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NONPARAMETRIC IDENTIFICATION OF HAMMERSTEIN SYSTEMS USING ORTHOGONAL BASIS FUNCTIONS AS ERSATZ NONLINEARITIES

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ABSTRACT

In this paper, we present a technique for estimating the input nonlinearity of a Hammerstein system by using multiple orthogonal ersatz nonlinearities. Theoretical analysis shows that by replacing the unknown input nonlinearity by an ersatz nonlinearity, the estimates of the Markov parameters of the plant are correct up to a scalar factor, which is related to the inner product of the true input nonlinearity and the ersatz nonlinearity. These coefficients are used to construct and estimate the true nonlinearity represented as an orthogonal basis expansion. We demonstrate this technique by using a Fourier series expansion as well as orthogonal polynomials. We show that the kernel of the inner product associated with the orthogonal basis functions must be chosen to be the density function of the input signal.

1 INTRODUCTION

Hammerstein systems represent a practically meaningful class of nonlinear systems that directly generalize linear systems. In a Hammerstein plant, the input to a linear model is distorted by an input nonlinearity. The input nonlinearity may be one-to-one and onto, as in the case of a cubic nonlinearity; it may be one-to-one but not onto, as in the case of the arctangent nonlinearity; it may be onto but not one-to-one, as in the case of the polynomial $f(u) = u^3 + u^2$; and it may be neither one-to-one nor onto, as in the case of an on-off or saturation nonlinearity. A bounded input nonlinearity is not onto, an even input nonlinearity is not one-to-one, and a nonlinearity that is strictly increasing or strictly decreasing is one-to-one. An odd nonlinearity may or may not be one-to-one, but must be zero at zero.

Since all real control inputs are subject to saturation, all realistic input nonlinearities are bounded [1]. Consequently, the literature on control of linear systems with saturation represents a practical special case of control of Hammerstein systems [2].

Aside from control, identification of Hammerstein systems represents a specialized nonlinear system identification problem. The literature on identification of Hammerstein systems is extensive, and this problem has been considered under various assumptions [3–6]. For example, the input nonlinearity may be unknown, and the objective is to identify both the linear system dynamics and the input nonlinearity. In other cases, partial information about the input nonlinearity may be available; for example, the input nonlinearity may be assumed to be odd or its value at zero may be assumed to be known or zero.

In the present paper we revisit the approach to Hammerstein system identification given in [7], where both the input nonlinearity and linear dynamics are assumed to be unknown. Least squares techniques are used in [7] by preprocessing the input through an *ersatz* nonlinearity, which takes the place of the true nonlinearity. Although the ersatz nonlinearity may differ from the true nonlinearity, it is shown in [7] that, in the presence of output noise, the estimates of the Markov parameters of the linear dynamics are semiconsistent, that is, asymptotically unbiased up to an unknown scale factor. The unknown scale factor is to be expected due to the fact that the input nonlinearity and the plant can be scaled by an arbitrary constant and its reciprocal without affecting the input-output data. The eigensystem realization algorithm [8] can be used to construct an estimate of the linear dynamics up to an unknown scale factor. However, the approach of [7] does not provide an estimate of the input nonlinearity.

In the present paper we extend the results of [7] by developing a technique for identifying Hammerstein systems that uses ersatz nonlinearities to estimate both the input nonlinearity and

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the linear dynamics. The novel aspect of this method is the use of multiple ersatz nonlinearities to estimate the coefficients of an expansion of the true nonlinearity. More specifically, for a given data set, we apply the technique of [7] with a collection of orthogonal ersatz nonlinearities. Theoretical analysis shows that the unknown scale factor in the estimates of the Markov parameters of the plant are related to the inner product of the true input nonlinearity and the ersatz nonlinearity. These coefficients can then be used to construct and estimate the true nonlinearity represented as an orthogonal basis expansion. We show that the kernel of the inner product associated with the orthogonal basis functions must be chosen to be the density function of the input signal.

In the present paper, we briefly review the approach of [7] for identifying Hammerstein systems using ersatz nonlinearities. We then present the technique for estimating the input nonlinearity by using multiple orthogonal ersatz nonlinearities, and we demonstrate this technique by using a Fourier series expansion as well as orthogonal polynomials. We demonstrate the effect of both input and output noise in an errors-in-variables setting. We consider examples in which the input nonlinearity is not necessarily either one-to-one, onto, odd, even, monotonic, or zero at zero.

2 Problem Formulation

Consider the Hammerstein structure shown in Figure 1, where u_0 is the input signal, $\mathcal{N}:\mathbb{R}\to\mathbb{R}$ is the static nonlinearity, $\mathcal{N}(u_0)$ is the intermediate signal, y_0 is the output signal, and G is the asymptotically stable, SISO, causal, discrete-time system

$$A(\mathbf{q})y_0(k) = B(\mathbf{q})\mathcal{N}\Big(u_0(k)\Big), \tag{1}$$

where $\mathbf{q}$ is the forward shift operator, and A and B are polynomials in $\mathbf{q}$. We assume that the nonlinearity $\mathcal{N}$, and hence the intermediate signal $\mathcal{N}(u_0)$, is unknown. Furthermore, we assume that the measurement of u_0 is corrupted with additive white noise, and the measurement of y_0 is corrupted with additive white or colored noise.

