matrix. Section VI presents application of the FTRE and FPRE to a periodic system on an example of an elastic beam under the action of periodic axial load. Finally, conclusions are summarized in section VII.

## II. FTRE Control

FTRE control is related to the SDRE technique, which involves factorization of the nonlinear dynamics (1) to the state-dependent coefficient (SDC) form,

$$\dot{x}(t) = f(x(t), u(t)), x(0) = x_0, \tag{1}$$

where $x \in \mathbf{R}^n$ is the state vector, $u \in \mathbf{R}^m$ is the input vector, function $f : \mathbf{R}^n \to \mathbf{R}^n$, and $B : \mathbf{R}^n \to \mathbf{R}^{n\times m}$. Factorization yields the linear structure with SDC matrices given by

$$\dot{x}(t) = A(x(t), u(t))x(t) + B(x(t), u(t))u(t), x(0) = x_0. \tag{2}$$

In addition, state-dependent weighting matrices provide flexibility in controller design.

The goal is to find a state feedback control law of the form $u(t) = -K(x(t))x(t)$ that minimizes the infinite-horizon performance cost function

$$J(x_0, u) = \frac{1}{2}\int_0^\infty [x^{\mathrm{T}}(t)R_1(x(t))x(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t, \tag{3}$$

where $R_1(x) \in \mathbf{R}^{n\times n}$ is positive semidefinite, $R_2(x) \in \mathbf{R}^{m\times m}$ is positive definite. The state feedback control law is given by

$$\begin{aligned} u(t) &= -K(x(t))x(t) \\ &= -R_2^{-1}(x(t))B^{\mathrm{T}}(x(t), u(t))P(x(t))x(t), \end{aligned} \tag{4}$$

where $P(x)$ is the solution of the state-dependent algebraic Riccati equation

$$A^T(x)P(x) + P(x)A(x) - P(x)B(x)R_2^{-1}(x)B^{\mathrm{T}}(x)P(x) + R_2(x) = 0. \tag{5}$$

SDRE is heuristic since the control law is not necessarily optimal and may not be stabilizing.

In this work we aim to apply the SDRE methodology to LTV systems. In the FTRE method, we “freeze” the state and input matrices at every time instant, and treat them as matrices with constant entries. A solution $P(t)$ of the frozen-time algebraic Riccati equation can be found at each time instant $t$ by solving

$$A^{\mathrm{T}}(t)P(t) + P(t)A(t) - P(t)B(t)R_2^{-1}(t)B^T(t)P(t) + R_1(t) = 0. \tag{6}$$

The control law at each time step is computed in the same manner as for a classical linear quadratic regulator problem

$$u(t_i) = -R_2^{-1}(t)B^T(t)P(t)x(t). \tag{7}$$

It is expected that FTRE control inherits the stability properties of the SDRE controller. In [24] global asymptotic stability of SDRE controllers is shown for second-order systems with single input and constant matrix $B$. For higher order systems, conditions for global stability are discussed in [18].

## III. FPRE Control

In this section, we discuss the problem of full-state feedback stabilizing control design for LTV systems, using the forward-in-time Riccati-based control law. For this purpose, we consider a linear system of the form [22]

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{8}$$

and the quadratic cost function

$$J(u) = \frac{1}{2}\int_0^{t_f} [x^{\mathrm{T}}(t)R_1(x(t))x(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t. \tag{9}$$

To regulate the system to the origin, consider the feedback control law

$$u(t) = R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x(t), \tag{10}$$

where $P(t)$ is the solution of the control Riccati equation (11), obtained by forward-in-time integration for a specified initial-time boundary condition, that is,

$$\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t) + R_1(t) = 0, P(0) = P_0. \tag{11}$$

The control law (10) is referred to as the FPRE method. Ref. [22] applies FPRE to LTV systems and also provides conditions under which FPRE is stabilizing.

