ABSTRACT

We apply an extension of retrospective cost adaptive control (RCAC) to a command-following problem for the uncertain electromagnetically controlled oscillator (ECO). We assume that an estimate of the first Markov parameter of the discretized and linearized plant is known, but RCAC does not require knowledge of the inertia, damping, or stiffness of the plant. RCAC uses a setpoint feedback path and an auxiliary nonlinearity to stabilize the unstable ECO at the commanded equilibria.

INTRODUCTION

Inverse square laws are ubiquitous in physics, for example, in gravitational, electromagnetic, and electrostatic fields. Although the first of these is not (yet) useful for actuation in control, electromagnetic and electrostatic fields are widely used as a means of actuation. When applied over a fixed gap, electromagnetic actuation is easy to manage; this is the basis of rotary motors. When applied over a variable gap, however, electromagnetic actuation can be challenging to work with. The electromagnetically levitated ball is a staple of control labs [1]. However, the restoring force in this case is uniform gravity and thus is independent of displacement. If, however, the restoring force is provided by a stiffness, then the restoring force depends on the displacement, and this dependence leads to extremely challenging dynamics. We call this system the electromagnetically controlled oscillator (ECO).

Control of the ECO is considered in [2–7] with applications to linear motors in [8]. As shown in [6] the presence of the stiffness leads to unstable equilibria; in fact, for a linear spring, all equilibria beyond one-third of the initial gap are unstable, and these equilibria become increasingly unstable as the gap increases. In addition, as shown in Figure 3, for each equilibrium current, the ECO has two equilibria; consequently, the domain of attraction and transient response of the adaptive controller can lead to convergence to the “wrong” equilibrium. Another complicating factor is the fact that the applied force is proportional to the square of the current, which introduces a quadratic input nonlinearity [9]. A consequence of this quadratic nonlinearity is the fact that the electromagnetic force is able to pull but not push (assuming a nonmagnetic target mass) and thus the actuation is one-sided. The same observations apply to electrostatic actuation, which is used in MEMS devices [10, 11] and flexible antennas [12, 13].

The goal of the present paper is to develop a control law for the ECO that is applicable to the case in which the mass, damping, and stiffness parameters are uncertain and, in addition, does not use detailed knowledge of the quadratic dependence on current and the inverse-quadratic dependence on the distance between the mass and the electromagnet. This goal is motivated by the realistic situation in which estimates of these parameters are uncertain due to measurement, identification, and calibration errors. Consequently, we do not attempt to invert the input nonlinearities as in [6].

The approach that we take in the present paper is based on retrospective cost adaptive control (RCAC). RCAC is a direct digital control approach that requires minimal modeling information about the plant. RCAC was developed for linear systems, but is extended in [14] to the case of Hammerstein systems with uncertain memoryless input nonlinearities. For the ECO we modify the approach of [14] to account for the fact that, for each equilibrium current, the ECO has two equilibria. Consequently, the domain of attraction and transient response of the adaptive controller can lead to convergence to the “wrong” equilibrium.
To counteract this possibility, we introduce a setpoint feedback path to assist RCAC in reaching the desired equilibrium as the position command increases and thus the mass is moved farther into the unstable region.

The contents of the paper are as follows. In Section II, we present the dynamic model of the ECO and illustrate conditions under which the ECO may have zero, one, or two equilibria. In Section III, we linearize the ECO and analyze its local stability. In Section IV, we construct a feedback controller to help the plant output follow the command signal. We apply an extension of RCAC using auxiliary nonlinearities, and we employ a setpoint feedback path to help RCAC adapt to the new commanded equilibrium. Numerical results are presented in Section V, and conclusions are given in Section VI.