The ARX model of (1) is given by

$$y_0(k) = \sum_{j=0}^{n} b_j\mathcal{N}\Big(u_0(k-j)\Big) - \sum_{j=1}^{n} a_j y_0(k-j). \tag{2}$$

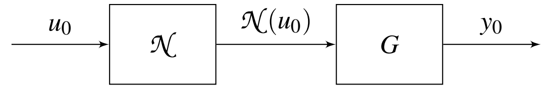

Figure 1. Block-structured Hammerstein model where u_0 is the input, $\mathcal{N}\Big(u_0(k)\Big)$ is the intermediate signal, and y_0 is the output.

3 μ-Markov Model and Least Squares Estimates

For all $k \geq 0$, $n_{\rm mod} \geq n$, $\mu \geq 1$, the signals $y_0(k)$ and $\mathcal{N}(u_0(k))$ satisfy the μ-Markov model

$$\begin{aligned} y_0(k) = &\sum_{j=0}^{\mu-1} H_j\mathcal{N}(u_0(k-j)) + \sum_{j=\mu}^{n_{\rm mod}+\mu-1} b'_j\mathcal{N}(u_0(k-j)) \\ &- \sum_{j=\mu}^{n_{\rm mod}+\mu-1} a'_j y_0(k-j), \end{aligned} \tag{3}$$

where $H_0,\ldots,H_{\mu-1}$ are the first μ Markov parameters of G. Note that the μ-Markov model is an overparameterization of the ARX model (2), where μ Markov parameters are explicitly displayed [9]. Furthermore, setting $n_{\rm mod} = n$ and $\mu = 1$ gives the ARX model (2).

Next, let $n_{\rm mod} \geq n$ and $\mu \geq 1$. Then the μ-Markov model (3) of (1) can be expressed as

$$y_0(k) = \theta_\mu\mathcal{N}(\phi_{\mu_0})(k) + \theta_u\mathcal{N}(\phi_{u_0})(k) - \theta_y\phi_{y_0}(k), \tag{4}$$

where

$$\begin{aligned} \theta_\mu &\triangleq \big[\, H_0 \;\cdots\; H_{\mu-1} \,\big], \\ \theta_u &\triangleq \big[\, b_\mu \;\cdots\; b_{n_{\rm mod}+\mu-1} \,\big], \\ \theta_y &\triangleq \big[\, a_\mu \;\cdots\; a_{n_{\rm mod}+\mu-1} \,\big], \\ \mathcal{N}(\phi_{\mu_0})(k) &\triangleq \big[\, \mathcal{N}(u_0(k)) \;\cdots\; \mathcal{N}(u_0(k-\mu+1)) \,\big]^{\rm T}, \\ \mathcal{N}(\phi_{u_0})(k) &\triangleq \big[\mathcal{N}(u_0(k-\mu)) \;\cdots\; \mathcal{N}(u_0(k-n_{\rm mod}-\mu+1))\big]^{\rm T}, \\ \phi_{y_0}(k) &\triangleq \big[\, y_0(k-\mu) \;\cdots\; y_0(k-n_{\rm mod}-\mu+1) \,\big]^{\rm T}. \end{aligned}$$

Furthermore, the least squares estimate $\hat{\theta}^{\mathcal{N}}_{\mu,\ell}$, $\hat{\theta}^{\mathcal{N}}_{u,\ell}$, $\hat{\theta}^{\mathcal{N}}_{y,\ell}$ of θ_μ, θ_u, θ_y are given by

$$\begin{aligned} &\Big[\, \hat{\theta}^{\mathcal{N}}_{\mu,\ell} \; \hat{\theta}^{\mathcal{N}}_{u,\ell} \; \hat{\theta}^{\mathcal{N}}_{y,\ell} \,\Big] \\ &\quad= \underset{[\,\theta_\mu\;\theta_u\;\theta_y\,]}{\arg\min} \Big\|\Psi_{y_0,\ell} - \theta_\mu\mathcal{N}(\Phi_{\mu_0,\ell}) - \theta_u\mathcal{N}(\Phi_{u_0,\ell}) + \theta_y\Phi_{y_0,\ell}\Big\|_{\rm F}, \end{aligned} \tag{5}$$

