Proceedings of the 41st IEEE
Conference on Decision and Control
Las Vegas, Nevada USA, December 2002

Energy Equipartition and the Emergence of Damping in Lossless Systems[1]

Dennis S. Bernstein[2] Sanjay P. Bhat[3]

1 Introduction

Thermodynamics is an inherently empirical branch of science based on "laws" that govern the thermal behavior of physical systems. The development of these laws and associated concepts has a long and tortuous history, see [1-6]. From a systems perspective, thermodynamics is a theory of large scale systems, whose properties, in modern terminology, are a manifestation of emergent behavior.

There have been many attempts to develop the laws of thermodynamics from first principles, that is, in terms of discrete, coupled entities. In classical statistical thermodynamics, a crystalline solid is modeled as a lattice of vibrating molecules, whose degrees of freedom satisfy the principle of equipartition of energy, which implies that the temperature of each subsystem converges to the same value. Therefore, the temperature of a subsystem whose initial temperature is greater than the average temperature will ultimately decrease, while the temperature of a subsystem whose initial temperature is less than the average temperature will ultimately increase. This is a statement of the zeroth law of thermodynamics, that is, that heat flows from hot to cold. These properties hold for systems with linear dynamics; for nonlinear systems it is well known that equipartition does not generally occur (see, for example, [7] and the references cited therein).

One explanation for the emergence of damping can be given in terms of Poincare's recurrence theorem, which implies that every finite-dimensional, isolated system will return arbitrarily close to its initial state infinitely often [4]. As the dimensionality increases, the time for recurrence increases (presumably at a very fast rate), and thus the practical inability for energy to return to its starting configuration (flowing from hot to cold and back to hot) is intuitively clear. The shortcoming of this classical view is that no decay-like damping model per se emerges.

The principle of equipartition of energy is usually viewed as a statistical result formulated in terms of the probability distribution of configurations, that is, entropy. The derivation of the empirical laws of thermodynamics from large scale systems of discrete subsystems thus depends on the statistical sense in which the former are valid. Stochastic averaging can be performed with respect to the statistics of initial conditions, exogenous disturbances, or subsystem and coupling parameters. Whichever technique is used, the ultimate objective is to approximate the high-order dynamics of an oscillatory system with the low-order dynamics of a non-oscillatory system. Consequently, wave dynamics are approximated by diffusion dynamics. However, there is a distinction between the response of an averaged model, which is an intellectual construct, and the physical response that is actually observed in the real world.

In the area of mechanical and acoustic vibrations, there has been extensive work on thermodynamically motivated energy flow modeling [10-19]. This kind of analysis is often used when finite element and modal modeling are ineffective due to dimensionality or uncertainty.

A thermodynamic model is concerned with the flow of energy among subsystems. This is a special case of a compartmental system, which involves the exchange of nonnegative quantities such as energy, mass, and chemical reactants [20-22]. An interesting aspect of compartmental systems is the fact that they possess a continuum of equilibria and thus cannot be asymptotically stable. Nevertheless, compartmental models have convergent trajectories and Lyapunov stable equilibria, a property known as semistability [23, 24]. Equipartition of energy in compartmental models was studied in [22].

In this paper we use deterministic linear systems techniques to analyze the vibrational energy of systems of undamped coupled oscillators with identical coupling. Our approach is based on time averaging of squared outputs of the system and thus avoids both recurrence and statistical arguments. We first consider a single undamped oscillator and show that the time-averaged potential energy and the time-averaged kinetic energy converge to the same value. This result, which is completely intuitive, follows from the classical virial

[1]This research was supported in part by the Air Force Office of Scientific Research under grant F49620—01-1-0094.

[2]Department of Aerospace Engineering The University of Michigan Ann Arbor, MI 48109-2140

[3]Department of Aerospace Engineering Indian Institute of Technology Powai, Mumbai 400076, India

theorem of mechanics [25], p. 23. However, we provide a novel proof of this result by direct computation to establish notation and techniques for energy equipartition in multiple oscillators.

Next, we consider a collection of n identical undamped oscillators with lossless coupling. As in the case of a single oscillator, equipartition of energy holds for the total system kinetic and potential energies. Again, this is a consequence of the virial theorem. However, this is not the problem we are interested in. Rather, we focus on the equipartition of *oscillator* energy, that is, the equal distribution of energy among oscillators, regardless of the form of energy.