1 Equations of Motion and Equilibria of the ECO

Consider the ECO shown in Figure 1, where \( m \) is the mass, \( i \) is the manipulated input current to the electromagnet, \( c > 0 \) is the damping constant, and \( k > 0 \) is the spring constant. The displacement \( q = 0 \) corresponds to the position of the mass where the spring is relaxed, and \( \ell \) is the gap between the electromagnet and the relaxed position of the mass. The dynamics of the oscillator are given by

\[
mq + cq + kq = \frac{\varepsilon i^2}{(\ell - q)^2},
\]

which can be written as

\[
\begin{bmatrix}
\dot{q} \\
\dot{\xi}
\end{bmatrix} = A_c \begin{bmatrix}
q \\
\xi
\end{bmatrix} + B_c \frac{\varepsilon i^2}{(\ell - q)^2},
\]

where

\[
A_c \triangleq \begin{bmatrix}
0 & 1 \\
-k & -c/m
\end{bmatrix}, \quad B_c \triangleq \begin{bmatrix}
0 \\
1/m
\end{bmatrix}.
\]

The parameter \( \varepsilon \) is a force constant needed to render (1) dimensionally correct. For simplicity, we assume \( \varepsilon = 1 \text{ N-m}^2/\text{A}^2 \).

Next, let \( q_{eq} \in (0, \ell) \) denote the desired equilibrium of the ECO. The corresponding equilibrium current \( i_{eq} \) satisfies

\[
\frac{\ell^2}{(\ell - q_{eq})^2} = k q_{eq}.
\]

Conversely, given a constant current \( i_{eq} \), (1) may have zero, one, or two equilibria depending on whether (4) has either zero, one, or two solutions.

**Proposition 1.1.** The following statements hold:

1. If \( i_{eq}^2 > \frac{4}{27} k \ell^3 \), then (1) has no equilibria.
2. If \( i_{eq}^2 = \frac{4}{27} k \ell^3 \), then (1) has a unique equilibrium, which is given by \( q_{eq} = \ell/3 \).
3. If \( 0 < i_{eq}^2 < \frac{4}{27} k \ell^3 \), then (1) has two equilibria, namely, \( q_{eq1} = \frac{\ell}{2} (1 - \cos \frac{\alpha}{2}) \) and \( q_{eq2} = \frac{\ell}{2} (1 + \cos \left( \frac{\alpha}{2} + \frac{\pi}{6} \right) ) \), where \( \alpha = \cos ^{-1} \left( \frac{27 \ell^3}{2k q_{eq}^2} - 1 \right) \).

**Proof.** Let \( f_1(q_{eq}) \triangleq k q_{eq} \) and \( f_2(q_{eq}) \triangleq \frac{i_{eq}^2}{(\ell - q_{eq})^2} \). Then it follows that \( f_1'(q_{eq}) = k \) and \( f_2'(q_{eq}) = 2 k q_{eq} / \ell - q_{eq} \in (0, \infty) \). Furthermore, \( f_1'(q_{eq}) = f_2'(q_{eq}) = k \) if and only if \( q_{eq} = \ell/3 \). Therefore, \( f_1(q_{eq}) \) has no intersection with \( f_2(q_{eq}) \) if and only if \( f_1(\ell/3) < f_2(\ell/3) \), one intersection point if and only if \( f_1(\ell/3) = f_2(\ell/3) \), and two intersection points if and only if \( f_1(\ell/3) > f_2(\ell/3) \).

**Proposition 1.2.** Assume that \( 0 < i_{eq}^2 < \frac{4}{27} k \ell^3 \). Then, \( q_{eq1} < \ell/3 \) and \( q_{eq2} > \ell/3 \).

**Proof.** Since \(-1 < \left( \frac{27 \ell^3}{2k q_{eq}^2} - 1 \right) < 1 \). It follows that \( 0 < \alpha < \pi \). Hence \( \alpha / 2 \in (0, \pi / 2) \) and thus \( \alpha / 2 + \pi / 6 \in (\pi / 2, 7 \pi / 6) \). Furthermore, \( \cos \frac{\alpha}{2} \in (1/2, 1) \) and \( \cos \left( \frac{\alpha}{2} + \frac{\pi}{6} \right) \in (-1/2, 1/2) \). Therefore \( q_{eq1} = \frac{\ell}{2} (1 - \cos \frac{\alpha}{2}) \in (0, \ell/3) \) and \( q_{eq2} = \frac{\ell}{2} (1 + \cos \left( \frac{\alpha}{2} + \frac{\pi}{6} \right) ) \in (\ell/3, \ell) \).