## IV. Example 1: Mathieu Equation

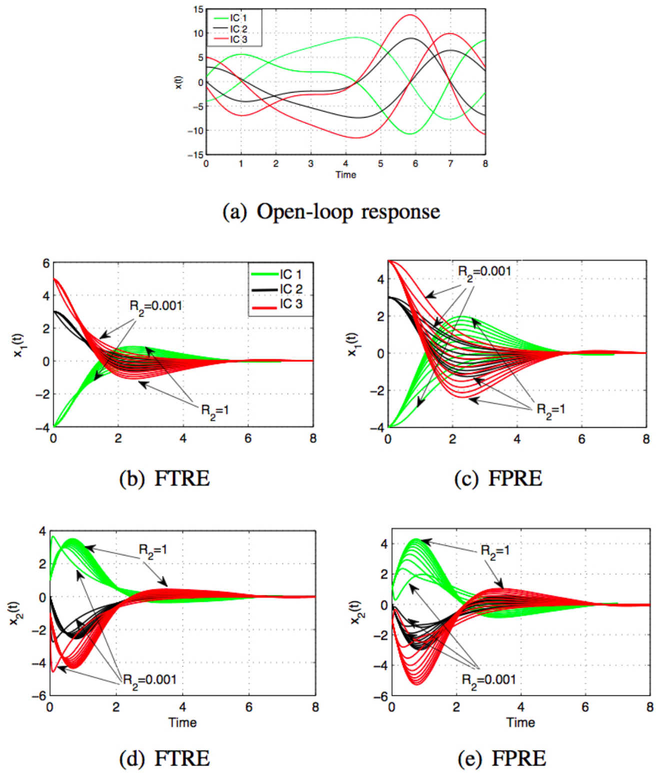
Consider stabilization of the Mathieu equation (12)

$$\ddot{q}(t) + (\alpha + \beta\cos(\omega t))q(t) = bu(t). \tag{12}$$

The open-loop system is described by

$$x(t) = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}, A(t) = \begin{bmatrix} 0 & 1 \\ -(\alpha + \beta\cos(\omega t)) & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ b \end{bmatrix}. \tag{13}$$

Let $q(0) = q_0$ and $\dot{q}(0) = \dot{q}_0$ denote the initial conditions. Fig. 1 shows unstable open-loop responses to three sets of initial conditions: [3; 0], [4; -1], [-5; 1]. The task is to bring the states to the equilibrium $[0; 0]$. In this example we use the parameter values $\alpha = 1$, $\beta = 1$, $b = 1$, $\omega = 1$. Let $R_1 = I$, $R_2$ take values from 0.001 to 1, and $P_0 = 0$ for FPRE, and apply the FTRE and FPRE controls. As seen in Fig. 1 FTRE provides faster stabilization than FPRE, especially for large values of $R_2$.




(a) Open-loop response

(b) FTRE

(c) FPRE

(d) FTRE

(e) FPRE

Fig. 1: Mathieu equation. (a) shows open-loop responses to initial conditions. State trajectories are shown in (b), (d) for FTRE and (c), (e) for FPRE for three sets of initial conditions and $R_2$ ranging from 0.001 to 1. For this example, FTRE provides better stabilizing performance than FPRE control.

## V. Example 2: Rotating Disc

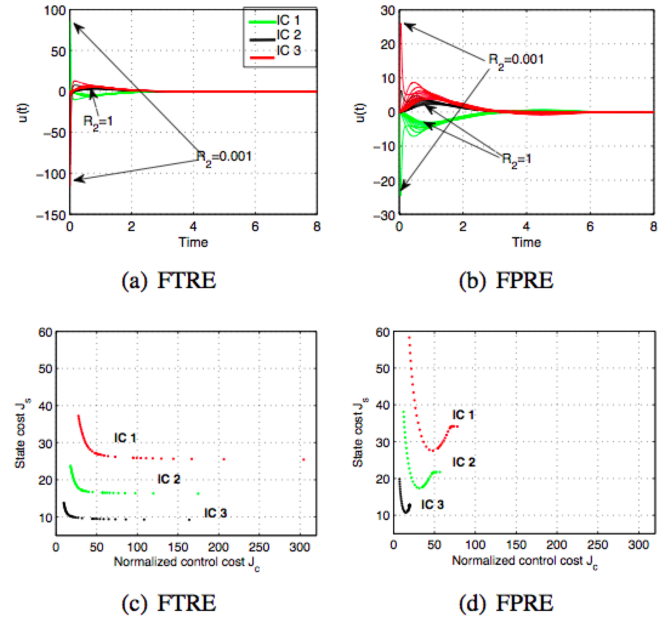
We consider a disk that translates on a horizontal plane while rotating at a constant rate. Control is performed by a thruster, located at the center of mass. The direction of the thrust is fixed with respect to the disk body frame. The goal is to bring the center of mass of the disk to a specified point. The dynamics are given by

