1. Introduction

In [1] active energy flow control techniques were considered for interconnected modal subsystems. These techniques are now applied to interconnected structural subsystems. For this purpose we extend results given in [2] and derive two energy flow models for structures interconnected either conservatively or dissipatively. In the modal subsystem model considered in [1], each mode is viewed as a subsystem, while in the structural subsystem model each substructure is treated as a subsystem. For the modal subsystem model we can directly apply the control techniques considered in [1]. The structural subsystem model, however, requires special care. In particular, a dissipation filter and a disturbance filter are required since now the real part of the substructure impedance and the disturbance spectral density are frequency-dependent.

Two distinct energy flow control techniques developed in [3] are applied to the modal subsystem model and the structural subsystem model. Specifically, the controller is designed either as an additional sub-system or as a dissipative coupling to minimize energy flow entering a specified substructure. The goal in [1] was to maximize the energy flow from a specified substructure in the modal subsystem model and thus reduce the vibration of this substructure.

In previous works [3,4] $H_2$ and $H_{\infty}$ control techniques were used to regulate energy flow in a certain frequency band. In this paper, as in [1], controllers are designed according to a specialized LQG positive real control approach [5] that yields positive real controllers. Thus the resulting controller minimizes an $H_2$ performance index and guarantees asymptotic stability of the closed-loop system in spite of modeling uncertainty.

2. Structural Model

We consider $r$ one- or two-dimensional structures under vibration by means of pointwise external disturbance forces. Each pair of structures is assumed to be mutually interconnected by means of conservative or dissipative couplings. For convenience, we make the simplifying assumption that all couplings to a given structure are connected to a single point on that structure. The case of structures interconnected at multiple points is more complicated and is outside the scope of this paper.

The partial differential equation for the response of the $i$th structure is given by

$$\rho_i(\xi) \frac{d^2 x_i(\xi,t)}{d\xi^2} + C_i(\xi,t) = \bar{w}_i(t)\delta(\xi - \xi_i) - h_i(\xi,\xi_i,t),$$

where $\xi \in \Omega_i$ denotes the spatial coordinate defined in the region of space $\Omega_i$ for the $i$th structure. Furthermore, $\rho_i(\xi)$ is the mass density, $C_i$ is the self-adjoint stiffness operator for the $i$th structure, and $\bar{w}_i(t)$ is the external disturbance force acting on the $i$th substructure at the point $\xi_i$. We assume that $\bar{w}_i(t)$, $i = 1, \ldots, r$, are mutually uncorrelated white noise disturbances with unit intensity. Additionally, the coupling effect $h_i(\xi,\xi_i,t)$ at the coupling position $\xi_i$ is given by

$$h_i(\xi,\xi_i,t) \triangleq f_i(t)(\xi - \xi_i),$$

for an interaction force $f_i(t)$.

We consider a modal decomposition of the $i$th structure of the form

$$x_i(\xi_i, t) = \sum_{j=1}^{\infty} q_j(\xi_i) \psi_{ij}(\xi_i), \quad i = 1, \ldots, r, \quad (3)$$

where $q_j(t)$ and $\psi_{ij}(\xi_i)$ denote modal coordinates and normalized eigenfunctions, respectively, and the double subscript $ij$ denotes the $j$th mode of the $i$th substructure. The normalized eigenfunctions $\psi_{ij}(\xi)$ satisfy the orthogonality properties

$$\int_{\Omega_i} q_j(\xi_i) \psi_{ij}(\xi_i) d\xi_i = \delta_{ij}, \quad \int_{\Omega_i} C_i(\xi_i) \psi_{ij}(\xi_i) d\xi_i = \omega_{ij}^2 \delta_{ij}, \quad (4)$$

where $\omega_{ij}$ is the natural frequency of the $j$th mode of the $i$th structure and $\delta_{ij}$ is the Kronecker delta. From (3), (4) and appropriate boundary conditions, it follows that the modal coordinates $q_j(t)$ satisfy the equations of motion

$$\ddot{q}_j(t) + 2\zeta_j \omega_{ij} q_j(t) + \omega_{ij}^2 q_j(t) = a_j \bar{w}_j(t) - b_j v_j(t), \quad (5)$$

