Extended Matrix Pencils for the Delta-Operator Riccati Equation

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1 Introduction

Modern optimal control techniques such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control rely on the solution of algebraic Riccati equations for controller synthesis. Reliable numerical techniques for numerical computation of the solution of these equations have been proposed using eigenvector or Schur decompositions of Hamiltonian matrices for continuous-time algebraic Riccati equations (CARE), or symplectic matrices for discrete-time (Z-transform) algebraic Riccati equations (DARE) [12, 6]. An improved solution method for the DARE was proposed in [9] in terms of a generalized eigenvalue problem which is equivalent to the symplectic decomposition and which does not require that the state dynamics matrix be invertible. This approach provides a computationally efficient algorithm for singular and ill-conditioned problems.

The generalized eigenvalue problem technique was further improved in [2, 1] where extended Hamiltonian and symplectic pencils were proposed for the CARE and DARE, respectively. This result extends the advantages of the generalized eigenvalue technique one step further in the sense that it eliminates the need to invert any of the matrices involved in the equation, making it ideal for numerically ill-conditioned systems.

The present paper is concerned with delta operator (or difference operator) descriptions of discrete-time systems. The difference operator formulation has been shown [8, 3] to provide a framework that is less numerically sensitive than the equivalent shift operator framework in analyzing discrete-time problems with high sampling frequencies relative to the bandwidth of the underlying system. In particular, computed solutions of difference operator Riccati equations have been shown to be more accurate than the corresponding computed solution of the shift operator Riccati equation [8, p. 511–515].

The purpose of the present work is to derive an extended matrix pencil generalized eigenvalue problem for the difference operator Riccati equation. This eigenvalue problem thus exploits the superior numerical properties of both the difference operator formulation and the generalized eigenvalue technique.

The paper is organized as follows. Section 2 introduces the concept of ζ-Hamiltonian matrices for differential, difference, and shift operator systems and then reviews the properties of these matrices. Section 3 reviews differential, difference, and shift operator versions of the algebraic Riccati equation and provides a ζ-Hamiltonian matrix for each version. Section 4 discusses the generalized eigenvalue problems associated with the standard eigenvalue problems of ζ-Hamiltonian matrices. Section 5 then reviews the extended matrix pencil generalized eigenvalue problem and presents extended matrix pencils associated with the algebraic Riccati equations for all three operators.

2 ζ-Hamiltonian Matrices

We require the following notation and definitions. We denote the set of eigenvalues of a square matrix A by spec(A) (including multiplicities), and we denote differential-, shift-, or difference-operator versions of equations by $\zeta = s$, q, or δ, respectively. The corresponding stability regions for these operators are given by

$$\begin{aligned} \mathcal{S}(\zeta) &\triangleq \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta < 0\}, & \zeta = s, \quad (1)\\ &\triangleq \{\zeta \in \mathbb{C} : |\zeta| < 1\}, & \zeta = q, \quad (2)\\ &\triangleq \{\zeta \in \mathbb{C} : \frac{h}{2}|\zeta| + \mathrm{Re}\,\zeta < 0\}, & \zeta = \delta, \quad (3) \end{aligned}$$

where $h > 0$ represents the sampling period for δ-domain systems. The associated stability boundary is denoted by $\partial\mathcal{S}(\zeta)$.

For the following definition, we define the matrix

$$J \triangleq \begin{bmatrix} 0_{n\times n} & -I_n \\ I_n & 0_{n\times n} \end{bmatrix}, \quad (4)$$

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and note that $J^{\mathrm{T}} = J^{-1} = -J$.