$$x(t) = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix}, A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B(t) = \begin{bmatrix} 0 & \cos(\omega t) & 0 & \sin(\omega t) \end{bmatrix}^{\mathrm{T}}. \quad (14)$$

The structure of the state matrix $A(t)$ indicates unstable open-loop dynamics.

Consider the two sets of the initial conditions: [0.5; 0; -0.5; 0] and [0.3; 0.1; 0.5; -0.1]. Let $P_0 = 0$, and apply FTRE and FPRE controls for $\omega$ given by 1 rad/sec and 10 rad/sec, letting $R_1 = 1$ and $R_2$ change in the range from 10 to 100. Fig. 4 shows that for $\omega = 1$ rad/sec FPRE provides better stabilizing performance than FTRE with less control effort (Fig. 6) for all values of $R_2$ within the defined range. If we increase $\omega$ to 10 rad/sec, Fig. 5 shows that the response of FPRE is improved due to higher frequency of



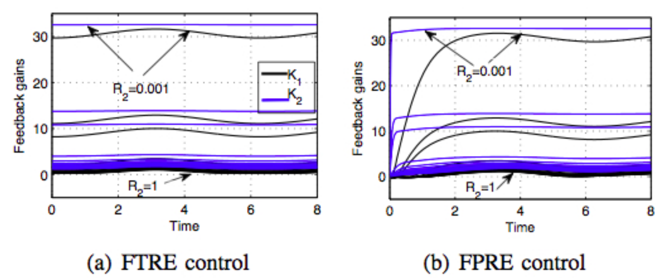

(a) FTRE

(b) FPRE

(c) FTRE

(d) FPRE

Fig. 2: Mathieu equation. (a) and (b) show the control inputs, while (c) and (d) show Pareto performance tradeoff curves for FTRE and FPRE for $R_2$ ranging from 0.001 to 1 and for three choices of initial conditions. For this example, FTRE produces a more efficient tradeoff curve than FPRE.




(a) FTRE control

(b) FPRE control

Fig. 3: Mathieu equation. This figure shows how the feedback gains change while increasing $R_2$ from 0.001 to 1.

the control action signal as seen in Fig. 7. The phenomenon of stabilization via fast time variation for FPRE control is discussed in [22]. However, for FTRE, increasing $\omega$ does not lead to improvement in the response. Pareto performance curves and feedback gains for this example are illustrated in Fig. 8.

## VI. Example 3: Elastic Beam

We consider the axially loaded simply supported beam, shown in Fig. 9. The equation of motion is given by [25]

$$EI\frac{\partial^4 v}{\partial x^4} + (P_0 + P_1 \cos\omega t)\frac{\partial^2 v}{\partial x^2} + m\frac{\partial^2 v}{\partial t^2} = 0. \quad (15)$$

For $i = 1, 2, 3, \ldots$ the mode shapes $\phi_i(x)$ and the natural frequencies $\omega_i$ are given by

$$\phi_i(x) = \sin(\tfrac{i\pi x}{L}), \quad \omega_i = \frac{i^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}. \quad (16)$$