For this problem, we derive expressions for the transient and steady-state behavior of each oscillator. Then we prove that equipartition of energy holds for a pair of identical, coupled oscillators with *distinct* coupled frequencies. This result shows that, in terms of time-averaged quantities, energy flows from the initially higher-energy oscillator to the initially lower-energy oscillator, in agreement with the zeroth law of thermodynamics, that is, that energy flows from hot to cold, for a pair of coupled oscillators. The assumption of distinct coupled frequencies is necessary. In fact, somewhat surprisingly, numerical results show that equipartition of energy can fail if the coupled system has repeated frequencies (see Figure 4). Finally, we give numerical evidence to suggest that the analogous result holds for a collection of coupled oscillators. The requirement for distinct coupled frequencies, without which equipartition fails, seems to have been overlooked in the literature.

Implications of these results for the emergence of damping in lossless systems are discussed in Section 4.

For convenience we use the following standard notation: "vec" denotes the column stacking operator, $\otimes$ denotes Kronecker product, $\oplus$ denotes Kronecker sum, and $(\cdot)^+$ denotes the Moore-Penrose generalized inverse, which, for normal matrices, coincides with the Drazin and group generalized inverses [26].

2 Equipartition of Energy for a Single Oscillator

We begin by considering a single oscillator of the form

$$\dot{x}(t) = Ax(t), \tag{2.1}$$

where the matrix A given by

$$A \triangleq \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \tag{2.2}$$

represents the dynamics of an undamped oscillator with position and velocity states

$$x(t) \triangleq \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}. \tag{2.3}$$

The natural frequency is given by $\omega \triangleq \sqrt{k/m}$, where $k > 0$ and $m > 0$ denote stiffness and inertia parameters.

Next consider the instantaneous potential energy $E_{\rm pot}(t)$ and the instantaneous kinetic energy $E_{\rm kin}(t)$ given by

$$E_{\rm pot}(t) = \frac{1}{2}kq^2(t),\ E_{\rm kin}(t) = \frac{1}{2}m\dot{q}^2(t). \tag{2.4}$$

Since $m\ddot{q}(t)+kq(t) = 0$, it follows that the total energy $E_{\rm pot}(t)+E_{\rm kin}(t)$ is constant for all $t \geq 0$. We thus define the total energy by

$$E_{\rm tot} \triangleq E_{\rm pot}(0) + E_{\rm kin}(0). \tag{2.5}$$

Hence, for all $t \geq 0$,

$$E_{\rm tot} = E_{\rm pot}(t) + E_{\rm kin}(t). \tag{2.6}$$

Analysis of the kinetic and potential energy is simplified by performing a change of basis. Hence define

$$K \triangleq \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix},$$

$$\hat{A} \triangleq K^{1/2}AK^{-1/2} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix},$$

$$\hat{x}(t) \triangleq K^{1/2}x(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{k}q(t) \\ \sqrt{m}\dot{q}(t) \end{bmatrix}$$

so that

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t),\ t \geq 0, \tag{2.7}$$

$$E_{\rm pot}(t) = \frac{1}{2}\hat{x}_1^2(t),\ E_{\rm kin}(t) = \frac{1}{2}\hat{x}_2^2(t), \tag{2.8}$$

$$E_{\rm tot} = \frac{1}{2}\hat{x}_1^2(t) + \frac{1}{2}\hat{x}_2^2(t),\ t \geq 0. \tag{2.9}$$

Next, define the 2×2 symmetric nonnegative-definite matrices $\hat{H}(t)$ and $\hat{M}(t)$ by

$$\hat{H}(t) \triangleq \int_0^t \hat{x}(s)\hat{x}^{\rm T}(s){\rm d}s,\ \hat{M}(t) \triangleq \frac{1}{t}\hat{H}(t). \tag{2.10}$$

Note that

$$\frac{1}{t}\int_0^t E_{\rm pot}(s){\rm d}s = \frac{1}{2}\hat{M}_{11}(t) \tag{2.11}$$

$$\frac{1}{t}\int_0^t E_{\rm kin}(s){\rm d}s = \frac{1}{2}\hat{M}_{22}(t). \tag{2.12}$$