Definition 1 A matrix $H \in \mathbb{R}^{2n\times 2n}$ is ζ-Hamiltonian if

$$
\begin{aligned}
-JH^{\mathrm{T}}J &= -H, & \zeta = s, \quad (5)\\
H^{\mathrm{T}}JH &= J, & \zeta = q, \quad (6)\\
-JH^{\mathrm{T}}J(I+hH) &= -H, & \zeta = \delta. \quad (7)
\end{aligned}
$$

Definition 1 implies that if H is ζ-Hamiltonian then

$$
\begin{aligned}
\lambda \neq 0 \text{ for all } \lambda \in \operatorname{spec}(H), &\qquad \zeta = q, \quad (8)\\
\lambda \neq -1/h \text{ for all } \lambda \in \operatorname{spec}(H), &\qquad \zeta = \delta. \quad (9)
\end{aligned}
$$

The following results summarize the properties of ζ-Hamiltonian matrices.

Proposition 1 Let $H \in \mathbb{R}^{2n\times 2n}$ be ζ-Hamiltonian. Then $\lambda \in \operatorname{spec}(H)$ if and only if

$$-\lambda^* \in \operatorname{spec}(H), \qquad \zeta = s, \qquad (10)$$

$$1/\lambda^* \in \operatorname{spec}(H), \qquad \zeta = q, \qquad (11)$$

$$\frac{-\lambda^*}{1+h\lambda^*} \in \operatorname{spec}(H), \qquad \zeta = \delta. \qquad (12)$$

Proof: See [14, p. 327, 538], [8, p. 317]. $\square$

Figure 1 illustrates the symmetries described in Proposition 1 for the spectra of ζ-Hamiltonian matrices.

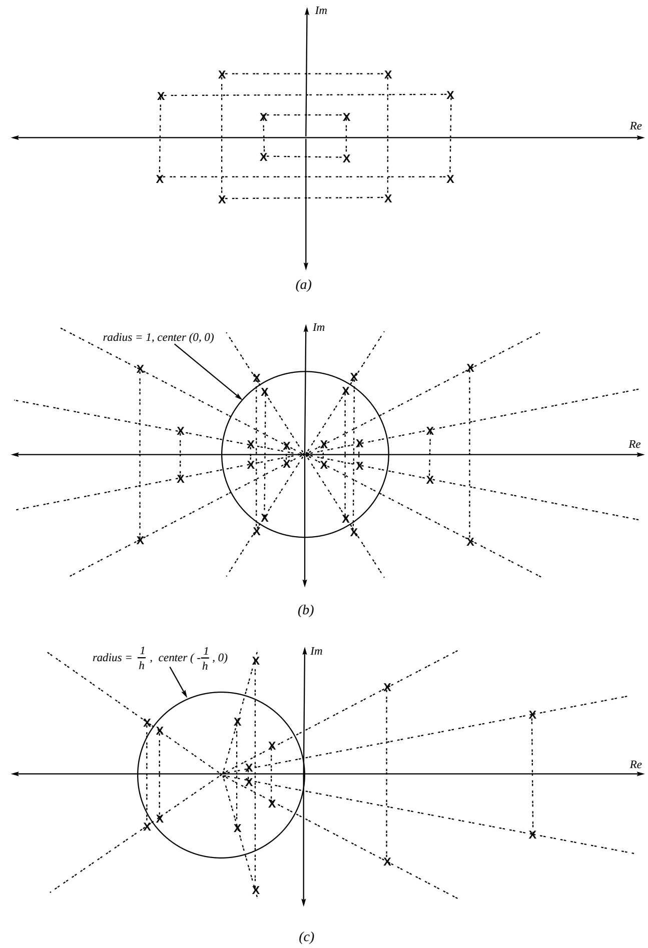

Figure 1: Reflection Patterns for Spectra of *(a)* s-Hamiltonian, *(b)* q-Hamiltonian, and *(c)* δ-Hamiltonian Matrices.