where $||\ .\ ||_{\mathrm{F}}$ denotes the Frobenius norm,

$$\Psi_{y_0,\ell} \triangleq \left[\, y_0(n_{\mathrm{mod}}+\mu-1) \cdots y_0(\ell) \,\right],$$
$$\Phi_{y_0,\ell} \triangleq \left[\, \phi_{y_0}(n_{\mathrm{mod}}+\mu-1) \cdots \phi_{y_0}(\ell) \,\right],$$
$$\mathcal{N}(\Phi_{\mu_0,\ell}) \triangleq \left[\, \mathcal{N}(\phi_{\mu_0})(n_{\mathrm{mod}}+\mu-1) \cdots \mathcal{N}(\phi_{\mu_0})(\ell) \,\right],$$
$$\mathcal{N}(\Phi_{u_0,\ell}) \triangleq \left[\, \mathcal{N}(\phi_{u_0})(n_{\mathrm{mod}}+\mu-1) \cdots \mathcal{N}(\phi_{u_0})(\ell) \,\right],$$

and ℓ is the number of samples. In this case, if $\mathcal{N}(u_0)$ is persistently exciting, then $\lim_{\ell\to\infty} \hat{\theta}^{\mathcal{N}}_{\mu,\ell} \overset{\mathrm{wp1}}{=} \theta_\mu$.

4 Ersatz Nonlinearity

Since $\mathcal{N}$ is assumed to be unknown, we cannot construct $\mathcal{N}(\Phi_{\mu_0,\ell})$ and $\mathcal{N}(\Phi_{u_0,\ell})$. Hence, it is not possible to solve the least squares problem (5) for the coefficients of the μ-Markov model. We thus consider a least squares problem of the form (5) in which we replace the unknown nonlinearity $\mathcal{N}$ by a nonlinearity $\mathcal{E}:\mathbb{R}\to\mathbb{R}$. The identification architecture is shown in Figure 2. The ersatz nonlinearity $\mathcal{E}$ is not intended to be an approximation of $\mathcal{N}$. Rather, $\mathcal{E}$ serves as a substitute for $\mathcal{N}$ that has the ability to render the solution of the least squares problem useful for estimating the coefficients of the μ-Markov model.

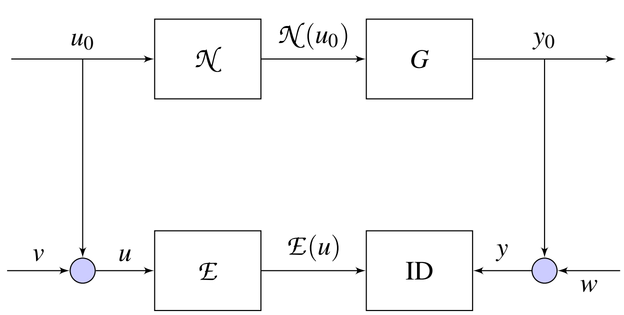

Figure 2. Identification of a Hammerstein system using the ersatz nonlinearity $\mathcal{E}$.

5 CONSISTENCY ANALYSIS

The consistency analysis for this problem is shown in [7]. For convenience we list the assumptions and the main result here.

We use the following assumptions.

Assumption 5.1. u_0 and v are realizations of the stationary white processes U_0 and V.

Assumption 5.2. w is a realization of the stationary white or colored process W.

Assumption 5.3. U_0, W, and V are mutually independent.

Assumption 5.4. $U(k) = U_0(k) + V(k)$.

Assumption 5.5. $Y(k) = Y_0(k) + W(k)$.

Assumption 5.6. $\theta_\mu \neq 0_{1\times\mu}$.

Assumption 5.7. For all $k \geq 0$, $\mathbb{E}\Big[\mathcal{E}\Big(U(k)\Big)\Big] = 0$ and $\mathbb{E}\Big[\mathcal{N}\Big(U_0(k)\Big)\mathcal{E}\Big(U(k)\Big)\Big] \neq 0$.

Assumption 5.8. For all $k \geq 0$ and $p \geq -k$, $\mathcal{N}\big(U(k)\big)$, $\mathcal{E}\big(U(k)\big)$, $\mathcal{N}\big(U(k)\big)\mathcal{E}\big(U(k+p)\big)$, $\mathcal{E}\big(U(k)\big)\mathcal{E}\big(U(k+p)\big)$, and $W(k)$ have finite mean and variance.

Under the above assumptions it is shown in [7] that by replacing $\mathcal{N}$ with the ersatz nonlinearity $\mathcal{E}$, the vector of estimated Markov parameters $\hat{\theta}^{\mathcal{E}}_{\mu,\ell}$ is a consistent estimator of θ_μ up to a scalar factor, that is,

$$\lim_{\ell\to\infty} \hat{\theta}^{\mathcal{E}}_{\mu,\ell} \overset{\mathrm{wp1}}{=} \frac{\mathbb{E}\left[\mathcal{N}\Big(U_0(k)\Big)\mathcal{E}\Big(U(k)\Big)\right]}{\mathbb{E}\left[\mathcal{E}\Big(U(k)\Big)\mathcal{E}\Big(U(k)\Big)\right]}\theta_\mu. \tag{6}$$

The result (6) was used in [7] to identify only the linear plant G. Here we use (6) to also identify the nonlinearity $\mathcal{N}$. Note that the coefficient of θ_μ in (6) is unknown since it depends on $\mathcal{N}$.