where $v_j(t)$ is the coupling interaction and the modal damping term $2\zeta_j \omega_{ij} q_j(t)$ is now included. In (5), the modal coefficient $a_j$ is defined by

$$a_j \triangleq \psi_{ij}(\xi_i), \quad (6)$$

while

$$b_j \triangleq \psi_{ij}(\xi_i), \quad v_j(t) \triangleq f_j(t),$$

for force interaction and

$$b_j \triangleq \frac{\partial \psi_{ij}(\xi_i)}{\partial \xi_i}, \quad v_j(t) \triangleq g_j(t),$$

for torque interaction.

The modal velocity $y_j(t)$ of the $j$th mode of the $i$th structure and the velocity $y_i(t)$ of the $i$th substructure at the coupling point are given by

$$y_j(t) = b_j q_j(t), \quad (9)$$

$$y_i(t) = \sum_{j=1}^{\infty} y_j(t), \quad (10)$$

where $n_i$ is the number of modes of the $i$th structure in the frequency range of interest. For later use we note that the modal impedance $z_j(s)$, $i = 1, \ldots, r$, $j = 1, \ldots, n_i$, is given by

$$z_j(s) = \frac{s^2 + 2\zeta_j \omega_{ij} s + \omega_{ij}^2}{s} \frac{1}{s}. \quad (11)$$

3. Energy Flow Modeling: Modal Subsystem Model

First we obtain the modal subsystem model by considering each mode as a subsystem. Let $w_{ij}(t)$ denote the disturbance force exciting the $j$th mode of the $i$th structure, that is,

$$w_{ij}(t) = a_j \bar{w}_j(t), \quad i = 1, \ldots, r, \quad j = 1, \ldots, n_i, \quad (12)$$

and we assume that the coupling interaction $v(t)$ and the structural velocity $y_i(t)$ are related by a coupling transfer function $L(s)$, that is,

$$v_i = L(s) y_i, \quad (13)$$
where $y_m(t) \triangleq [y_1(t) \cdots y_r(t)]^T$ and $v_m(t) \triangleq [v_1(t) \cdots v_r(t)]^T$.

To obtain a feedback representation of the interconnected systems, we define the model impedance matrix

$$Z_m(s) \triangleq \text{diag}(z_1(s), z_2(s), \ldots, z_m(s), \ldots, z_n(s), \ldots, z_n(s)),$$

and the vectors

$$y_m(t) \triangleq [y_1(t) \cdots y_m(t) \cdots y_m(t) \cdots y_m(t)]^T,$$

$$v_m(t) \triangleq [v_1(t) \cdots v_m(t) \cdots v_m(t) \cdots v_m(t)]^T,$$

$$w_m(t) \triangleq [w_1(t) \cdots w_m(t) \cdots w_m(t) \cdots w_m(t)]^T,$$

$$\bar{w}(t) \triangleq [\bar{w}_1(t) \cdots \bar{w}_r(t)]^T.$$

Note that $w_m(t) = D_m \bar{w}(t)$, $y_m(t) = E_m v_m(t)$ and $u_m(t) = E_m u(t)$, where the matrices $D_m$ and $E_m$ are defined by

$$D_m \triangleq \begin{bmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a_{21} & \cdots & a_{2n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & a_{r1} & \cdots & a_{rn} \end{bmatrix},$$

$$E_m \triangleq \begin{bmatrix} b_{11} & \cdots & b_{1n} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{21} & \cdots & b_{2n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & b_{r1} & \cdots & b_{rn} \end{bmatrix}.$$

With this notation the interconnected system (5) can be expressed as the feedback system shown in Fig. 3-1, where $u_m(t) \triangleq u_m(t) - u_m(t)$ and the coupling matrix $L_m(s)$ for the model subsystem model satisfying $u_m = L_m y_m$ is defined by

$$L_m(s) \triangleq E_m L(s) E_m^T.$$

Note that if $L(s)$ is conservative, that is, $L(x(u)) + L^T(x(u)) = 0$, it follows that $L_m(x_m) + L_m^T(x_m) = 0$ so that $L_m(s)$ is also conservative. In the same manner if $L(s)$ is dissipative, that is, $L(x(u)) + L^T(x(u)) \geq 0$, then $L_m(s)$ is also dissipative, $L_m(x_m) + L_m^T(x_m) \geq 0$. Since now $Z_m(s)$ is strictly positive real, the feedback system in Fig. 3-1 is asymptotically stable.