Lemma 2.1. The matrix $\hat{M}(t)$ satisfies

$$\operatorname{tr}\hat{M}(t) = 2E_{\rm tot},\ t \geq 0. \tag{2.13}$$

Proof. Note that (2.9) implies that, for all $t \geq 0$,

$$\begin{aligned}\operatorname{tr}\hat{M}(t) &= \operatorname{tr}\frac{1}{t}\int_0^t \hat{x}(s)\hat{x}^{\rm T}(s){\rm d}s \\ &= \frac{1}{t}\int_0^t \operatorname{tr}[\hat{x}(s)\hat{x}^{\rm T}(s)]{\rm d}s = \frac{1}{t}\int_0^t \hat{x}^{\rm T}(s)\hat{x}(s){\rm d}s \\ &= \frac{1}{t}\int_0^t [\hat{x}_1^2(s) + \hat{x}_2(s)^2]{\rm d}s = \frac{1}{t}\int_0^t 2E_{\rm tot}{\rm d}s = 2E_{\rm tot}.\end{aligned}$$

Lemma 2.2. The matrix $\hat{M}(t)$ satisfies

$$\hat{M}(t) = \hat{M}_{\text{trans}}(t) + \hat{M}_\infty, \ t \geq 0, \tag{2.14}$$

where

$$\hat{M}_{\text{trans}}(t) \triangleq \frac{1}{t}\text{vec}^{-1}\Big[(\hat{A} \oplus \hat{A})^+\Big(e^{(\hat{A}\oplus\hat{A})t} - I\Big)\text{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\Big],$$
$$\hat{M}_\infty \triangleq \text{vec}^{-1}\Big[[I - (\hat{A} \oplus A)(\hat{A} \oplus \hat{A})^+]\text{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\Big].$$

Furthermore,

$$\lim_{t\to\infty} \hat{M}_{\text{trans}}(t) = 0, \tag{2.15}$$

$$\lim_{t\to\infty} \hat{M}(t) = \hat{M}_\infty, \tag{2.16}$$

$$\text{tr}\ \hat{M}_{\text{trans}}(t) = 0, \ t > 0, \tag{2.17}$$

$$\text{tr}\ \hat{M}_\infty = 2E_{\text{tot}}. \tag{2.18}$$

Proof. Using Theorem 9.2.4 of [26] it follows that

$$\begin{aligned}\hat{M}(t) &= \frac{1}{t}\int_0^t e^{\hat{A}s}\hat{x}(0)\hat{x}^{\mathrm{T}}(0)e^{\hat{A}^{\mathrm{T}}s}\mathrm{d}s \\ &= \text{vec}^{-1}\frac{1}{t}\int_0^t \Big(e^{\hat{A}s}\otimes e^{\hat{A}s}\Big)\text{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\mathrm{d}s \\ &= \text{vec}^{-1}\frac{1}{t}\int_0^t e^{(\hat{A}\oplus\hat{A})s}\text{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\mathrm{d}s \\ &= \text{vec}^{-1}\frac{1}{t}\Big[(\hat{A}\oplus\hat{A})^+\Big(e^{(\hat{A}\oplus\hat{A})t} - I\Big) \\ &\quad + t[I - (\hat{A}\oplus\hat{A})(\hat{A}\oplus\hat{A})^+]\text{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\mathrm{d}s\Big] \\ &= \hat{M}_{\text{trans}}(t) + \hat{M}_\infty.\end{aligned}$$

Since $e^{(\hat{A}\oplus\hat{A})t}$ is bounded, it follows that (2.15) holds. Now (2.14) and (2.15) imply (2.16). Finally, (2.13) and (2.16) imply (2.18), while (2.13), (2.14), and (2.18) imply (2.17). □

Lemma 2.3. The matrices $\hat{H}(t)$, $\hat{M}(t)$, and $\hat{M}_\infty$ satisfy

$$\hat{A}\hat{H}(t) + \hat{H}(t)\hat{A}^{\mathrm{T}} = \hat{x}(t)\hat{x}^{\mathrm{T}}(t) - \hat{x}(0)\hat{x}^{\mathrm{T}}(0), \tag{2.19}$$

$$\hat{A}\hat{M}(t) + \hat{M}(t)\hat{A}^{\mathrm{T}} = \frac{1}{t}[\hat{x}(t)\hat{x}^{\mathrm{T}}(t) - \hat{x}(0)\hat{x}^{\mathrm{T}}(0)], \tag{2.20}$$

$$\hat{A}\hat{M}_\infty + \hat{M}_\infty\hat{A}^{\mathrm{T}} = 0. \tag{2.21}$$

Proof. Integrating by parts yields

$$\hat{H}(t) = \hat{A}^{-1}\Big[e^{\hat{A}t}\hat{x}(0)\hat{x}^{\mathrm{T}}(0)e^{-\hat{A}t} - \hat{x}(0)\hat{x}^{\mathrm{T}}(0)\Big] - \hat{A}^{-1}\hat{H}(t)\hat{A},$$

which implies (2.19). Now (2.20) and (2.21) are immediate. □

The following result shows that the time-averaged potential and kinetic energies are asymptotically equal.