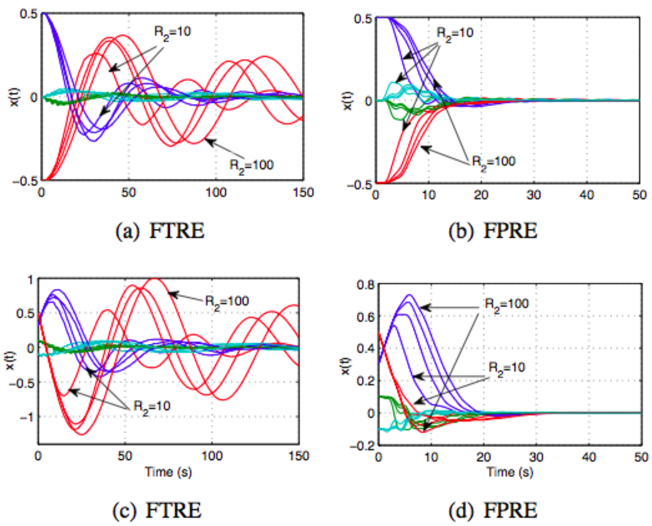



Fig. 4: Rotating disk for $\omega = 1$ rad/sec. State responses for two choices of initial conditions are shown in (a), (c) for FTRE and (b), (d) for FPRE. $R_2$ changes from 10 to 100. FPRE provides fast stabilization for the full range of $R_2$. FTRE controller is much less effective, especially for large values of $R_2$.

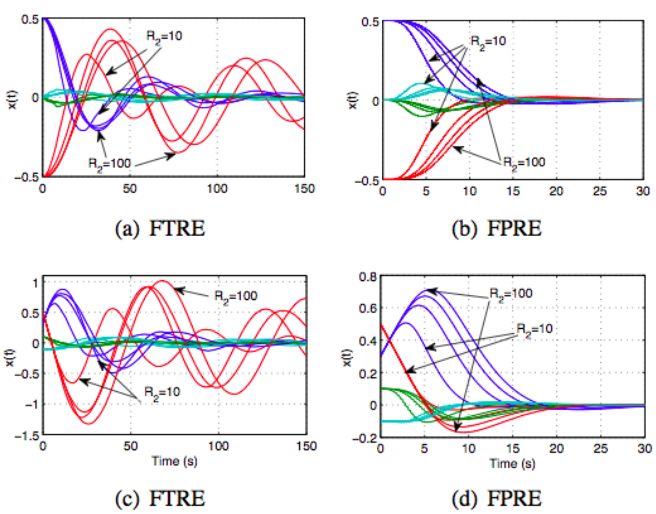



Fig. 5: Rotating disk for $\omega = 10$ rad/sec. This figure shows that, for increasing values of $\omega$, FPRE provides less oscillatory response than FTRE with shorter settling time.

Considering a solution of the form

$$v(x,t) = \sum_{i=1}^{r} f_i(t)\phi_i(x), \tag{17}$$

the differential equation for $f_i(t)$ has the form

$$\frac{\partial^2 f_i}{\partial t^2} + \omega_i^2 \left(1 - \frac{P_0 + P_1 \cos \omega t}{P_i^*}\right) f_i = 0, \tag{18}$$

where $i = 1, 2, ...$ and

$$P_i^* \triangleq i^2 EI\pi^2/L^2, \tag{19}$$

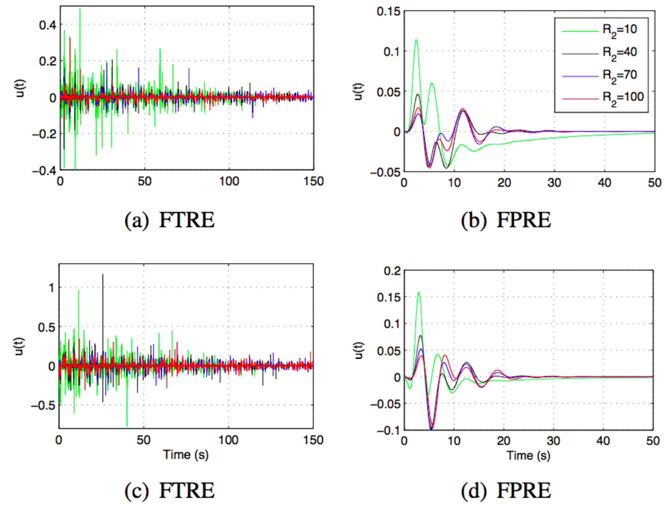



Fig. 6: Rotating disk for $\omega = 1$ rad/sec. (a), (c) illustrate controls for $R_2$ ranging from 10 to 100 for FTRE, while (b), (d) show the control inputs for FPRE. Top and bottom subfigures correspond to different initial conditions.