6 IDENTIFICATION OF THE HAMMERSTEIN SYSTEM

For the identification process of the Hammerstein system, only one experiment is needed, where all the work related to the ersatz nonlinearity is performed offline using computer simulations. We perform the identification in two separate steps, first identification of the linear plant G and second identification of the Hammerstein nonlinearity $\mathcal{N}$. For the rest of the paper we assume that $v(k) = 0$ for all $k \geq 0$.

6.1 Identification of the Linear Plant G

To identify the linear plant G we choose an ersatz nonlinearity $\mathcal{E}$ that satisfies Assumptions 5.7 and 5.8. The identified Markov parameters are asymptotically correct up to a scalar factor as shown in (6). Examples are given in [7].

6.2 Identification of the Hammerstein Nonlinearity $\mathcal{N}$

To identify the Hammerstein nonlinearity we represent $\mathcal{N}$ by a series expansion of orthogonal basis functions, that is,

$$\mathcal{N}(u) = \sum_{n=0}^{\infty} h_n f_n(u), \tag{7}$$

where the basis functions $f_n(\cdot)$, $n = 0,1,2,\ldots$ are orthogonal over an interval with respect to some inner product, that is $< f_i, f_j >= 0$, if $i \neq j$. In the following we consider two different orthogonal bases, namely, sines and cosines from the Fourier series and the Laguerre polynomials.

6.2.1 Fourier Series Expansion To identify the Hammerstein nonlinearity we represent $\mathcal{N}$ by its Fourier series expansion, that is,

$$\mathcal{N}(u) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi u}{L}) + \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi u}{L}). \quad (8)$$

Note that using a stationary input uniformly distributed in $[-L, L]$, that is,

$$P_{U(k)}(u) = \begin{cases} \frac{1}{2L}, & \text{if } |u| \le L, \\ 0, & \text{if } |u| > L, \end{cases} \quad (9)$$

we obtain

$$a_0 \triangleq \frac{1}{2L} \int_{-L}^{L} \mathcal{N}(u) du = \mathbb{E}\left[\mathcal{N}(U(k))\right], \quad (10)$$

and for $n = 1, 2, \dots$

$$\begin{aligned} a_n &\triangleq \frac{1}{L} \int_{-L}^{L} \cos(\frac{n\pi u}{L}) \mathcal{N}(u) du \\ &= 2 \int_{-L}^{L} \cos(\frac{n\pi u}{L}) \mathcal{N}(u) \frac{1}{2L} du \\ &= 2\mathbb{E}\left[\cos(\frac{n\pi U(k)}{L}) \mathcal{N}(U(k))\right], \end{aligned} \quad (11)$$

and

$$\begin{aligned} b_n &\triangleq \frac{1}{L} \int_{-L}^{L} \sin(\frac{n\pi u}{L}) \mathcal{N}(u) du \\ &= 2 \int_{-L}^{L} \sin(\frac{n\pi u}{L}) \mathcal{N}(u) \frac{1}{2L} du \\ &= 2\mathbb{E}\left[\sin(\frac{n\pi U(k)}{L}) \mathcal{N}(U(k))\right]. \end{aligned} \quad (12)$$

Next, we truncate (8) to obtain an approximation $\hat{\mathcal{N}}$ of $\mathcal{N}$, that is,

$$\hat{\mathcal{N}}(u) = a_0 + \sum_{n=1}^{N} a_n \cos(\frac{n\pi u}{L}) + \sum_{n=1}^{N} b_n \sin(\frac{n\pi u}{L}), \quad (13)$$

where N is chosen large enough such that $\hat{\mathcal{N}}(u)$ is an acceptable approximation of $\mathcal{N}(u)$. Since the coefficient in (6) is unknown, it follows that the coefficients in (13) are not implementable, however we show next that we can determine these coefficients up to a scalar factor.

Let $\mathcal{E}(u)$ and $\mathcal{Y}(u)$ be different ersatz nonlinearities such that $\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))] \neq 0$. Define

$$r(\mathcal{E}, \mathcal{Y}) \triangleq \frac{\lim_{\ell\to\infty} \hat{H}_{i,\ell}^{\mathcal{E}}}{\lim_{\ell\to\infty} \hat{H}_{i,\ell}^{\mathcal{Y}}}, \quad (14)$$

where, for $i \in \{0, 1, \dots, \mu - 1\}$, $H_i \neq 0$. It follows from (6) that

$$\lim_{\ell\to\infty} \hat{H}_{i,\ell}^{\mathcal{E}} \triangleq \frac{\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}(U(k))]}{\mathbb{E}[\mathcal{E}^2(U(k))]} H_i, \quad (15)$$

and

$$\lim_{\ell\to\infty} \hat{H}_{i,\ell}^{\mathcal{Y}} \triangleq \frac{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]}{\mathbb{E}[\mathcal{Y}^2(U(k))]} H_i. \quad (16)$$