Theorem 2.1. The undamped oscillator (2.1) satisfies the property

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t E_{\text{pot}}(s)\mathrm{d}s = \lim_{t\to\infty}\frac{1}{t}\int_0^t E_{\text{kin}}(s)\mathrm{d}s = \frac{1}{2}E_{\text{tot}}. \tag{2.22}$$

Proof. It follows from (2.21) that

$$\begin{bmatrix} 2\hat{M}_{\infty12} & \hat{M}_{\infty22} - \hat{M}_{\infty11} \\ \hat{M}_{\infty22} - \hat{M}_{\infty11} & -2\hat{M}_{\infty12}\end{bmatrix} = 0.$$

Therefore, $\hat{M}_{\infty11} = \hat{M}_{\infty22}$. Next, it follows from (2.18) that $\hat{M}_{\infty11} + \hat{M}_{\infty22} = 2E_{\text{tot}}$. Therefore, $\hat{M}_{\infty11} = \hat{M}_{\infty22} = E_{\text{tot}}$. Finally, using (2.11) we have $\lim_{t\to\infty}\frac{1}{t}\int_0^t E_{\text{pot}}(s)\mathrm{d}s = \lim_{t\to\infty}\frac{1}{2}\hat{M}_{11}(t) = \frac{1}{2}\hat{M}_{\infty11} = \frac{1}{2}E_{\text{tot}}$ and likewise for E_{kin}. □

3 Equipartition of Energy for a Collection of Oscillators

In this section we consider the dynamics of n identical coupled oscillators modeled by

$$\dot{x}(t) = \mathcal{A}x(t), \tag{3.1}$$

where $\mathcal{A}$ is the $2n \times 2n$ matrix

$$\begin{bmatrix} A & C & C & \cdots & C \\ -K^{-1/2}C^{\mathrm{T}}K^{1/2} & A & C & \cdots & C \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K^{-1/2}C^{\mathrm{T}}K^{1/2} & -K^{-1/2}C^{\mathrm{T}}K^{1/2} & -K^{-1/2}C^{\mathrm{T}}K^{1/2} & \cdots & A\end{bmatrix}.$$

The position and velocity of the ith oscillator are given by $x_{2i-1}(t)$ and $x_{2i}(t)$, respectively. The 2×2 coupling matrix C is written as $C = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$.

Next, define $\mathcal{K} \triangleq \text{diag}(K, \dots, K)$, $\hat{\mathcal{A}} \triangleq \mathcal{K}^{1/2}\mathcal{A}\mathcal{K}^{-1/2}$, and $\hat{x}(t) \triangleq \mathcal{K}^{1/2}x(t)$. Note that

$$\hat{\mathcal{A}} = \begin{bmatrix} \hat{A} & \hat{C} & \hat{C} & \cdots & \hat{C} \\ -\hat{C}^{\mathrm{T}} & \hat{A} & \hat{C} & \cdots & \hat{C} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\hat{C}^{\mathrm{T}} & -\hat{C}^{\mathrm{T}} & -\hat{C}^{\mathrm{T}} & \cdots & \hat{A}\end{bmatrix}, \tag{3.2}$$

which is skew symmetric, where

$$\hat{C} \triangleq K^{-1/2}CK^{1/2} = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d}\end{bmatrix}. \tag{3.3}$$

Next, note that the energy $\mathcal{E}_{\text{osc}i}(t)$ of the ith oscillator is given by

$$\mathcal{E}_{\text{osc}i}(t) = \frac{1}{2}kx_{2i-1}^2(t) + \frac{1}{2}mx_{2i}^2(t) = \frac{1}{2}\hat{x}_{2i-1}^2(t) + \frac{1}{2}\hat{x}_{2i}^2(t),$$

while the total energy of the system is given by

$$\mathcal{E}_{\text{tot}} = \sum_{i=1}^n \mathcal{E}_{\text{osc}i}(t) = \frac{1}{2}x^{\mathrm{T}}(t)\mathcal{K}x(t) = \frac{1}{2}\hat{x}^{\mathrm{T}}(t)\hat{x}(t).$$

Proposition 3.1. $\mathcal{E}_{\text{tot}}$ is constant for all initial conditions $x(0)$.