Now we consider three steady-state average energy flows $P_{ij}^s$, $P_{ij}^d$ and $P_{ij}^e$ which symbolize $P_{ij}^s$ is the steady-state average energy flow entering the $j$th mode of $i$th structure through the coupling $L_m(s)$,

$P_{ij}^d$ is the steady-state average energy dissipation rate of the $j$ mode of $i$th structure,

$P_{ij}^e$ is the steady-state average external power entering the $j$th mode of $i$th structure.

To evaluate these steady-state average energy flows consider state-space realizations of $Z_m(s)$ and $L(s)$ in Fig. 3-1 given by

$$\dot{z}_m(t) = A_m z_m(t) + B_m y_m(t),$$

$$y_m(t) = C_m z_m(t),$$

$$\dot{x}_L(t) = A_L x_L(t) + B_L u(t),$$

$$u(t) = C_L x_L(t),$$

respectively. Since $u_m(t) = u_m(t) - u_m(t) = D_m \bar{w}(t) - E_m v_m(t)$ and $y_m(t) = E_m y_m(t)$, the augmented system (21)-(24) is given by

$$\dot{z}_m(t) = A z_m(t) + D \bar{w}(t),$$

$$A \triangleq \begin{bmatrix} A_m & B_m E_m C_L \\ B_L E_m C_m & A_L \end{bmatrix}, \quad D \triangleq \begin{bmatrix} B_m D_m & 0 \end{bmatrix}.$$

Also define $C_m$ and $C_{n_m}$ by

$$C_m \triangleq [C_m \ 0], \quad C_{n_m} \triangleq [0 \ E_m C_L],$$

so that $y_m(t) = C_m x_m(t)$ and $u_m(t) = C_{n_m} x_m(t)$.

Next we define the diagonal damping matrix

$$C_{sd} \triangleq \text{Re}[Z_m(s)],$$

and note that (11) and (14) imply

$$C_{sd} w_j(t) = 2C_{sd} \bar{w}_j(t),$$

where $A_{sd}$ denotes $A_{n_m n_m}$ and $n_m \triangleq (\sum_{k=1}^{m} n_k) + j$. With this notation the steady-state average energy flows $P_{ij}^s$, $P_{ij}^d$ and $P_{ij}^e$ are given by

$$P_{ij}^s = -E_{ij} y_m(t) u_m(t) = -(C_m y_m(t) C_{sd})_{ij},$$

$$P_{ij}^d = E_{ij} y_m(t) u_m(t) = (C_{n_m} y_m(t) C_{sd})_{ij},$$

$$P_{ij}^e = E_{ij} y_m(t) u_m(t) = \frac{1}{2} (D_m D^T C_{sd})_{ij},$$

where $u_m(t)$, $u_m(t)$ and $u_m(t)$ are the $n_i$th element of $u_m(t)$, $v_m(t)$ and $w_m(t)$, respectively, and the steady-state covariance $Q_m \triangleq \lim_{s \to 0} E[x_m(t) x_m^T(t)]$ satisfies the algebraic Lyapunov equation

$$\dot{Q}_m + Q_m A^T + A Q_m = 0.$$

As shown in [2], $P_{ij}^s$, $P_{ij}^d$ and $P_{ij}^e$ satisfy

$$P_{ij}^s + P_{ij}^d + P_{ij}^e = 0, \quad i = 1, \ldots, r, \quad j = 1, \ldots, n_i.$$