Substituting (15) and (16) in (14) yields

$$r(\mathcal{E}, \mathcal{Y}) = \frac{\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}(U(k))]\mathbb{E}[\mathcal{Y}^2(U(k))]}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]\mathbb{E}[\mathcal{E}^2(U(k))]}. \quad (17)$$

Define

$$R(\mathcal{E}, \mathcal{Y}) \triangleq \frac{r(\mathcal{E}, \mathcal{Y})\mathbb{E}[\mathcal{E}^2(U(k))]}{\mathbb{E}[\mathcal{Y}^2(U(k))]}. \quad (18)$$

Note that $R(\mathcal{E}, \mathcal{Y})$ is known since $r(\mathcal{E}, \mathcal{Y})$ can be obtained from (14) and $\mathbb{E}[\mathcal{E}^2(U(k))]$ and $\mathbb{E}[\mathcal{Y}^2(U(k))]$ can be calculated since we choose $\mathcal{E}(u)$, $\mathcal{Y}(u)$, and the distribution of $U(k)$. Therefore, (17) can be rearranged as

$$\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}(U(k))] = R(\mathcal{E}, \mathcal{Y})\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]. \quad (19)$$

Note that the right hand side of (19) is unknown since it depends on the unknown value of $\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]$.

Suppose now that we fix $\mathcal{Y}(u)$ and consider the ersatz nonlinearities $\mathcal{E}_1(u)$ and $\mathcal{E}_2(u)$. It follows from (19) that

$$\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}_1(U(k))] = R(\mathcal{E}_1, \mathcal{Y})\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))], (20)$$
$$\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}_2(U(k))] = R(\mathcal{E}_2, \mathcal{Y})\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]. (21)$$

Note that $\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]$ is the same for both $\mathcal{E}_1(u)$ and $\mathcal{E}_2(u)$, i.e. for both ersatz nonlinearities $\mathcal{E}_1$ and $\mathcal{E}_2$, $\mathbb{E}[\mathcal{N}(U(k))\mathcal{E}_i(U(k))]$ is known up to the same scalar factor.

Define

$$\begin{aligned} \hat{\mathcal{N}}^{\mathcal{Y}}(u) &\triangleq \frac{1}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]} \hat{\mathcal{N}}(u) \\ &= \hat{a}_0^{\mathcal{Y}} + \sum_{n=1}^{N} \hat{a}_n^{\mathcal{Y}} \cos(\frac{n\pi u}{L}) + \sum_{n=1}^{N} \hat{b}_n^{\mathcal{Y}} \sin(\frac{n\pi u}{L}), \end{aligned} \quad (22)$$

where

$$\hat{a}_0^{\mathcal{Y}} \triangleq \frac{a_0}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]} = R(1,\mathcal{Y}(u)), \tag{23}$$

and for $n = 1,2,\ldots,$

$$\hat{a}_n^{\mathcal{Y}} \triangleq \frac{a_n}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]} = 2R(\cos(\frac{n\pi u}{L}),\mathcal{Y}(u)), \tag{24}$$

$$\hat{b}_n^{\mathcal{Y}} \triangleq \frac{b_n}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]} = 2R(\sin(\frac{n\pi u}{L}),\mathcal{Y}(u)). \tag{25}$$

That is, $\hat{\mathcal{N}}^{\mathcal{Y}}(u)$ is an approximation of $\mathcal{N}(u)$ up to an unknown scalar factor.

6.2.2 Laguerre Series Expansion

We represent $\mathcal{N}$ by its Laguerre series expansion, that is,

$$\mathcal{N}(u) = \sum_{n=0}^{\infty} q_n L_n(u), \tag{26}$$

where q_n is defined as

$$q_n \triangleq \int_0^{\infty} L_n(u)\mathcal{N}(u)e^{-u}du, \tag{27}$$

and the Laguerre polynomial $L_n(u)$ is defined as

$$L_n(u) \triangleq \sum_{i=0}^{n} \binom{n}{i} \frac{(-1)^i}{i!} u^i. \tag{28}$$

Note that using a stationary and exponentially distributed input $U(k)$ with unity mean it follows from (27) that

$$q_n = \mathbb{E}[L_n(U(k))\mathcal{N}(U(k))]. \tag{29}$$

We truncate (26) to obtain an approximation $\hat{\mathcal{N}}$ of $\mathcal{N}$, that is,

$$\hat{\mathcal{N}}(u) = \sum_{n=0}^{N} q_n L_n(u), \tag{30}$$

where N is chosen large enough such that $\hat{\mathcal{N}}(u)$ is an acceptable approximation of $\mathcal{N}(u)$. Since $\mathcal{N}$ is unknown, it follows that q_n is not implementable using (27). We apply the procedure discussed in the previous subsection to obtain q_n for $n = 0,1,2,\ldots$ up to a scalar factor, that is,

$$\begin{aligned}\hat{\mathcal{N}}^{\mathcal{Y}}(u) &\triangleq \frac{1}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]}\hat{\mathcal{N}}(u)\\ &= \sum_{i=0}^{n} q_n^{\mathcal{Y}} L_n(u),\end{aligned} \tag{31}$$

where

$$\hat{q}_n^{\mathcal{Y}} \triangleq \frac{q_n}{\mathbb{E}[\mathcal{N}(U(k))\mathcal{Y}(U(k))]} = R(L_n,\mathcal{Y}). \tag{32}$$

It follows that $\hat{\mathcal{N}}^{\mathcal{Y}}(u)$ is a an approximation of $\mathcal{N}(u)$ up to an unknown scalar factor.