Proof. For all $t \geq 0$, $\dot{\mathcal{E}}_{\text{tot}}(t) = \hat{x}^{\mathrm{T}}(t)\dot{\hat{x}}(t) = \hat{x}^{\mathrm{T}}(t)\hat{\mathcal{A}}\hat{x}(t) = 0$. □

Next as in (2.10) we define the $2n \times 2n$ symmetric nonnegative-definite matrices $\hat{\mathcal{H}}(t)$ and $\hat{\mathcal{M}}(t)$ by

$$\hat{\mathcal{H}}(t) \triangleq \int_0^t \hat{x}(s)\hat{x}^{\mathrm{T}}(s)\mathrm{d}s, \quad \hat{\mathcal{M}}(t) \triangleq \frac{1}{t}\hat{\mathcal{H}}(t). \tag{3.4}$$

We thus obtain the following extensions of Lemma 2.1, Lemma 2.2, and Lemma 2.3.

Lemma 3.1. The matrix $\hat{\mathcal{M}}(t)$ satisfies

$$\operatorname{tr} \hat{\mathcal{M}}(t) = 2\mathcal{E}_{\text{tot}}, \quad t \geq 0. \tag{3.5}$$

Lemma 3.2. The matrix $\hat{\mathcal{M}}(t)$ satisfies

$$\hat{\mathcal{M}}(t) = \hat{\mathcal{M}}_{\text{trans}}(t) + \hat{\mathcal{M}}_{\infty}, \quad t \geq 0, \tag{3.6}$$

where

$$\hat{\mathcal{M}}_{\text{trans}}(t) \triangleq \frac{1}{t}\operatorname{vec}^{-1}\left[(\hat{\mathcal{A}} \oplus \hat{\mathcal{A}})^{+}\left(e^{(\hat{\mathcal{A}}\oplus\hat{\mathcal{A}})t} - I\right)\operatorname{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\right]$$

$$\hat{\mathcal{M}}_{\infty} \triangleq \operatorname{vec}^{-1}\left[[I - (\hat{\mathcal{A}} \oplus \mathcal{A})(\hat{\mathcal{A}} \oplus \hat{\mathcal{A}})^{+}]\operatorname{vec}[\hat{x}(0)\hat{x}^{\mathrm{T}}(0)]\right].$$

Furthermore,

$$\lim_{t\to\infty} \hat{\mathcal{M}}_{\text{trans}}(t) = 0, \quad \lim_{t\to\infty} \hat{\mathcal{M}}(t) = \hat{\mathcal{M}}_{\infty}, \tag{3.7}$$

$$\operatorname{tr} \hat{\mathcal{M}}_{\text{trans}}(t) = 0, \quad t > 0, \quad \operatorname{tr} \hat{\mathcal{M}}_{\infty} = 2\mathcal{E}_{\text{tot}}. \tag{3.8}$$

Lemma 3.3. Assume that $\mathcal{A}$ is nonsingular. Then $\hat{\mathcal{H}}(t)$, $\hat{\mathcal{M}}(t)$, and $\hat{\mathcal{M}}_{\infty}$ satisfy

$$\hat{\mathcal{A}}\hat{\mathcal{H}}(t) + \hat{\mathcal{H}}(t)\hat{\mathcal{A}}^{\mathrm{T}} = \hat{x}(t)\hat{x}^{\mathrm{T}}(t) - \hat{x}(0)\hat{x}^{\mathrm{T}}(0), \tag{3.9}$$

$$\hat{\mathcal{A}}\hat{\mathcal{M}}(t) + \hat{\mathcal{M}}(t)\hat{\mathcal{A}}^{\mathrm{T}} = \frac{1}{t}[\hat{x}(t)\hat{x}^{\mathrm{T}}(t) - \hat{x}(0)\hat{x}^{\mathrm{T}}(0)], \tag{3.10}$$

$$\hat{\mathcal{A}}\hat{\mathcal{M}}_{\infty} + \hat{\mathcal{M}}_{\infty}\hat{\mathcal{A}}^{\mathrm{T}} = 0. \tag{3.11}$$