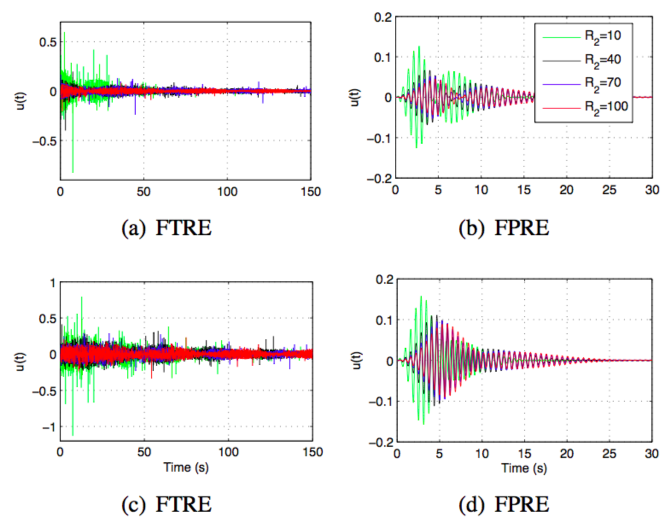



Fig. 7: Rotating disk for $\omega = 10$ rad/sec. This figure shows the control inputs for FTRE - (a), (c), and FPRE - (b), (d) for $R_2$ ranging from 10 to 100. Top and bottom plots are for different initial conditions.

is the $i^{th}$ Euler buckling load. Defining

$$\Omega_i \triangleq \omega_i \sqrt{1 - \frac{P_0}{P_i^*}} \quad \mu_i \triangleq \frac{P_1}{2(P_i^* - P_0)}, \tag{20}$$

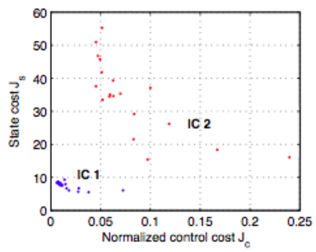
after some manipulations (18) can be written in the form

$$\frac{\partial^2 f_i}{\partial t^2} + \Omega_i^2 (1 - 2\mu_i \cos \omega t) f_i = 0. \tag{21}$$

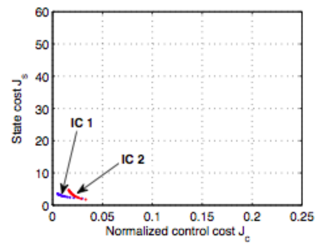
To obtain a finite-dimensional state space model involving $r$ modes of the form