Proposition 1.1 and Proposition 1.2 are illustrated in Figure 2.

2 Linearization, Local Stability Analysis, and Discretization of the ECO

In this section, we linearize (1) around an equilibrium \( q_{eq} \), analyze the local stability, and discretize the linearized plant. Linearizing (1) around \( q = q_{eq} \) yields

\[
\begin{bmatrix}
\dot{\xi} \\
\dot{\xi}
\end{bmatrix} = A_1 \begin{bmatrix}
\xi \\
\xi
\end{bmatrix} + B_1 \delta i,
\]

Figure 1. Schematic of the electromagnetically controlled oscillator.
$q_{eq} = \ell/3 = 1 \text{ m}; \text{ and } i_{eq} \text{ decreases as the mass moves farther into the unstable region to the right of } \ell/3.$ Meanwhile, note that the unstable equilibria become increasingly unstable as the mass moves farther to the right of $\ell/3$.

3 Command-Following Problem for the ECO

We now consider the ECO command-following problem shown in Figure 4. We apply a feedforward/feedback controller to have the output $y$ follow the command signal $r$. The goal is to develop an adaptive feedforward/feedback controller that minimizes the command-following error $z$ in the presence of the command signal $r$ with minimal modeling information about the dynamics of the ECO. For the feedforward path, the controller uses a measurement of the command $r$. For the feedback path, we apply RCAC to the ECO assuming that the state $q$ is available for feedback.
\[ A = e^{A_1 T_s} = \begin{bmatrix} e^{-\omega n T_s} & \cos \gamma T_s + \omega n \sin \gamma T_s & \frac{1}{2} \sin \gamma T_s \\ \frac{\ell - 3 q_{eq}}{\ell - q_{eq}} & \frac{\omega n T_s}{\ell - q_{eq}} & \cos \gamma T_s - \omega n \sin \gamma T_s \end{bmatrix}, \quad \xi^2 < \frac{\ell - 3 q_{eq}}{\ell - q_{eq}}, \]

\[ B = \left( \int_0^T e^{A_1 t} \right) B_1 = \begin{bmatrix} \frac{m}{c} & 0 & m \varepsilon \\ 0 & -1 & 0 \end{bmatrix} (A - I) B_1 + \begin{bmatrix} \sqrt{3k/\ell^2 T_s} \\ 0 \end{bmatrix}, \quad q_{eq} = \ell/3. \]

### 3.1 Auxiliary Nonlinearity

Define the saturation function \( \text{sat}_a \) by

\[ \text{sat}_a(x) = \begin{cases} -a, & x < -a, \\ x, & -a \leq x \leq a, \\ a, & x > a, \end{cases} \]

where \( a > 0 \) is the saturation level.

#### 3.2 Offset Current \( i_{\text{offset}} \)

Let \( r \) be a nondecreasing sequence of step commands, that is, \( r(k_1) \leq r(k_2) \) for all \( k_1 < k_2 \). Then \( i_{\text{offset}}(k) \) is given by

\[ i_{\text{offset}}(k) = \begin{cases} 0, & 0 < r(k) \leq \ell/3, \\ pe^{-\frac{\alpha}{|q(k) - r(k)|}}, & \ell/3 < r(k) < \ell, \end{cases} \]

where \( p \geq 0, \alpha > 0, \beta > 0, \) and \( q(k) \) is the position of the mass at time step \( k \).