For Gaussian $U(k)$ Hermite polynomials are the appropriate basis functions.

7 Numerical Examples

In this section we show examples with odd, even, neither odd nor even, saturation, and deadzone nonlinearities. Define $M(e^u)$ to be the mean of the realization of the random process e^U, and define $M(|u|)$ to be the mean of the realization of the random process $|U|$. Moreover, define the normalized Markov parameters of the transfer function G to be

$$H_i^{\mathrm{n}} \triangleq \frac{H_i}{H_d},$$

where H_d is the first nonzero Markov parameter of G . The estimated Markov parameters obtained by replacing $\mathcal{N}$ by the ersatz nonlinearity $\mathcal{Y}$, namely $\hat{H}_{i,\ell}^{\mathcal{Y}}$, are normalized by $\hat{H}_{d,\ell}^{\mathcal{Y}}$ to obtain $\hat{H}_{i,\ell}^{\mathcal{Y},\mathrm{n}}$, where $\hat{H}_{d,\ell}^{\mathcal{Y}}$ is the first nonzero estimated Markov parameter. The least squares estimates are computed for 200 independent realizations of U. We also define the error metric

$$\varepsilon \triangleq \frac{1}{200}\sum_{i=0}^{200} \frac{|H_i^n - \hat{H}_{i,\ell}^{\mathcal{Y},\mathrm{n}}|}{|H_i^n|}. \tag{33}$$

Example 7.1. Consider the transfer function

$$G(\mathbf{q}) = \frac{(\mathbf{q}-0.5)(\mathbf{q}-0.3)}{(\mathbf{q}+0.5)(\mathbf{q}+0.8)} \tag{34}$$

and the odd Hammerstein nonlinearity $\mathcal{N}(u) = u^3$. We choose $\mathcal{Y}(u) = e^u - M(e^u)$. Figure 3 indicates that the estimates of the Markov parameters are correct up to a scalar factor. In (22) we choose the basis functions to be $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ with $n = 0,1,2,\ldots,50$. White noise input uniformly distributed in $[-5,5]$ is used with $\ell = 50{,}000$ data points. Figure 4 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[U^3(k)(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

Example 7.2. Consider the transfer function (34) and the even Hammerstein nonlinearity $\mathcal{N}(u) = u^2$. In (22) we choose $\mathcal{Y}(u) = |u| - M(|u|)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ with $n = 0,1,2,\ldots,50$. White noise input uniformly distributed in $[-5,5]$ is used with $\ell = 50,000$ data points. Output signal is corrupted with Gaussian noise of SNR=10. Figure 5 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[U^2(k)(|U(k)| - \mathbb{E}[|U(k)|])]$.

Example 7.3. Consider the transfer function (34) and the neither even nor odd Hammerstein nonlinearity $\mathcal{N}(u) = e^u$. In (22) we choose $\mathcal{Y}(u) = |u| - M(|u|)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ with $n = 0,1,2,\ldots,50$. White noise input uniformly distributed in $[-5,5]$ is used with $\ell = 50,000$ data points. Figure 6 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[e^{U(k)}(|U(k)| - \mathbb{E}[|U(k)|])]$.

Example 7.4. Consider the transfer function (34) and the saturation Hammerstein nonlinearity

$$\mathcal{N}(u) = \begin{cases} -1, & \text{if } u < -1 \\ u, & \text{if } -1 \le u \le 1 \\ 1, & \text{if } u > 1. \end{cases} \tag{35}$$

In (22) we choose $\mathcal{Y}(u) = \sin(u)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ with $n = 0,1,2,\ldots,50$. White noise input uniformly distributed in $[-5,5]$ is used with $\ell = 50,000$ data points. Figure 7 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[\mathcal{N}(U(k))\sin(U(k))]$ where $\mathcal{N}(u)$ is as shown in (35).

Example 7.5. Consider the transfer function (34) and the deadzone Hammerstein nonlinearity

$$\mathcal{N}(u) = \begin{cases} u+1, & \text{if } u < -1 \\ 0, & \text{if } -1 \le u \le 1 \\ u-1, & \text{if } u > 1. \end{cases} \tag{36}$$

In (22) we choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ with $n = 0,1,2,\ldots,50$. White noise input uniformly distributed in $[-5,5]$ is used with $\ell = 50,000$ data points. Figure 8 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[\mathcal{N}(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$ where $\mathcal{N}(u)$ is as shown in (36).