Furthermore, if $L(s)$ is conservative, then [2]

$$\sum_{j=1}^{n_i} \sum_{i=1}^{r} P_{ij}^s = 0,$$

while if $L(s)$ is dissipative, then [2]

$$\sum_{j=1}^{n_i} \sum_{i=1}^{r} P_{ij}^s \leq 0.$$

4. Energy Flow Modeling: Structural Subsystem Model

Now we obtain the structural subsystem energy flow model by treating each substructure as a subsystem. In this model the energy flows are evaluated at the coupling points of the substructures. Hence the total impedance $z_i(s)$ of the $i$th structure at the coupling point is given by

$$\frac{1}{z_i(s)} = \sum_{j=1}^{n_i} \frac{1}{z_{ij}(s)}.$$
Since \( z_i(s) \) is strictly positive real, it follows that
\[
\alpha_i(\omega) \triangleq \text{Re} [z_i(j\omega) > 0, \ i = 1, \ldots, r, \ \omega \in \mathbb{R},
\]  
(40)
where \( \alpha_i(\omega) \) is the frequency-dependent resistance or damping. For convenience, define the \( r \times r \) diagonal transfer function
\[
Z(s) \triangleq \text{diag}(z_1(s), \ldots, z_r(s)),
\]  
(41)
and the frequency-dependent resistance or damping matrix
\[
C_0(\omega) \triangleq \text{Re} [Z(j\omega)] = \text{diag}(\alpha_1(\omega), \ldots, \alpha_r(\omega)).
\]  
(42)
With this notation the interconnected system in (39) can be expressed as a feedback system in Fig. 4.1. In Fig. 4.1 \( u_i(t) \triangleq [u_1(t) \cdots u_r(t)]^T, \ u_i(t) \triangleq [u_1(t) \cdots u_r(t)]^T = w_i(t) - u_i(t) \) and \( u_i(t), \ u_i(t) \) and \( L(s) \) satisfy (13).
Now we consider the steady-state average energy flow among substructures. In a similar manner to the previous section the steady-state average energy flows \( P^e_i, P^e_0 \) and \( P^e_T \) are defined for \( i = 1, \ldots, r \) by
\[
P^e_i \triangleq -E[u_i(t)u_i(t)],
\]  
(43)
\[
P^e_0 \triangleq -E[u_i(t)u_i(t)],
\]  
(44)
\[
P^e_T \triangleq E[y_i(t)y_i(t)],
\]  
(45)
where \( u_i(t) \) is the \( i \)-th element of \( u_i(t) \). The meaning of these energy flow quantities corresponds to the meanings of \( P^e_i, P^e_0 \) and \( P^e_T \) in the previous section, respectively, but now \( P^e_i, P^e_0 \) and \( P^e_T \) are the energy flows for the \( i \)-th substructure and \( P^e_T \) is the energy flow through the coupling \( L(s) \) in Fig. 4.1.
In the previous section we expressed \( P^e_i, P^e_0, P^e_T \) in terms of the steady-state covariance \( Q_0(s) \) according to the approach in [2,3]. In the structural energy flow model, however, the real part \( \alpha_i(s) \) of \( z_i(s) \), is not constant and the disturbance \( w_i(t) \) entering \( z_i(s) \) is no longer white noise. Thus to obtain the steady-state covariance corresponding to \( Q_0(s) \) we now introduce two filter transfer function matrices \( T(s) \) and \( R(s) \) as shown in Fig. 4.1, where the disturbance filter \( T(s) \) is defined by
\[
T(s) \triangleq \text{diag}(T_1(s), T_2(s), \ldots, T_r(s)),
\]  
(46)
and the stable dissipation filter \( R(s) \) satisfying [4]
\[
R(s)\Phi_0(s) = C_0(s).
\]  
(47)
Now let \( z^*_0(s), T(s) \) and \( R(s) \) have the realizations
\[
\hat{x}_i(t) = A_0 x_i(t) + B_0 y_i(t),
\]  
(48)
\[
\hat{w}_0(t) = C_0 x_i(t),
\]  
(49)
\[
\hat{x}_0(t) = A_0 \hat{x}_i(t) + B_0 \hat{w}_0(t),
\]  
(50)
\[
\hat{w}_0(t) = C_0 \hat{x}_i(t) + D_0 \hat{w}_0(t),
\]  
(51)
\[
\hat{x}_0(t) = A_0 \hat{x}_0(t) + B_0 \hat{w}_0(t),
