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Cumulative Retrospective Cost Adaptive Control with RLS-Based Optimization

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***Abstract*— We present a discrete-time adaptive control algorithm that is effective for multi-input, multi-output systems that are either minimum phase or nonminimum phase. The adaptive control algorithm requires limited model information, specifically, the first nonzero Markov parameter and the nonminimum-phase zeros of the transfer function from the control signal to the performance measurement. Furthermore, the adaptive control algorithm is effective for stabilization as well as command following and disturbance rejection, where the command and disturbance spectrum is unknown. The novel aspect of this adaptive controller is the use of a retrospective performance function which is optimized using a recursive least-squares algorithm.**

I. INTRODUCTION

One of the major challenges in direct adaptive control is the existence of nonminimum-phase zeros. More specifically, many direct adaptive control methodologies rely on the assumption that the plant is minimum phase [1]–[5], while other invoke the stronger assumption that the plant is passive or positive real [1]–[3]. With regard to command following and disturbance rejection, many adaptive controllers rely on assumptions regarding the spectrum of the commands to be followed and disturbances to be rejected. More specifically, it is commonly assumed that the commands and disturbances have known spectrum and/or the disturbances are measured directly [6], [7]. Furthermore, for disturbance rejection problems, many adaptive control methods require that the range of the disturbance input matrix is contained in the range of the control input matrix, meaning that the disturbance can be rejected directly by the input without using the system dynamics [5], [6].

In the present paper, we present a discrete-time adaptive control algorithm that addresses several of these common challenges in adaptive control. More specifically, the adaptive controller presented in this paper is effective for plants that are either minimum phase or nonminimum phase, provided that we have estimates of the nonminimum-phase zeros. Furthermore, this adaptive controller does not require that the disturbance input matrix is matched to the control input matrix. Finally, this adaptive controller is effective for command following and disturbance rejection where the spectrum of the commands and disturbances is unknown and the disturbance is unmeasured.

Although the discrete-time adaptive control literature is less extensive than the continuous-time literature, discrete-time versions of many continuous-time algorithms are available [2], [4], [8]–[10]. In addition, there are adaptive control algorithms that are unique to discrete-time [4], [11]–[13]. In [4], [11], discrete-time adaptive control laws are presented for stabilization and command following of minimum-phase systems based on the assumption that the commands are known a priori and that an ideal tracking controller exists. An extension is given in [12], which addresses the combined stabilization, command following, and disturbance rejection problem. Note that the results of [4], [11], [12] are restricted to minimum-phase systems. For nonminimum-phase systems, [13] shows that periodic control may be used; however, this adaptive control scheme requires periods of open-loop operation.

Another class of discrete-time adaptive controllers use a retrospective cost [14], [15]. These retrospective cost adaptive controllers are known to be effective for systems that are either minimum phase or nonminimum phase provided that knowledge of the nonminimum-phase zeros is available. Retrospective cost adaptive control uses a retrospective performance measurement, in which the performance measurement is modified based on the difference between the actual past control inputs and the recomputed past control inputs, assuming that the current controller had been used in the past. Retrospective cost adaptive controllers have been demonstrated on various experiments and applications, including the Air Force's deployable optical telescope testbed in [16], the NASA generic transport model in [17], and flow control problems in [18].

The adaptive laws of [14], [15] are derived by minimizing an instantaneous retrospective cost, which is a function of the retrospective performance at the current time. In this paper, we present an adaptive control algorithm that is based on a cumulative retrospective cost function. This cumulative retrospective cost is a function of the retrospective performance at the current time step and all previous time steps. Using a cumulative retrospective cost function, which is minimized by a recursive least-squares algorithm, can result in improved transient performance as compared to the instantaneous retrospective cost used in [14], [15].

II. PROBLEM FORMULATION

Consider the multi-input, multi-output discrete-time system

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k), \tag{1}$$

$$y(k) = Cx(k) + Du(k) + D_2 w(k), \tag{2}$$

$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k), \tag{3}$$

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where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^{l_y}$, $z(k) \in \mathbb{R}^{l_z}$, $u(k) \in \mathbb{R}^{l_u}$, $w(k) \in \mathbb{R}^{l_w}$, and $k \geq 0$. Our goal is to develop an adaptive output feedback controller that generates a control signal u that minimizes the performance z in the presence of the exogenous signal w. We assume that measurements of the output y and the performance z are available for feedback; however, we assume that a direct measurement of the exogenous signal w is not available.

Note that w can represent either a command signal to be followed, an external disturbance to be rejected, or both. For example, if $D_1 = 0$, $E_2 = 0$, and $E_0 \neq 0$, then the objective is to have the output $E_1 x$ follow the command signal $-E_0 w$. On the other hand, if $D_1 \neq 0$, $E_2 = 0$, and $E_0 = 0$, then the objective is to reject the disturbance w from the performance measurement $E_1 x$. The combined command following and disturbance rejection problem is addressed when D_1 and E_0 are block matrices. Lastly, if D_1 and E_0 are empty matrices, then the objective is output stabilization, that is, convergence of z to zero.

Furthermore, note that the performance variable z can include the feedthrough term $E_2 u$. This term allows us to design an adaptive controller where the performance z to be minimized can include a weighting on control authority.