As an example, consider \( p = 1, \alpha = 1, \beta = 1 \), and \( r(k) = \ell/2 \), where \( \ell = 3 \) m. Figure 6 shows the offset current \( i_{\text{offset}} \) corresponding to each mass position \( q(k) \). Note that the offset current is nonzero except for \( q(k) = r(k) \). The offset current increases as the distance between current mass position and commanded mass position increases.

### 4 Numerical Examples

We now use RCAC with the auxiliary nonlinearity \( N_1 \) and the offset current \( i_{\text{offset}} \) to control the position of the mass. In particular, we consider the command-following problem with the step command \( r = q_{eq} \geq \ell/3 \).
The adaptive controller requires an estimate of the first nonzero Markov parameter of the linearized plant (9). This Markov parameter is used to implement the retrospective optimization (26). RCAC generates the control signal $u_c$, which is added to the offset current $i_{\text{offset}}$.

For simulation we consider $m = 1$ kg, $k = 5$ N/m, $c = 5$ N-s/m, and $\ell = 3$ m with a sample time $T_s = 0.01$ sec. Hence $\omega_0 = 2.2361$ rad/s and $\zeta = 1.1180$. First, numerical simulations are performed for the constant command input $q_{eq} = \ell/3 = 1.0$ m. The first nonzero Markov parameter of (5) is $H_1(q_{eq}) \triangleq CB$, where $B$ is defined in (11) and $C \triangleq \begin{bmatrix} 1 & 0 \end{bmatrix}$. We choose $H_1(q_{eq}) = H_1(1) = 1.0996 \times 10^{-4}$ mA. Figure 7 shows the dependence of $H_1$ on the equilibria of the ECO. We initialize the control gains to zero, and we choose the controller order $n_c = 8$ and the covariance matrix $P(0) = 10^{-9}I_{3n_c}$. Furthermore, since the linearized model is minimum phase, we choose the regularization $\eta = 0$. Finally, we set $\rho = 0$ so that $i_{\text{offset}} = 0$, and we do not use a forgetting factor in the adaptive controller, that is, $\lambda = 1$. Figure 8 shows that the controller stabilizes the plant and follows the command input. Figure 9 shows the time history of the control input $u_c$. It follows from Proposition 1.1 that the steady-state value of the current $i = u_c$ is the maximum current such that (1) has an equilibrium.

Next, we do not assume that $H_1(1)$ is known exactly [15]. Figure 10 shows the position of the mass with various estimates $\hat{H}_1(1)$ of $H_1(1)$, the RCAC controller is able to stabilize the plant and follow the step command with erroneous estimates of $H_1(1)$. However, the best overall performance for both the transient response and the convergent time is obtained for $\hat{H}_1(1) = H_1(1).

Now, we implement the adaptive controller with a nondecreasing sequence of setpoint commands as shown by Figure 11. To do this, we set $i_{\text{offset}}$ based on (6) when $r(k) > \ell/3$. In particular, we choose $\rho = 1$, $\alpha = 1$, $\beta = 1$, $\hat{H}_1(1) = H_1(1)$, $n_c = 8$, and initialize the control gains to zero. Figure 11 shows that the control algorithm is able to stabilize the system up to $q_{eq} = 1.79$. Figure 12(a) shows the time history of the current offset $i_{\text{offset}}$, and Figure 12(b) shows the time history of the control input $u_c$ from the RCAC.

Finally, we reduce the damping coefficient so that $c = 4$ N-s/m, and thus the ECO is underdamped with $\zeta = 0.8944$. Following the same procedure, and using the same parameters for initializing RCAC, Figure 13 shows that RCAC is able to stabilize the underdamped system up to $q_{eq} = 1.79$. Figure 14(a) shows the time history of the current offset $i_{\text{offset}}$, and Figure 14(b) shows the time history of the control input $u_c$ from RCAC. Note that, in this case, the transient response for the open-loop underdamped ECO system is worse than the response in the open-loop overdamped case. Figure 15 shows the largest distance the mass...
can be moved by the feedforward/feedback controller versus the open-loop damping ratio of the ECO system. Note that, in all those cases, we choose $\hat{H}_1 = H(1)$.