\]  
(52)
\[
\hat{w}_0(t) = C_0 \hat{x}_0(t) + D_0 \hat{w}_0(t),
\]  
(53)
respectively. By considering the state space model of \( L(s) \) given in (23) and (24) the augmented system is given by
\[
\dot{\hat{x}}_m(t) = \hat{A} \hat{x}_m(t) + \hat{D} \hat{w}_0(t),
\]  
(54)
where \( \hat{A} \triangleq \begin{bmatrix} A_0 & B_0C_0 \\ 0 & A_0 \end{bmatrix}, \ \hat{x} \triangleq \begin{bmatrix} x_0(t) \\ x_1(t) \end{bmatrix} \) and
\[
\hat{D} \triangleq \begin{bmatrix} B_0D_0 \\ 0 \end{bmatrix}.
\]
Furthermore, define
\[
\lambda_1 \triangleq [C_0 \ 0 \ 0 \ 0], \ \lambda_2 \triangleq [C_0 \ 0 \ 0 \ C_L], \ \lambda_3 \triangleq [D_0C_0 \ 0 \ 0 \ 0],
\]  
(55)
so that \( y_k(t) = C_0 x_m(t), v(t) = C_0 x_m(t) \) and \( y_R(t) = C_0 x_m(t) \).
With the above notation, \( P^e_i, P^e_0 \) and \( P^e_T \) are given by [2]
\[
P^e_i = -(C_0 \hat{Q}_0 C_0^T)_{ii}(0,0),
\]  
(56)
\[
P^e_0 = -(C_0 \hat{Q}_0 C_0^T)_{i0}(0,0),
\]  
(57)
where the steady-state covariance \( \hat{Q}_0 \triangleq \lim_{t \to \infty} \mathbb{E}[x_m(t)x_m^T(t)] \) satisfies
\[
0 = \hat{A} \hat{Q}_0 + \hat{Q}_0 \hat{A}^T + \hat{D} \hat{D}^T.
\]  
(58)
As in the modal subsystem energy flow model, \( P^e_i, P^e_0 \) and \( P^e_T \)
satisfy
\[
P^e_i + P^e_0 + P^e_T = 0,
\]  
(59)
Furthermore if \( L(s) \) is conservative, then
\[
\sum_{i=1}^r P^e_i = 0,
\]  
(60)
while if \( L(s) \) is dissipative, then
\[
\sum_{i=1}^r P^e_i \leq 0.
\]  
(61)
5. Design of an Energy Flow Controller as an Additional Subsystem: Modal Subsystem Model
In this section we consider a control problem involving \( r-1 \) structures interconnected by a stiffness (lossless) coupling and design the controller as an additional subsystem. Now we connect the single-input single-output controller \( z^*_i(s) \) to the structures \( z^*_0(s) \) whose state space model is given by (21) and (22). The additional subsystem, that is, the controller \( z^*_i(s) \) is assumed to be expressed by
\[
z_i(t) = A_0 x_i(t) + B_0 y(t),
\]  
(62)
\[
u(t) = C_0 x_i(t),
\]  
(63)
where \( u(t) \) and \( y(t) \) are scalars and we now assume that the disturbance does not directly enter into the controller \( z^*_i(s) \). Then the augmented feedback representation of the feedback system corresponding to Fig. 3.1 is shown in Fig. 5.1. In Fig. 5.1
\[
E = \begin{bmatrix} E_m & 0 \\ 0 & 1 \end{bmatrix}, \ D = \begin{bmatrix} D_m \\ 0 \end{bmatrix},
\]  
(64)
where \( D_m \) and \( E_m \) are defined by (18), (19), respectively.
As shown in Fig. 5.1 the admittance matrix corresponding to \( z^*_0(s) \) in Fig. 3.1 is now comprised of \( z^*_0(s) \) and \( z^*_i(s) \). In this case the augmented vectors \( u_m(t), u_m(t), w_m(t) \) and \( u_i(t) \) in Fig. 5.1 are defined by
\[
u_m(t) = \begin{bmatrix} y_m(t) \\ u(t) \end{bmatrix}, \ u_m(t) = \begin{bmatrix} u_m(t) \\ y(t) \end{bmatrix},
\]  
(65)
\[
w_m(t) = \begin{bmatrix} w_m(t) \\ 0 \end{bmatrix}, \ \hat{u}_i(t) = \begin{bmatrix} \hat{u}(t) \\ 0 \end{bmatrix},
\]  
(66)
respectively.
On the other hand the stiffness coupling \( L(s) \) is now expressed by
\[
L(s) = \frac{1}{s} C_L.
\]  
(67)
where the symmetric matrix \( C_L \in \mathbb{R}^{r \times r} \) is partitioned as
\[
C_L = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{bmatrix}.
\] (64)