Example 7.6. Consider the transfer function (34) and the odd Hammerstein nonlinearity $\mathcal{N}(u) = u^3$. In (31) we choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ with $n = 0,1,2,\ldots,50$. Exponentially distributed input with unity mean is used with $\ell = 50,000$ data points. Figure 9 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[U^3(k)(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

Example 7.7. Consider the transfer function (34) and the even Hammerstein nonlinearity $\mathcal{N}(u) = \cos(u)$. In (31) we choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ with $n = 0,1,2,\ldots,50$. Exponentially distributed input with unity mean is used with $\ell = 50,000$ data points. Figure 10 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[\cos(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

Example 7.8. Consider the transfer function (34) and the saturation Hammerstein nonlinearity (35). In (31) we choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ with $n = 0,1,2,\ldots,50$. Exponentially distributed input with unity mean is used with $\ell = 50,000$ data points. Figure 11 shows the actual nonlinearity and estimated nonlinearity scaled by $\mathbb{E}[\mathcal{N}(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$ where $\mathcal{N}(u)$ as in (35).

8 CONCLUSIONS

In this paper, we briefly reviewed the approach for identifying Hammerstein systems using ersatz nonlinearities. Then we showed that using multiple orthogonal ersatz nonlinearities, the Hammerstein nonlinearity can be estimated correctly up to a scalar factor. Two different orthogonal bases, namely, sines and cosines from the Fourier series and the Laguerre polynomials were used to construct and estimate the true nonlinearity represented as an orthogonal basis expansion. This method was demonstrated on several numerical examples including odd, even, neither odd nor even, saturation and deadzone nonlinearities. Finally, we showed that the kernel of the inner product associated with the orthogonal basis functions must be chosen to be the density function of the input signal.

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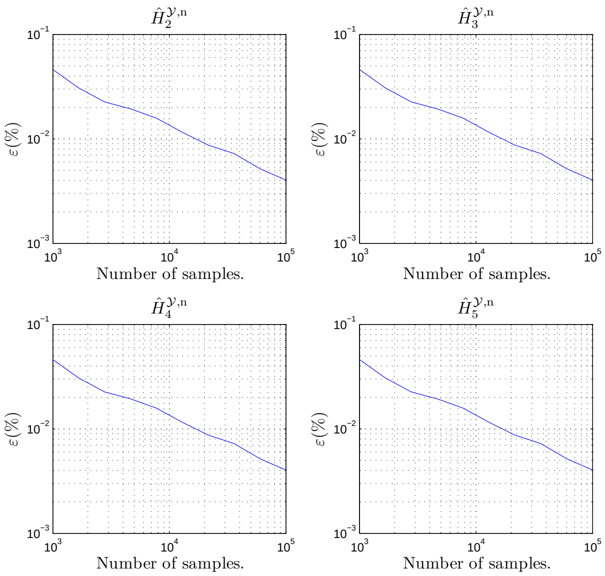

Figure 3. Error between true and estimated Markov parameters after normalization. The ersatz nonlinearity $\mathcal{Y}(u) = e^u - M(e^u)$ is used to replace the unknown nonlinearity $\mathcal{N}(u) = u^3$.

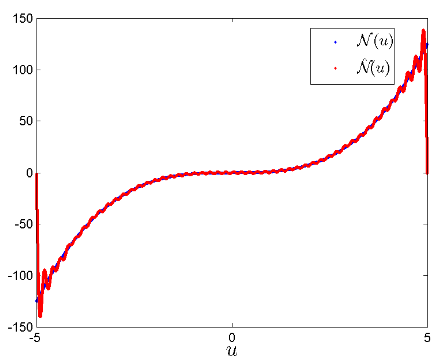

Figure 4. Actual nonlinearity $\mathcal{N}(u) = u^3$ and estimated nonlinearity after scaling for example 7.1. We choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[U^3(k)(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

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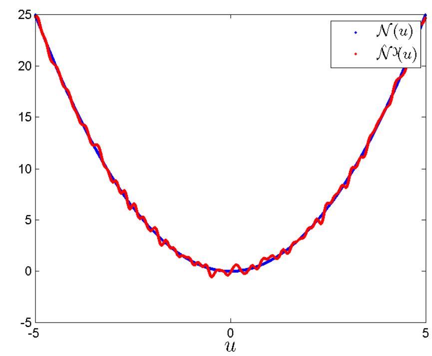

Figure 5. Actual nonlinearity $\mathcal{N}(u) = u^2$ and estimated nonlinearity after scaling for example 7.2. Gaussian distributed output noise w with SNR=10 is added. We choose $\mathcal{Y}(u) = |u| - M(|u|)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[U^2(k)(|U(k)| - \mathbb{E}[|U(k)|])]$.