It thus remains to derive a generalization of the energy equipartition result (2.22) given by Theorem 2.1. We first consider the case of a pair of oscillators, where $\hat{\mathcal{A}}$ is given by

$$\hat{\mathcal{A}} = \begin{bmatrix} 0 & \omega & \hat{a} & \hat{b} \\ -\omega & 0 & \hat{c} & \hat{d} \\ -\hat{a} & -\hat{c} & 0 & \omega \\ -\hat{b} & -\hat{d} & -\omega & 0 \end{bmatrix}. \tag{3.12}$$

Since Lemma 3.3 requires that $\mathcal{A}$ be nonsingular, it is useful to note that, for $n = 2$, $\det \hat{\mathcal{A}} = (\omega^2 - \det \hat{C})^2$. Therefore, $\hat{\mathcal{A}}$ is nonsingular if and only if $\det \hat{C} \neq \omega^2$. Note that if $\hat{\mathcal{A}}$ is singular, then, since $\hat{\mathcal{A}}$ is skew symmetric and has even order 4, its eigenvalue 0 is repeated. The following result gives necessary and sufficient conditions under which $\hat{\mathcal{A}}$ has a repeated eigenvalue.

Lemma 3.4. The matrix $\hat{\mathcal{A}}$ has a repeated eigenvalue if and only if either *i)* $\hat{\mathcal{A}}$ is singular or *ii)* $\hat{a} = -\hat{d}$ and $\hat{b} = \hat{c}$. In case *i)*, $\hat{\mathcal{A}}$ has the repeated eigenvalue 0, while in case *ii)*, $\hat{\mathcal{A}}$ has the repeated eigenvalues $\jmath\sqrt{\omega^2 + \hat{a}^2 + \hat{b}^2}$ and $-\jmath\sqrt{\omega^2 + \hat{a}^2 + \hat{b}^2}$.

Proof. First note that $\hat{\mathcal{A}}$ has a repeated eigenvalue if and only if there exists $\alpha \geq 0$ such that $\det(sI - \hat{\mathcal{A}})$ has either the form $\det(sI - \hat{\mathcal{A}}) = s^2(s^2 + \alpha)$, in which case $\hat{\mathcal{A}}$ has the repeated eigenvalue 0, or the form $\det(sI - \hat{\mathcal{A}}) = s^4 + 2\alpha s^2 + \alpha^2 = (s^2 + \alpha)^2$, in which case $\hat{\mathcal{A}}$ has the repeated imaginary eigenvalues $\jmath\sqrt{\alpha}$ and $-\jmath\sqrt{\alpha}$. The former case occurs if and only if $\det \hat{\mathcal{A}} = 0$.

Next note that the characteristic polynomial of $\hat{\mathcal{A}}$ is $\det(sI - \hat{\mathcal{A}}) = s^4 + (\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2 + 2\omega^2)s^2 + \det \hat{\mathcal{A}}$. Hence set $\alpha = \det \hat{C} - \omega^2$ and $\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2 + 2\omega^2 = 2\alpha$. This condition implies $(\hat{a} - \hat{d})^2 + (\hat{b} + \hat{c})^2 + 2\omega^2 = 0$, which yields $\omega = 0$, which contradicts $\omega \neq 0$. Alternatively, set $\alpha = \omega^2 - \det \hat{C}$ and $\hat{a}^2 + \hat{b}^2 + \hat{c}^2 + \hat{d}^2 + 2\omega^2 = 2\alpha$. This condition is equivalent to $(\hat{a} + \hat{d})^2 + (\hat{b} - \hat{c})^2 = 0$, which is equivalent to $\hat{a} = -\hat{d}$ and $\hat{b} = \hat{c}$. Furthermore, in this case, $\alpha = \omega^2 - \det \hat{C} = \omega^2 + \hat{a}^2 + \hat{b}^2 > 0$. □

Lemma 3.4 implies that $\hat{\mathcal{A}}$ does not have a repeated eigenvalue if and only if $\hat{\mathcal{A}}$ is nonsingular and either $\hat{a} \neq -\hat{d}$ or $\hat{b} \neq \hat{c}$. Note that if $\hat{C} = 0$ then both $\hat{a} = -\hat{d}$ and $\hat{b} = \hat{c}$, and thus by Lemma 3.4 $\hat{\mathcal{A}}$ has a repeated eigenvalue. In fact, when $\hat{C} = 0$, $\hat{\mathcal{A}}$ has the repeated eigenvalues $\jmath\omega$ and $-\jmath\omega$.