Finally, to demonstrate the potential benefits of scheduling the Markov parameters as a function of $q_{eq}$, we consider the same example shown in Figure 11. Since the Markov parameter increases as the mass moves farther into the unstable region (in Figure 7), we thus let $\hat{H}_1 = H(1)$ for $q_{eq} \in (0,1.7)$ and $\hat{H}_1 = 1.2H(1)$ for $q_{eq} \geq 1.7$. Figure 16 shows that RCAC is able to stabilize the system up to $q_{eq} = 1.815$. 

"Figure 11. Position of the mass with a nondecreasing sequence of step commands for $m = 1$ kg, $\ell = 3$ m, $c = 5$ N-s/m, and $k = 5$ N/m, where $\zeta = 1.1180$. In order to stabilize the mass close to the electromagnet, the command signal is a nondecreasing sequence of step commands, which is shown as the red dash line. Note that all equilibria greater than $q_{eq} = 1$ are open-loop unstable. In this simulation, we choose $\hat{H}_1 = H(1)$."

"Figure 12. Current offset $i_{offset}$ (a) and control input $u_c$ (b) corresponding to the closed-loop response shown in Figure 11."

"Figure 13. Position of the mass with a nondecreasing sequence of step commands for $m = 1$ kg, $\ell = 3$ m, $c = 4$ N-s/m, and $k = 5$ N/m, where $\zeta = 0.8944$. In order to stabilize the mass close to the electromagnet, the command signal is a nondecreasing sequence of step commands, which is shown as the red dash line. Note that we choose $\hat{H}_1 = H(1)$, and all equilibria greater than $q_{eq} = 1$ are open-loop unstable. In this case, which is underdamped, the transient response is worse than the response in Figure 11."
was used with limited modeling information, namely, an estimate of the first nonzero Markov parameter of the linearized system. To handle the effect of the nonlinearities and the unstable region of the ECO, RCAC was augmented by an auxiliary nonlinearity. An equilibrium feedback path was also used to assist RCAC in reaching the desired unstable equilibrium. Future research will focus on the effect of noise and sample rate as well as the potential benefits of scheduling the Markov parameters as a function of $q_{eq}$.

5 Conclusions

In this paper, we considered a command-following problem for the electromagnetically controlled oscillator (ECO). RCAC

REFERENCES

the Markov parameter \( H_j \triangleq E_1 A^{j-1} B \), and

\[
\bar{U}(k-1) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-m) \end{bmatrix}
\]

Next, we present an adaptive control algorithm for the general control problem represented by (14)–(16). The control \( u(k) \) is given by the strictly proper time-series controller of order \( n_c \)

\[
u(k) = \sum_{i=1}^{n_c} M_i(k) u(k-i) + \sum_{i=1}^{n_c} N_i(k) z(k-i), \quad (18)
\]

where, for all \( i = 1, \ldots, n_c \), \( M_i(k) \in \mathbb{R}^{l_u \times l_v} \) and \( N_i(k) \in \mathbb{R}^{l_v \times l_u} \). The control (18) can be expressed as

\[

u(k) = \theta(k) \phi(k-1), 
\]

where

\[

\theta(k) \triangleq \left[ M_1(k) \cdots M_{n_c}(k) \; N_1(k) \cdots N_{n_c}(k) \right] 
\in \mathbb{R}^{l_v \times (l_u + l_c)}
\]

and

\[

\phi(k-1) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ z(k-1) \\ \vdots \\ z(k-n_c) \end{bmatrix} \in \mathbb{R}^{n_c (l_u + l_v)}.
\]