$$\dot{x}(t) = A(t)x(t) + Bu(t) + D_1 w, \tag{22}$$

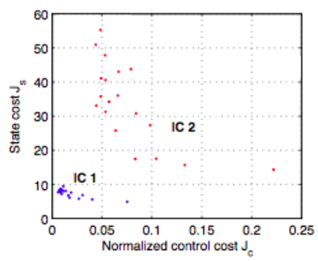
$$y(t) = Cx(t), \tag{23}$$



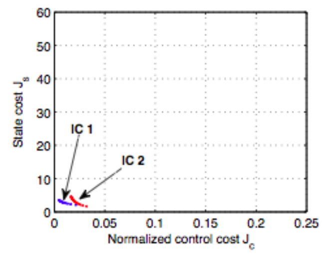

(a) FTRE; $\omega = 1$ rad/sec



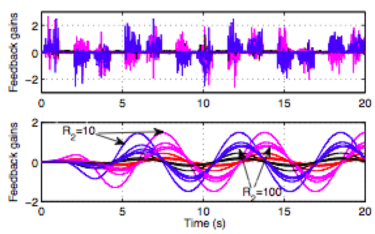

(b) FPRE; $\omega = 1$ rad/sec




(c) FTRE; $\omega = 10$ rad/sec




(d) FPRE; $\omega = 10$ rad/sec




(e) Top: FTRE. Bottom: FPRE. $\omega = 1$ rad/sec

Fig. 8: Rotating disk. In (a) and (b) for $\omega = 1$ rad/sec Pareto performance tradeoff curves for FTRE and FPRE for $R_2$ ranging from 10 to 100 are shown for two choices of initial conditions. Corresponding curves for $\omega = 10$ are illustrated in (c), (d). In this example a more efficient tradeoff curve is produced by FPRE than FTRE. In (e) feedback gains are shown for $\omega = 1$ for FTRE and FPRE for $R_2$ increasing from 10 to 100. In (c) changes in feedback gains are illustrated for the case where $\omega = 1$ rad/sec and given range for $R_2$.

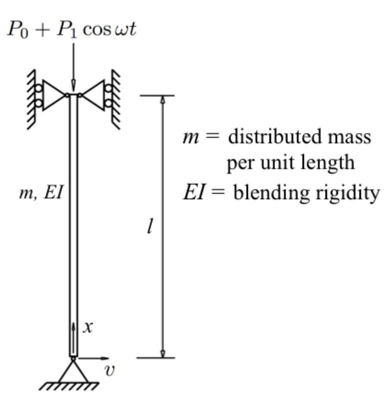



Fig. 9: Simply supported beam.

we define the state vector

$$x(t) = \left[ \begin{array}{cccccc} f_1(t) & \cdots & f_r(t) & \dot{f}_1(t) & \cdots & \dot{f}_r(t) \end{array} \right]^{\mathrm{T}}. \tag{24}$$

Consider that a transverse force $u(t)$ is concentrated at a single interior point $x_{\rm a}$, a measurement is available at a single interior point $x_{\rm s}$, and disturbance acts at $x_{\rm d}$ yields

$$A(t) = \left[ \begin{array}{cc} 0 & I \\ M(t) & 0 \end{array} \right],$$

$$B = \left[ \begin{array}{cccccc} 0 & \cdots & 0 & \phi_1(x_{\rm a}) & \cdots & \phi_r(x_{\rm a}) \end{array} \right]^{\mathrm{T}},$$
$$C = \left[ \begin{array}{cccccc} \phi_1(x_{\rm s}) & \cdots & \phi_r(x_{\rm s}) & 0 & \cdots & 0 \end{array} \right],$$
$$D_1 = \left[ \begin{array}{cccccc} 0 & \cdots & 0 & \phi_1(x_{\rm d}) & \cdots & \phi_r(x_{\rm d}) \end{array} \right]^{\mathrm{T}},$$

where

$$M(t) \triangleq \left[ \begin{array}{ccc} \Omega_1^2(1 - 2\mu_1 \cos \omega t) & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Omega_r^2(1 - 2\mu_r \cos \omega t) \end{array} \right]. \tag{25}$$

Assuming full-state measurement is not available, we implement an observer-based compensator, [22]. We consider two scenarios. The first addresses stabilization of (22). In the second, we include a white noise disturbance with standard deviation 2 N, which acts at the interior point $x_{\rm d}$, which differs from $x_{\rm a}$ and $x_{\rm s}$

We consider an aluminum beam, with a length of 1 m, square cross-section of width 5 mm. Let $x_{\rm a} = L/4$, $x_{\rm s} = L/3$, $x_{\rm d} = 3L/4$, $P_0 = 30$ N, $P_1 = 10$ N, $\omega = 10$ rad/sec, $R_1 = 1$, and $R_2$ ranging between 0.001 and 1. In this example we apply FTRE and FPRE. The controller is enabled at time 0.5 sec. However, simulation results show that FTRE fails to stabilize the system for the defined range of $R_2$ and specified excitation parameters. Output response and the control action time histories of the FPRE for two simulation scenarios are shown in Fig.10. Fig. 11 shows the Pareto performance tradeoff curve for a range of $R_2$.