We define the position vectors \( y_m(t) = \int y_m(t) dt = C_{pm} x_m(t) \) for \( Z_m^{-1}(a) \) and a scalar state \( x_p(t) \) by
\[
\dot{x}_p(t) = u(t),
\] (65)
for the controller \( Z_m^{-1}(a) \) so that the output vector \( v_m(t) \) of the stiffness coupling \( L_m(a) \) is given by
\[
v_m(t) = B_m \frac{1}{2} C_L E_{m}^T v_m = E_{C} C_L \begin{bmatrix} C_{pm} E_{m}^T x_m \\
x_p \end{bmatrix}.
\] (66)

By using (61) - (66) the feedback system shown in Fig. 5.1 is expressed as
\[
\dot{x}_m(t) = \tilde{A} x_m(t) + \tilde{B} u(t),
\] (67)
where
\[
\tilde{A} \triangleq \begin{bmatrix}
A_m - B_m E_{m} C_{121} E_{m}^T C_{pm} & -B_m E_{m} C_{132} & 0 \\
0 & 0 & C_c \\
-B_c C_{131} E_{m}^T C_{pm} & -B_c C_{131} & A_c
\end{bmatrix},
\]
\[
\tilde{B} \triangleq \begin{bmatrix}
B_m D_m \\
0 \\
0
\end{bmatrix}, \quad \tilde{x}_m(t) \triangleq \begin{bmatrix}
x_m(t) \\
x_p(t) \\
x_c(t)
\end{bmatrix}.
\]

Furthermore, define
\[
C_m \triangleq \begin{bmatrix} C_m & 0 & 0 \end{bmatrix},
\] (68)
so that \( y_m(t) = C_m x_m(t) \).

We now determine \((A_c, B_c, C_c)\) by means of the LQG positive real approach. By defining \( z(t) = \begin{bmatrix} x_m(t) \\ x_p(t) \end{bmatrix} \) and
\[
A \triangleq \begin{bmatrix}
A_m - B_m E_{m} C_{121} E_{m}^T C_{pm} & -B_m E_{m} C_{132} & 0 \\
0 & 0 & C_c \\
-B_c C_{131} E_{m}^T C_{pm} & -B_c C_{131} & A_c
\end{bmatrix},
\]
\[
C \triangleq -E_{m}^T C_{pm} - C_{122}, \quad D \triangleq \begin{bmatrix} B_m D_m \\
0 \\
0
\end{bmatrix},
\]

it follows that \( \tilde{A} \) and \( \tilde{D} \) in (67) can be expressed as
\[
\tilde{A} = \begin{bmatrix}
A & B C_c \\
B_c C & A_c
\end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D_1 \\ B_c D_0 \end{bmatrix},
\] (69)

where \( D_0 \) in \( \tilde{D} \) represents fictitious measurement noise required by the LQG approach.

Now the controller is required to reduce the vibration of a specified substructure. For this purpose we define the total energy flow through the coupling to all \( n_i \) modes of the \( i \)th structure \( P_i \) given by \( P_i = \sum_{j=1}^{n_i} P_{ij} \), while \( P_i \) and \( P_i^* \) defined by
\[
P_i^* = \sum_{j=1}^{n_i} P_{ij}, \quad P_i = \sum_{j=1}^{n_i} P_{ij}
\]
have a similar interpretation. Furthermore, from (33) it follows that
\[
P_i^* + P_i^* = -N_i^*.
\] (70)

Since \( P_i^* \) represents energy flow entering the \( i \)th structure through the coupling and \( P_i^* \) represents external energy flow entering the \( i \)th structure, it follows that the left hand side of (70) represents the total energy flow entering the \( i \)th structure. Hence by minimizing \( -P_i^* \), we can minimize the total energy flow entering the \( i \)th structure and as a result we can reduce the vibration of the \( i \)th structure.