3 Algebraic Riccati Equations

Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{n\times m}$, $R = R^{\mathrm{T}}$ sign definite $\in \mathbb{R}^{m\times m}$, and $Q = Q^{\mathrm{T}} \in \mathbb{R}^{n\times n}$. A matrix $X = X^{\mathrm{T}} \geq 0 \in \mathbb{R}^{n\times n}$ is called a solution of the ζ-domain algebraic Riccati equation if it satisfies

$$
\begin{aligned}
0 \;=\;& A^{\mathrm{T}}X + XA - (B^{\mathrm{T}}X + S^{\mathrm{T}})^{\mathrm{T}}R^{-1}(B^{\mathrm{T}}X + S^{\mathrm{T}})\\
&+ Q, \qquad \zeta = s, \qquad (13)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& A^{\mathrm{T}}XA - X - (B^{\mathrm{T}}XA + S^{\mathrm{T}})^{\mathrm{T}}(R + B^{\mathrm{T}}XB)^{-1}\\
&\cdot (B^{\mathrm{T}}XA + S^{\mathrm{T}}) + Q, \qquad \zeta = q, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& Q + A^{\mathrm{T}}X + XA + hA^{\mathrm{T}}XA\\
&- \left(B^{\mathrm{T}}X\left(I + hA\right) + S^{\mathrm{T}}\right)^{\mathrm{T}}\left(R + hB^{\mathrm{T}}XB\right)^{-1}\\
&\cdot \left(B^{\mathrm{T}}X\left(I+hA\right) + S^{\mathrm{T}}\right), \qquad \zeta = \delta, \qquad (15)
\end{aligned}
$$

The following result defines the relationship between ζ-Hamiltonian matrices and the algebraic Riccati equations (13), (14), and (15). Define the matrices

$$
\begin{aligned}
E &\triangleq A - BR^{-1}S^{\mathrm{T}},\\
G &\triangleq BR^{-1}B^{\mathrm{T}}, \qquad (16)\\
V &\triangleq Q - SR^{-1}S^{\mathrm{T}}.
\end{aligned}
$$

Proposition 2 Assume that

$$
\begin{aligned}
\det(E) &\neq 0, & \zeta = q, \quad (17)\\
\det(I + hE) &\neq 0, & \zeta = \delta, \quad (18)
\end{aligned}
$$

hold. Then a matrix $X = X^{\mathrm{T}} \geq 0 \in \mathbb{R}^{n\times n}$ satisfies (13), (14), or (15) if and only if X satisfies

$$\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = 0, \qquad (19)$$

where $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ is defined by

$$H = \begin{bmatrix} E & -G \\ -V & -E^{\mathrm{T}} \end{bmatrix}, \qquad \zeta = s, \quad (20)$$

$$H = \begin{bmatrix} E + GE^{-\mathrm{T}}V & -GE^{-\mathrm{T}} \\ -E^{-\mathrm{T}}V & E^{-\mathrm{T}} \end{bmatrix}, \qquad \zeta = q, \tag{21}$$

$$H = \begin{bmatrix} E + hG(I+hE^{\mathrm{T}})^{-1}V & -G(I+hE^{\mathrm{T}})^{-1} \\ -(I+hE^{\mathrm{T}})^{-1}V & -(I+hE^{\mathrm{T}})^{-1}E^{\mathrm{T}} \end{bmatrix},$$
$$\zeta = \delta. \tag{22}$$

Proof: [14, p. 335, 541], [8, p. 317]. □

It can be verified that H as given by (20), (21), (22) is ζ-Hamiltonian.

Definition 2 A matrix $Y \in \mathbb{R}^{n\times n}$ is *ζ-stable* if $\mathrm{spec}(Y) \subset \mathbb{S}(\zeta)$.

Definition 3 A solution X of (13), (14), or (15) is *ζ-stabilizing* solution if the matrix $H_{11} + H_{12}X$ is ζ-stable.

Numerical techniques for computing stabilizing solutions of the algebraic Riccati equations (13), (14), and (15) are based on decomposing the associated ζ-Hamiltonian matrix (20), (21), or (22), respectively. These techniques require an invariant subspace corresponding to the ζ-stable eigenvalues of H. Two such sets of vectors are the eigenvectors of H [12] and the Schur vectors of H [6]. These techniques can be summarized as follows.