## VII. CONCLUSIONS

We applied two heuristic techniques to LTV systems, namely, FTRE (frozen-time Riccati equation) and FPRE (forward-propagating Riccati equation). FTRE is inspired by SDRE, a heuristic technique for nonlinear systems, whereas FPRE is a heuristic technique for LTV systems. The goal was to compare these techniques on representative benchmark problems, all of which are periodically time-varying linear systems requiring stabilization. The chosen systems included second-order, 4th-order, and 10th-order dynamics, and were exponentially unstable, polynomially unstable, and exponentially unstable, respectively. The 10th-order system, in addition, was formulated as an output feedback problem requiring state estimation. These examples provide a representative benchmark collection for future studies of LTV system stabilization. For all three methods, both FTRE and FPRE worked reliably, although FPRE performed poorly on the second-order system with low frequency time-variation, as predicted by earlier studies on that approach. For the 4th-order system,

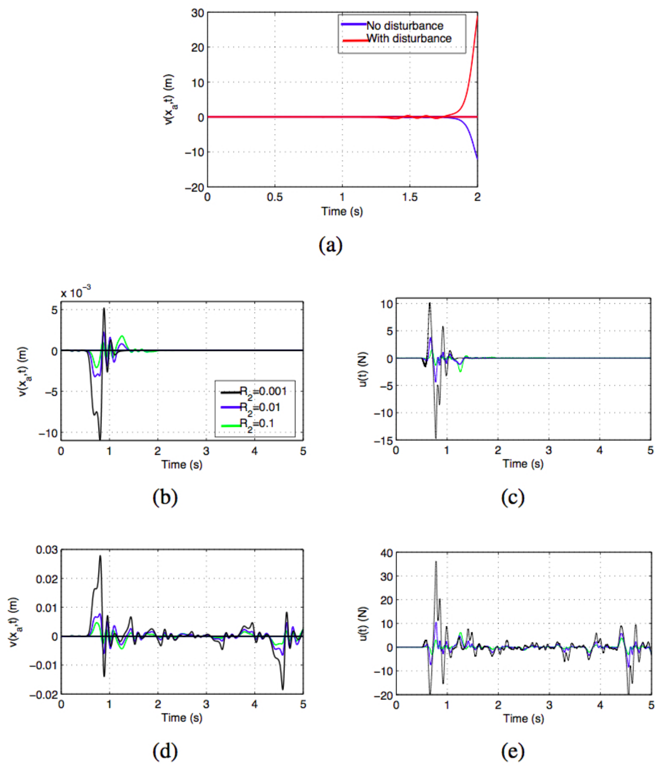



Fig. 10: Axially loaded simply supported beam. (a) shows open-loop responses of the beam with and without the disturbance. (b), (c) show the deflection of the beam and the control input, respectively, at the interior point $x_{\mathrm{a}}$. (d), (e) illustrate the same responses with a disturbance signal present.

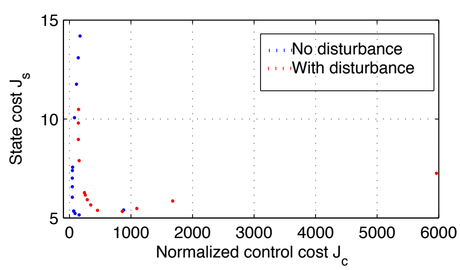



Fig. 11: Axially loaded simply supported beam. This figure shows the Pareto performance tradeoff curves for the FPRE simulations with and without a disturbance for $R_2$ ranging from 0.001 to 1.

FPRE outperformed FTRE. Both techniques are attractive due to their ease of implementation. However, much remains to be done to ascertain their reliability and suboptimality, not to mention their robustness to disturbances, sensor noise and model uncertainty. The present paper is a starting point in that regard.

## ACKNOWLEDGMENT

We thank P. Friedmann for suggesting the beam example.