Now defining the augmented diagonal matrix \( C_{am} \)
\[
C_{am} \triangleq \begin{bmatrix} C_m & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
\] (71)

where \( C_{am} \) is defined (27), and using (70) with (30) and (67) yields
\[
-P_i^* = E_c E_m C_{am} C_{am} E_c^T N_i^* E_c^T C_{am} C_{am} E_c^T N_i^* E_c^T C_{am} C_{am} E_c^T N_i^*,
\]
where the steady-state covariance \( \bar{Q}_m \triangleq \lim_{T \to \infty} E[x_m(t) x_m^T(t)] \) satisfies
\[
0 = A \bar{Q}_m + \bar{Q}_m A^T + D D^T,
\] (72)

and
\[
\bar{C}_1 \triangleq \sum_{j=1}^{n_i} E_{m}^T E_{m} \sum_{j=1}^{n_i} E_{m}^T E_{m} C_{am}.
\] (73)

Thus letting the performance variables have the form
\[z(t) = E_1 z(t) + E_2 u(t),\]

it follows that \( E_1 \) is given by \( E_1 = \bar{C}_1^{1/2} C_{am} \).

6. Example

We design an energy flow controller to serve as a dissipative controller for two simple cantilever beams as shown in Fig. 6.1. The beams are of lengths \( L_1, L_2 \), mass densities \( \rho_1, \rho_2 \), and bending stiessnesses \( E_1 L_1, E_2 L_2 \), respectively. Each beam is subjected to mutually uncorrelated white noise disturbances \( \eta_i(t), i = 1, 2, \) with unit intensity applied at \( \tilde{z}_i \) and with control force from the coupling controller \( f_i(t) \) applied at \( \tilde{z}_i \).

We consider the first three modes of each beam and let \( L_1 = 3, L_2 = 2.5, \rho_1 = \rho_2 = 1, E_1 L_1 = 1, E_2 L_2 = 1.2, \) \( \zeta_j = 0.01, \zeta_j = 0.02, j = 1, 2, 3, \tilde{z}_1 = 1, \tilde{z}_2 = 1.5 \) and \( \zeta_1 = \zeta_2 = 2.2. \)

To reduce the vibration of the \( i \)th beam, \( i = 1, 2 \), we design four controllers. Controllers 1 and 2 are designed by the modal subsystem model to minimize \( -P_i^* \) and \( -P_i^* \), respectively, while Controllers 3 and 4 are designed by the structural subsystem model to minimize \( -P_i^* \) and \( -P_i^* \), respectively. The resulting energy flow diagrams are illustrated in Figs. 6.2 and 6.3 for the modal subsystem model and the structural subsystem model, respectively, where OL denotes the open-loop system and \( G_{am} \) represents Controller \( i \). Figs. 6.2 and 6.3 show that the controller absorbs energy from all of the subsystems and minimizes the energy dissipation from each beam. Furthermore, Controllers 1 and 2 remove maximal energy from beams 1 and 2, respectively, while Controllers 3 and 4 minimize the total energy flow entering beams 1 and 2, respectively.


Fig. 3.1. Feedback Representation of Modal Subsystem Model.

Fig. 4.1. Feedback Representation of Structural Subsystem Model.

Fig. 5.1. Feedback Representation of Plant and Controller (Modal Subsystem Model).

Fig. 6.1. Two Cantilever Beams with Controller Coupling.

Fig. 6.2. Energy Flow between Beams with Controllers $G_{cl}$ and $G_{ct}$ based on the Modal Subsystem Model.

Fig. 6.3. Energy Flow between Beams with Controllers $G_{cl}$ and $G_{ct}$ based on the Structural Subsystem Model.