Proposition 3 Assume that $\mathrm{spec}(H) \cap \partial\mathbb{S}(\zeta) = \emptyset$. Let

$$\mathcal{H}^{-}(\zeta) \triangleq \mathrm{spec}(H) \cap \mathbb{S}(\zeta), \tag{23}$$

and let $\Lambda \in \mathbb{R}^{n\times n}$ be a matrix satisfying $\mathrm{spec}(\Lambda) = \mathcal{H}^{-}$. Let $\mathcal{X} \in \mathbb{R}^{2n\times n}$ be any nonzero matrix satisfying

$$H\mathcal{X} = \mathcal{X}\Lambda. \tag{24}$$

Let $\mathcal{X}$ be partitioned as $\mathcal{X} = \left[X_1^{\mathrm{T}} X_2^{\mathrm{T}}\right]^{\mathrm{T}}$, with $X_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$, and assume that that X_1 is nonsingular. Then $X = X_2X_1^{-1}$ is the unique, ζ-stabilizing solution of (19).

Proof: See [14, pp. 334, 539], [8, p. 319]. □

4 Matrix Pencils and Generalized Eigenvalue Problems

In this section, we consider use of a generalized eigenvalue problem to compute solutions to the algebraic Riccati equation. Generalized eigenvalue problems eliminate the need to assume the invertibility of either E (17) or $I + hE$ (18).

We require the following definitions for later developments

Definition 4 Let $A, B \in \mathbb{R}^{n\times n}$, and let $\lambda \in \mathbb{C}$. Then the linear matrix polynomial $\lambda B - A$ is called a *(linear) matrix pencil*, and is denoted by (A, B). If $\det(\lambda B - A) \not\equiv 0$, then (A, B) is said to be *regular*; otherwise, (A, B) is said to be *singular*.

We now extend the notions of eigenvalues and eigenvectors of a matrix to matrix pencils.

Definition 5 Let (A, B) be a regular matrix pencil, with $A, B \in \mathbb{R}^{n\times n}$. If there exist $\lambda \in \mathbb{C}$ and nonzero $x \in \mathbb{C}^n$ that satisfy

$$Ax = \lambda Bx, \tag{25}$$

then λ is a *generalized eigenvalue* of (A, B) and x is a *generalized eigenvector* of (A, B). The set of generalized eigenvalues of (A, B) (including multiplicities) is denoted by $\mathrm{spec}(A, B)$.

Remark 1 Note that in general $\mathrm{spec}(A, B) \neq \mathrm{spec}(B, A)$. If $\mathrm{rank}(B) = n$, then $\mathrm{spec}(A, B)$ has n elements; otherwise, $\mathrm{spec}(A, B)$ may be empty, finite, or infinite [4].

Definition 6 Let (A, B) be a regular matrix pencil. The subspace $\mathcal{X}$ is a *deflating subspace* of (A, B) if

$$\dim(B\mathcal{X} + A\mathcal{X}) \leq \dim(\mathcal{X}). \tag{26}$$

where "dim" denotes dimension.

Remark 2 For $B = I$, the definition of a deflating subspace specializes to that of an invariant subspace of the matrix A.

A numerically stable method for computing the eigenvalues and deflating subspaces of matrix pencils is given by the *QZ-algorithm*, which is based on the *generalized Schur decomposition.*

Theorem 1 ([4], pp. 253) Let $A, B \in \mathbb{R}^{2n\times 2n}$. Then there exist unitary matrices $Q, Z \in \mathbb{C}^{2n\times 2n}$ such that

$$Q^*AZ = T, \quad Q^*BZ = S, \tag{27}$$

where $T, S \in \mathbb{C}^{n\times n}$ are both upper triangular. The matrix pencil (A, B) is singular if and only if there exists $i \in \{1, 2, \ldots, n\}$ such that $s_{ii} = t_{ii} = 0$. In this case, $\mathrm{spec}(A, B) = \mathbb{C}$. Otherwise,

$$\mathrm{spec}(A, B) = \{\lambda : \lambda = t_{ii}/s_{ii}, s_{ii} \neq 0\}. \tag{28}$$

Furthermore, the columns of Z span a deflating subspace for (A, B).