This work is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) within 2215-Ph.D. Fellowship Program

## REFERENCES

[1] H. D'Angelo, *Linear Time-Varying Systems: Analysis and Synthesis*, Allyn and Bacon, 1970.
[2] K. S. Tsakalis and P. A. Ioannou, *Linear Time-Varying Systems: Control and Adaptation*, Prentice-Hall, 1992.
[3] P. DeWilde and A.-J. van der Veen, *Time-Varying Systems and Computation*, Springer, 1998.
[4] H. Bourles and B. Marinescu, *Linear Time-Varying Systems: Algebraic-Analytic Approach*, Springer, 2011.
[5] P. P. Friedmann, "Numerical Methods for the Treatment of Periodic Systems with Applications to Structural Dynamics and Helicopter Rotor Dynamics", *Computers and Structures*, vol. 35, no. 4, pp. 329-347, 1990.
[6] S. Shin, C.E.S. Cesnik, S.R. Hall, "Design and Simulation of Integral Twist Control for Helicopter Vibration Reduction", *International Journal of Control, Automation, and Systems*, vol. 5, no. 1, pp. 24-34, February 2007.
[7] V. A. Yakubovich and V. M. Starzhinskii, *Linear Differential Equations with Periodic Coefficients*, vol. 1, Wiley, 1975.
[8] J. A. Richards, *Analysis of Periodically Time-Varying Systems*, Springer, 1983.
[9] J. Dugundji and J. H. Wendell, "Some analysis methods for rotating systems with periodic coefficients", *AIAA Journal*, vol. 21, pp. 890-897, June 1983.
[10] N. M. Wereley, "Frequency response of linear time periodic systems", *Proc. 29th IEEE Conference on Decision and Control*, 1990.
[11] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems*, New York N.Y.: J. Wiley and Sons, 1972.
[12] M. A. Shayman, "On the Phase Portrait of the Matrix Riccati Equation Arising from the Periodic Control Problem", *SIAM Journal on Control and Optimization*, 23 (5), pp. 717-751, 1985.
[13] S. Bittanti and P. Colaneri, *Periodic Systems: Filtering and Control*, Springer, 2008.
[14] G. Tadmor, "Receding Horizon Revisited: An Easy Way to Robustly Stabilize an LTV System", *Systems and Control Letters*, vol. 18, Issue 4, pp. 285-294, April 1992.
[15] A. Loria, G. Chaillet, G. Besancon, and Y. Chitour, "On the Stabilization of Time-Varying Systems: Open Questions and Preliminary Answers", *IEEE Conference on Decision and Control*, pp. 6847-6852, December 2005.
[16] W. H. Kwon and S. H. Han, *Receding Horizon Control: Model Predictive Control for State Models*, Springer, 2005.
[17] A. Wernli and G. Cook, "Suboptimal control for the nonlinear quadratic regulator problem", *Automatica*, 11, pp. 75-84, 1975.
[18] C. P. Mracek and J. R. Cloutier, "Control designs for the nonlinear benchmark problem via the state-dependent Riccati equation method", *International Journal of Robust and Nonlinear Control*, 8, pp. 401-433, 1998.
[19] J. R. Cloutier and D. T. Stansbery, "The Capabilities and Art of State-Dependent Riccati Equation-Based Design", *Proc. Amer. Conf. Contr.*, pp. 86-91, Anchorage, AK, May 2002.
[20] T. Cimen, "State-Dependent Riccati Equation (SDRE) Control: A Survey", *Proc.17th IFAC World Congress*, pp. 3761-3771, Seoul, South Korea, 2008.
[21] M. -S. Chen and C.-Y. Kao, "Control of Linear Time-varying Systems Using Forward Riccati Equation", *Journal of Dynamic Systems, Measurement and Control*, vol. 119, no. 3, pp. 536-540, 1997.
[22] A. Weiss, I. Kolmanovsky, and D. S. Bernstein, "Forward-Integration Riccati-Based Output-Feedback Control of Linear Time-Varying Systems", *Proc. Amer. Conf. Contr.*, pp. 6708-6714, Montreal, Canada, June 2012.
[23] P. P. Friedmann and C.E. Hammond, "Efficient Numerical Treatment of Periodic Systems with Application to Stability Problems", *International Journal for Numerical Methods in Engineering*, vol. 11, pp. 1117-1136, 1977.
[24] E. B. Erdem and A.G. Alleyne, "Design of a class of nonlinear controllers via state dependent Riccati equations", *IEEE Transactions on Control Systems Technology*, 12, pp. 2986-2991, 2004.
[25] P. P. Friedmann, *Course Notes for AE543*, University of Michigan, 2013.