The following result concerns the equipartition of oscillator energy for a pair of coupled oscillators with nonrepeated coupled frequencies.

Theorem 3.1. Let $n = 2$ and assume that $\hat{\mathcal{A}}$ does not have a repeated eigenvalue. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{E}_{\text{osc1}}(s)\mathrm{d}s = \lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{E}_{\text{osc2}}(s)\mathrm{d}s = \frac{1}{2}\mathcal{E}_{\text{tot}}.$$

Proof. Omitted due to lack of space. □

The case of $n \geq 3$ oscillators is more involved. Here we state the following conjecture concerning the equipartition of oscillator energy.

Conjecture 3.1. Let $n \geq 3$ and assume that $\hat{\mathcal{A}}$ does not have a repeated eigenvalue. Then, for $i = 1, \ldots, n$,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \mathcal{E}_{\text{osc}i}(s)\mathrm{d}s = \frac{1}{n}\mathcal{E}_{\text{tot}}. \tag{3.13}$$

4 Examples

Example 4.1. Consider a single oscillator with $k = 7$ N/m and $m = 13$ kg and initial conditions $q(0) = 0$ m and $\dot{q}(0) = 1$ m/s. As can be seen in Figure 1, the energy is periodically exchanged between kinetic and potential. However, Figure 2 shows that the time-averaged kinetic and potential energies are asymptotically equal.

Example 4.2. Consider a pair of coupled oscillators with $k = 7$ N/m and $m = 13$ kg and initial conditions $x_1(0) = 1$ m, $x_2(0) = 1$ m/s, $x_3(0) = 0$ m, and $x_4(0) = 0$ m/s. Furthermore, the coupling matrix C is given by $C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. With this coupling matrix, the coupled system has distinct frequencies 1.388 rad/s and 0.38795 rad/s. As can be seen in Figure 3, the time-averaged oscillator energies are asymptotically equal.

Example 4.3. Consider a pair of coupled oscillators with $k = 7$ N/m and $m = 13$ kg and initial conditions $x_1(0) = 1$ m, $x_2(0) = 1$ m/s, $x_3(0) = 0$ m, and $x_4(0) = 0$ m/s. Furthermore, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. With this coupling matrix, the coupled system has the repeated frequency 1.24 rad/s. As can be seen in Figure 4, the time-averaged oscillator energies are not asymptotically equal, that is, equipartition fails. This is consistent with Theorem 3.1, which assumes that the coupled frequencies are distinct.

Example 4.4. Finally, as a test of Conjecture 3.1 we consider a collection of 10 coupled oscillators with $k = 7$ N/m and $m = 13$ kg, randomly generated initial conditions, and $C = \begin{bmatrix} 10 & 0 \\ 0 & 19 \end{bmatrix}$. As can be seen in Figure 5, the time-averaged oscillator energies are asymptotically equal, that is, equipartition of energy holds.

5 Implications for the Emergence of Damping

A fundamental implication of these results concerns the emergence of damping in lossless systems. In reality, energy is conserved, and, therefore, there can be no energy "lost" to damping. Rather, the notion of damping is merely a convenient fiction for modeling the removal of energy from a mechanical subsystem. Yet it is of intellectual interest to understand how the fiction of damping emerges from the lossless dynamics of the real world.

In view of Theorem 3.1 and Conjecture 3.1, we can explain this paradox as follows. Suppose that a single, low frequency oscillator is connected to a large collection of identical, coupled, high-frequency oscillators. The response of the high-frequency oscillators is effectively averaged over all finite intervals during which their dynamics exhibit "decay" as shown, for example, in Figure 5. However, this "decay" is not decay in the sense of convergence to zero; if it did converge to zero, then energy would truly be lost, which it is not. Rather, the equipartition energy level represents an average temperature, which is effectively the energy per oscillator. The simple act of redefining this energy level to be zero gives the impression that energy is lost through dissipation. Hence our analysis shows that damping is an artifact of averaging. In other words, *time averaging causes damping*. The derivation of an LTI approximation of this decay-like behavior is needed to complete this mechanism of emergence.

6 Acknowledgment

We thank Natalia B. Janson, Ernie Barany, and Larry Davis for helpful suggestions.