Generalized eigenvalue problems corresponding to (13), (14), or (15) are denoted by the matrix pencil (M, N), where $M, N \in \mathbb{R}^{2n\times 2n}$ are defined by

$$M = \begin{bmatrix} E & -G \\ -V & -E^{\mathrm{T}} \end{bmatrix}, N = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \zeta = s, \qquad (29)$$

$$M = \begin{bmatrix} E & 0 \\ -V & I \end{bmatrix}, \quad N = \begin{bmatrix} I & G \\ 0 & E^{\mathrm{T}} \end{bmatrix}, \; \zeta = q, \qquad (30)$$

$$M = \begin{bmatrix} E & -G \\ -V & -E^{\mathrm{T}} \end{bmatrix}, N = \begin{bmatrix} I & hG \\ 0 & I + hE^{\mathrm{T}} \end{bmatrix}, \quad \zeta = \delta, \qquad (31)$$

with E, G, and V as defined in (16). Note that in the case of nonsingular E or nonsingular $I + hE$, the ζ-Hamiltonian matrix H defined in Section 3 are recovered as $H = N^{-1}M$.

The algorithm for computing the stabilizing solution of the algebraic Riccati equations (13), (14), or (15) from the associated generalized eigenvalue problems can now be summarized as follows.

Proposition 4 Let (M, N) be given by (29), (30), or (31), and assume that $\mathrm{spec}(M, N) \cap \partial\mathcal{S}(\zeta) = \emptyset$. Define

$$\mathcal{H}^-(\zeta) \triangleq \mathrm{spec}(M, N) \cap \mathcal{S}(\zeta), \qquad (32)$$
$$\mathcal{H}^+(\zeta) \triangleq \mathrm{spec}(M, N) \setminus \mathcal{H}^-(\zeta). \qquad (33)$$

Then there exist orthogonal matrices $Q, Z \in \mathbb{R}^{2n\times 2n}$ such that

$$Q^*MZ = Q^*NZ \begin{bmatrix} \Lambda^- & 0 \\ 0 & \Lambda^+ \end{bmatrix}, \qquad (34)$$

where $\Lambda^- \in \mathbb{R}^{n\times n}$ satisfies $\mathrm{spec}(\Lambda^-) = \mathcal{H}^-$ and $\Lambda^+ \in \mathbb{R}^{n\times n}$ satisfies $\mathrm{spec}(\Lambda^+) = \mathcal{H}^+$. Partition Z as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}. \qquad (35)$$

where $Z_{ij} \in \mathbb{R}^{n\times n}$, $i, j = 1, 2$. If Z_{11} is nonsingular, then $X = Z_{21}Z_{11}^{-1}$ is the unique stabilizing solution to the corresponding algebraic Riccati equation.

Proof: The case $\zeta = s$ is simply a restatement of Propostion 3. The proof for the case of $\zeta = q$ is found in [9]. The proof for $\zeta = \delta$ is analogous.

5 Extended Matrix Pencils

While the generalized eigenvalue problems introduced in Section 4 eliminate the need for assuming the invertibility of either E (17) or $I + hE$ (18), formation of the matrix pencils (29), (30), and (31) still require inversion of the matrix R. In [2], an algorithm based on the concept of an *extended matrix pencil* was introduced which, while still requiring that R be nonsingular, does not explicitly require R^{-1}.

In the present context, an extended matrix pencil is a matrix pencil of dimension $(2n + m) \times (2n + m)$. An orthogonal transformation is selected that reduces the extended pencil to lower block-triangular form. This transformation also has the property that the first $2n\times 2n$ block of the transformed extended pencil has the same spectrum as the corresponding generalized eigenvalue problem (29), (30), or (31). The QZ-algorithm can then be used as discussed in Section 4 to obtain a basis for the deflating subspace of this block which corresponds to its ζ-stable generalized eigenvalues.