Next, we define the surrogate performance

\[

\bar{z}(k-k_j) \triangleq S_j (k-k_j) + H_j \bar{U}_j (k-k_j-1), \quad (21)
\]

where \( S_j \triangleq E_1 A^m x(k-m) - E_0 r(k) + H_j' U_j (k-1) \), and the past controls \( U_j (k-k_j-1) \) are replaced by the surrogate controls \( \bar{U}_j (k-k_j-1) \). Now, we express extended surrogate performance

\[

\bar{z}(k-k_j) \triangleq S_j (k-k_j) + \bar{H}_j \bar{U}_j (k-k_j-1),
\]

where \( \bar{H}_j \triangleq H_j + H_j' H_j \), and
as
\[
\hat{Z}(k) \triangleq \begin{bmatrix} \hat{z}(k-k_1) \\ \vdots \\ \hat{z}(k-k_s) \end{bmatrix} \in \mathbb{R}^{sl_z}
\]
and thus is given by
\[
\hat{Z}(k) = \tilde{S}(k) + \tilde{H} \hat{\tilde{U}}(k-1),
\]
where \(\hat{Z}(k-1)\) is the retrospective cost function. To ensure that (24) has a global minimizer, we consider the regularized cost
\[
J_R(\theta(k)) \triangleq \sum_{i=d+1}^{k} \lambda^{k-i} \| \theta^T(i-d-1) \theta^T(k-d) - \hat{u}^T(i-d) \| ^2 + \lambda^i \theta(k-\theta(0)) P^{-1}(0)(\theta(k)-\theta(0))^T,
\]
where \(\lambda \in (0,1)\) is the forgetting factor. Minimizing (27) yields
\[
\theta(k) = \theta(k-1) + P(k-1) \phi(k-d-1) \\
\cdot [\theta^T(k-d) P(k-1) \phi(k-d-1) + \lambda(k)]^{-1} \\
\cdot [\theta^T(k-d-1) \theta^T(k-1) - \hat{u}^T(k-d)],
\]
where \(\theta(k)\) is the unique global minimizer
\[
\hat{U}(k-1) = -\frac{1}{2} \mathcal{A}^{-1}(k) \mathcal{B}(k).
\]
Next, let \(d > 0\) be such that \(\hat{U}(k-1)\) contains \(u(k-d)\) and define the retrospective cost function
\[
\hat{J}(\hat{U}(k-1),k) \triangleq \hat{Z}^T(k) R(k) \hat{Z}(k),
\]
where \(R(k) \in \mathbb{R}^{l_1 \times l_1}\) is a positive-definite performance weighting. To ensure that (24) has a global minimizer, we consider the regularized cost
\[
J(\hat{U}(k-1),k) \triangleq \hat{Z}^T(k) R(k) \hat{Z}(k),
\]
where \(\eta(k) \geq 0\). Substituting (23) into (25) yields
\[
\hat{J}(\hat{U}(k-1),k) = \hat{U}(k-1)^T \mathcal{A}(k) \hat{U}(k-1) + \mathcal{B}(k) \hat{U}(k-1) \\
+ \mathcal{C}(k),
\]
where
\[
\mathcal{A}(k) \triangleq \tilde{H}^T R(k) \tilde{H} + \eta(k) I_{l_1}, \\
\mathcal{B}(k) \triangleq 2 \tilde{H}^T R(k) \hat{Z}(k-\tilde{H} \hat{U}(k-1)), \\
\mathcal{C}(k) \triangleq \hat{Z}^T(k) R(k) \hat{Z}(k-2 \tilde{H}^T(k) R(k) \tilde{H} \hat{U}(k-1) \\
+ \hat{U}(k-1)^T \tilde{H}^T R(k) \tilde{H} \hat{U}(k-1).
\]
If either \(\tilde{H}\) has full column rank or \(\eta(k) > 0\), then \(\mathcal{A}(k)\) is positive definite. In this case, \(\hat{J}(\hat{U}(k-1),k)\) has the unique global minimizer
\[
\hat{U}(k-1) = -\frac{1}{2} \mathcal{A}^{-1}(k) \mathcal{B}(k).
\]