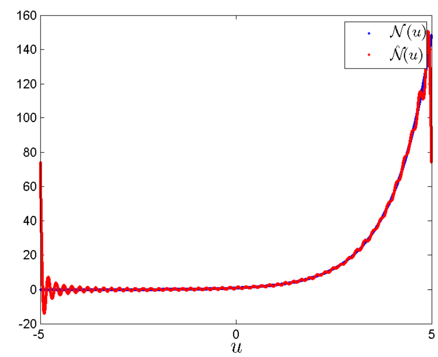

Figure 6. Actual nonlinearity $\mathcal{N}(u) = e^u$ and estimated nonlinearity after scaling for example 7.3. We choose $\mathcal{Y}(u) = |u| - M(|u|)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[e^{U(k)}(|U(k)| - \mathbb{E}[|U(k)|])]$.

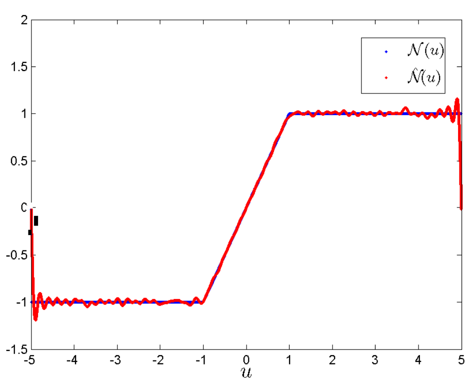

Figure 7. Actual nonlinearity (35) and estimated nonlinearity after scaling for example 7.4. We choose $\mathcal{Y}(u) = \sin(u)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[\mathcal{N}(U(k))\sin(U(k))]$, where $\mathcal{N}(U(k))$ is as shown in (35).

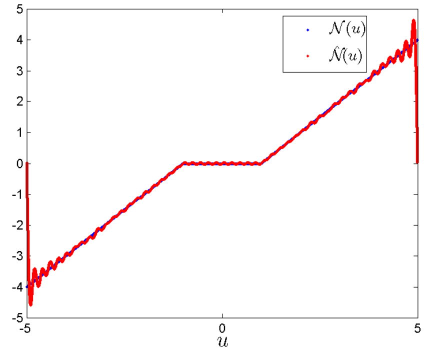

Figure 8. Actual nonlinearity (36) and estimated nonlinearity after scaling for example 7.5. We choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $\sin(\frac{n\pi u}{5})$ and $\cos(\frac{n\pi u}{5})$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[\mathcal{N}(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$, where $\mathcal{N}(U(k))$ is as shown in (36).

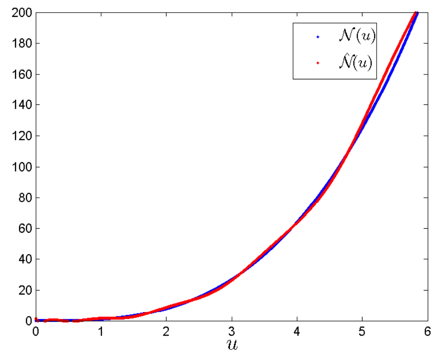

Figure 9. Actual nonlinearity $\mathcal{N}(u) = u^3$ and estimated nonlinearity after scaling for example 7.6. We choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[U^3(k)(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

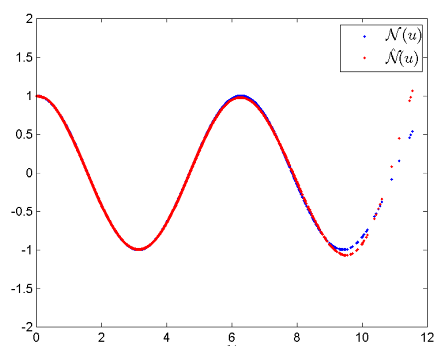

Figure 10. Actual nonlinearity $\mathcal{N}(u) = \cos(u)$ and estimated nonlinearity after scaling for example 7.7. We choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[\cos(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$.

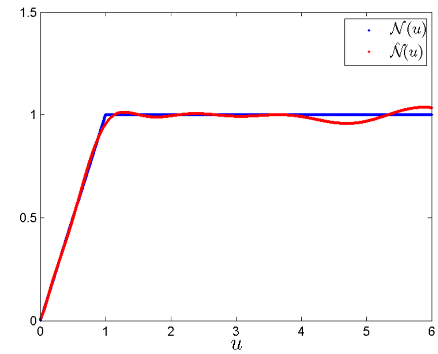

Figure 11. Actual nonlinearity (35) and estimated nonlinearity after scaling for example 7.8. We choose $\mathcal{Y}(u) = e^u - M(e^u)$, and the basis functions $L_n(u)$ for $n = 0, 1, 2, \ldots, 50$. The scaling factor is $\mathbb{E}[\mathcal{N}(U(k))(e^{U(k)} - \mathbb{E}[e^{U(k)}])]$, where $\mathcal{N}((U(k)))$ is as shown in (35).