For each of the generalized eigenvalue/eigenvector problems described in Section 4, an associated extended matrix pencil can be defined by $(\bar{M}, \bar{N})$, where

$$\begin{aligned} \bar{M} &= \begin{bmatrix} A & 0 & B \\ -Q & -A^{\mathrm{T}} & -S \\ S^{\mathrm{T}} & B^{\mathrm{T}} & R \end{bmatrix}, \\ \bar{N} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \end{aligned} \quad \zeta = s, \qquad (36)$$

$$\begin{aligned} \bar{M} &= \begin{bmatrix} A & 0 & B \\ -Q & I & -S \\ S^{\mathrm{T}} & 0 & R \end{bmatrix}, \\ \bar{N} &= \begin{bmatrix} I & 0 & 0 \\ 0 & A^{\mathrm{T}} & 0 \\ 0 & -B^{\mathrm{T}} & 0 \end{bmatrix}, \end{aligned} \quad \zeta = q, \qquad (37)$$

$$\begin{aligned} \bar{M} &= \begin{bmatrix} A & 0 & B \\ -Q & -A^{\mathrm{T}} & -S \\ S^{\mathrm{T}} & B^{\mathrm{T}} & R \end{bmatrix}, \\ \bar{N} &= \begin{bmatrix} I & 0 & 0 \\ 0 & I + hA^{\mathrm{T}} & 0 \\ 0 & -hB^{\mathrm{T}} & 0 \end{bmatrix}, \end{aligned} \quad \zeta = \delta. \qquad (38)$$

The algorithm for finding a basis for deflating subspaces of (29), (31), or (31) from (36), (37), or (38), respectively, may now be summarized as follows.

Proposition 5 Let $\bar{M}, \bar{N}$ be as defined in (36), (37), or (38), and partitioned as

$$\bar{M} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}, \qquad \bar{N} = \begin{bmatrix} \bar{N}_{11} & \bar{N}_{12} \\ \bar{N}_{21} & \bar{N}_{22} \end{bmatrix}. \qquad (39)$$

where $\bar{M}_{22}, \bar{N}_{2,2} \in \mathbb{R}^{m\times m}$. Let $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ be an orthogonal matrix that satisfies

$$\begin{bmatrix} U_{11} & U_{12} \end{bmatrix} \begin{bmatrix} \bar{M}_{12} \\ \bar{M}_{22} \end{bmatrix} = 0. \qquad (40)$$

Define $\hat{N} \triangleq U\bar{N}$, $\hat{M} \triangleq U\bar{M}$, and let $\hat{N}, \hat{M}$ be partitioned as

$$\hat{N} = \begin{bmatrix} \hat{N}_{11} & 0 \\ \hat{N}_{21} & \hat{N}_{22} \end{bmatrix}, \qquad \hat{M} = \begin{bmatrix} \hat{M}_{11} & 0 \\ \hat{M}_{21} & \hat{M}_{22} \end{bmatrix}. \qquad (41)$$

Then the deflating subspaces of the matrix pencils $(\hat{M}_{11}, \hat{N}_{11})$ and (M, N) as defined by (29), (30), or (31) are identical.

Proof: See [2] for the proofs of $\zeta = s$ and $\zeta = q$. The proof for $\zeta = \delta$ is analogous. $\square$

From this point, the algorithm of Section 4 can be applied to compute a basis for the deflating subspace of $(\hat{M}_{11}, \hat{N}_{11})$ corresponding to its ζ-stable generalized eigenvalues. This can then be used to compute the stabilizing solution of the Riccati equation (13), (14), or (15) as shown in Proposition 4.

6 Discussion and Conclusions

This paper has introduced an extended matrix pencil problem associated with the solution of the delta-operator (difference operator) algebraic Riccati equation. The result is integrated into a systematic treatment of solution algorithms for differential, shift, and difference operator algebraic Riccati equations. These algorithms included the standard eigenvalue problem for ζ-Hamiltonian matrices, a generalized eigenvalue problems for an associated matrix pencil, and the extended matrix pencil generalized eigenvalue problem. The relevant numerical solution techniques for obtaining the stabilizing solution of the corresponding algebraic Riccati equation from each of these problems were presented, along with a discussion of their relative advantages and disadvantages.

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