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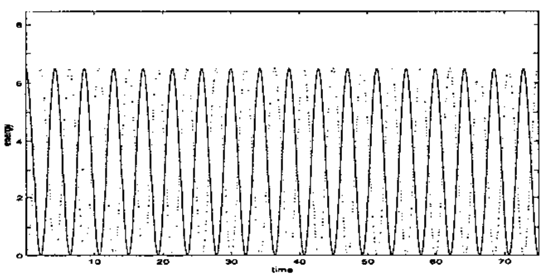

Figure 1: This plot of $E_{\rm pot}(t)$ and $E_{\rm kin}(t)$ shows the actual potential and kinetic energies for the single oscillator given by Example 4.1.

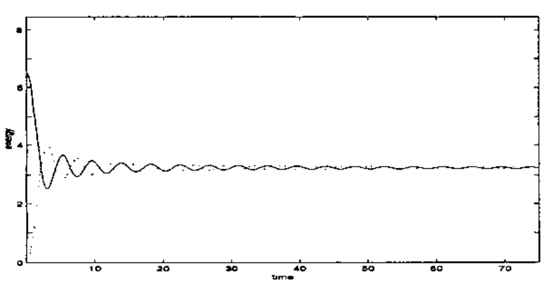

Figure 2: This plot of $\frac{1}{t}\int_0^t E_{\rm pot}(s)\mathrm{d}s$ and $\frac{1}{t}\int_0^t E_{\rm kin}(s)\mathrm{d}s$ shows that the time-averaged kinetic and potential energies for the single oscillator given by Example 4.1 satisfy energy equipartition.

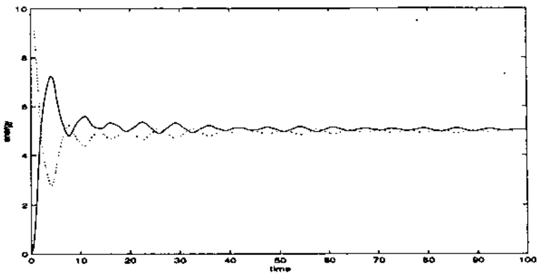

Figure 3: This plot of $\frac{1}{t}\int_0^t \mathcal{E}_{\rm osc1}(s)\mathrm{d}s$ and $\frac{1}{t}\int_0^t \mathcal{E}_{\rm osc2}(s)\mathrm{d}s$ shows energy equipartition for the pair of coupled oscillators given by Example 4.2.

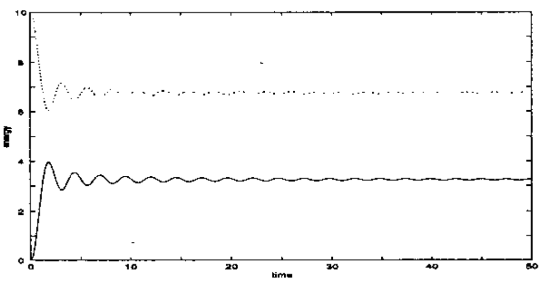

Figure 4: This plot of $\frac{1}{t}\int_0^t \mathcal{E}_{\rm osc1}(s)\mathrm{d}s$ and $\frac{1}{t}\int_0^t \mathcal{E}_{\rm osc2}(s)\mathrm{d}s$ shows time-averaged oscillator energies for a pair of coupled oscillators with the coupling chosen in Example 4.2 so that the coupled system has repeated frequencies. Consistent with Theorem 3.1, equipartition fails for this choice of coupling.

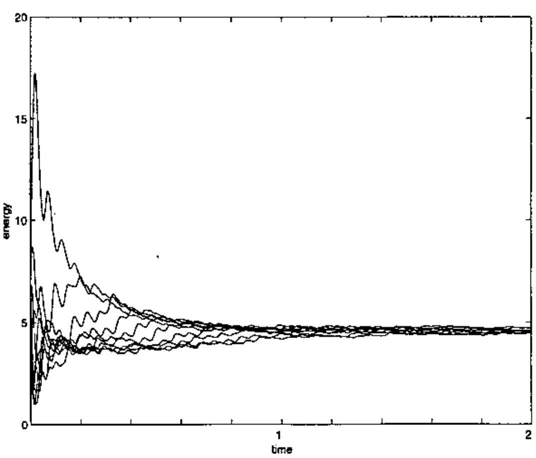

Figure 5: This plot shows energy equipartition of the averaged oscillator energies of $\frac{1}{t}\int_0^t \mathcal{E}_{{\rm osc}i}(s)\mathrm{d}s$ for a collection of 10 coupled oscillators.