We represent (1) and (3) as the time-series model from u and w to z given by

$$z(k) = \sum_{i=1}^{n} -\alpha_i z(k-i) + \sum_{i=d}^{n} \beta_i u(k-i) + \sum_{i=0}^{n} \gamma_i w(k-i), \tag{4}$$

where $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\beta_d, \ldots, \beta_n \in \mathbb{R}^{l_z \times l_u}$, $\gamma_0, \ldots, \gamma_n \in \mathbb{R}^{l_z \times l_w}$, and the relative degree d is the smallest non-negative integer i such that the ith Markov parameter, either $H_0 \triangleq E_2$ if $i = 0$ or $H_i \triangleq E_1 A^{i-1} B$ if $i > 0$, is nonzero. Note that $\beta_d = H_d$.

III. Controller Construction

In this section, we construct an adaptive control algorithm for the general control problem represented by (1)-(3). We use a strictly proper time-series controller of order n_c, such that the control $u(k)$ is given by

$$u(k) = \sum_{i=1}^{n_c} M_i(k) u(k-i) + \sum_{i=1}^{n_c} N_i(k) y(k-i), \tag{5}$$

where, for all $i = 1, \ldots, n_c$, $M_i : \mathbb{N} \to \mathbb{R}^{l_u \times l_u}$ and $N_i : \mathbb{N} \to \mathbb{R}^{l_u \times l_y}$ are determined by the adaptive control law presented below. The control (5) can be expressed as

$$u(k) = \theta(k)\phi(k),$$

where

$$\theta(k) \triangleq \begin{bmatrix} N_1(k) & \cdots & N_{n_c}(k) & M_1(k) & \cdots & M_{n_c}(k) \end{bmatrix},$$

and

$$\phi(k) \triangleq \begin{bmatrix} y^{\mathrm{T}}(k-1) & \cdots & y^{\mathrm{T}}(k-n_c) \\ u^{\mathrm{T}}(k-1) & \cdots & u^{\mathrm{T}}(k-n_c) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_c(l_u+l_y)}.$$

Next, we define the *retrospective performance*

$$\hat{z}(\hat{\theta}, k) \triangleq z(k) + \sum_{i=d}^{\nu} \bar{\beta}_i \left[\hat{\theta} - \theta(k-i)\right] \phi(k-i), \tag{6}$$

where $\nu \geq d$, $\hat{\theta} \in \mathbb{R}^{l_u \times (n_c(l_y+l_u))}$ is an optimization variable used to derive the adaptive law, and $\bar{\beta}_d, \ldots, \bar{\beta}_\nu \in \mathbb{R}^{l_z \times l_u}$. The choice of ν and $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ is discussed in sections IV and V. Defining $\hat{\Theta} \triangleq \mathrm{vec}\, \hat{\theta} \in \mathbb{R}^{n_c l_u (l_y+l_u)}$ and $\Theta(k) \triangleq \mathrm{vec}\, \theta(k) \in \mathbb{R}^{n_c l_u (l_y+l_u)}$, it follows that

$$\begin{aligned} \hat{z}(\hat{\Theta}, k) &= z(k) + \sum_{i=d}^{\nu} \Phi_i^{\mathrm{T}}(k) \left[\hat{\Theta} - \Theta(k-i)\right] \\ &= z(k) - \sum_{i=d}^{\nu} \Phi_i^{\mathrm{T}}(k)\Theta(k-i) + \Psi^{\mathrm{T}}(k)\hat{\Theta}, \end{aligned} \tag{7}$$

where, for $i = d, \ldots, \nu$, $\Phi_i(k) \triangleq \phi(k-i) \otimes \bar{\beta}_i^{\mathrm{T}} \in \mathbb{R}^{(n_c l_u (l_y+l_u)) \times l_z}$, where $\otimes$ represents the Kronecker product, and $\Psi(k) \triangleq \sum_{i=d}^{\nu} \Phi_i(k)$.

Now, define the *cumulative retrospective cost function*

$$\begin{aligned} J(\hat{\Theta}, k) \triangleq \sum_{i=0}^{k} \lambda^{k-i} \hat{z}^{\mathrm{T}}(\hat{\Theta}, i) R \hat{z}(\hat{\Theta}, i) \\ + \lambda^k (\hat{\Theta} - \Theta(0))^{\mathrm{T}} Q (\hat{\Theta} - \Theta(0)), \end{aligned} \tag{8}$$

where $\lambda \in (0, 1]$, and $R \in \mathbb{R}^{l_z \times l_z}$ and $Q \in \mathbb{R}^{(n_c l_u (l_y+l_u)) \times (n_c l_u (l_y+l_u))}$ are positive definite. Note that λ serves as a forgetting factor, which allows more recent data to be weighted more heavily than past data.

The cumulative retrospective cost function (8) is minimized by a recursive least-squares (RLS) algorithm with a forgetting factor [2], [4], [5]. Therefore, $J(\hat{\Theta}, k)$ is minimized by the adaptive law

$$\Theta(k+1) = \Theta(k) - P(k)\Psi(k)\Omega(k)^{-1} z_{\mathrm{R}}(k), \tag{9}$$

$$P(k+1) = \frac{1}{\lambda} P(k) - \frac{1}{\lambda} P(k)\Psi(k)\Omega(k)^{-1}\Psi^{\mathrm{T}}(k)P(k), \tag{10}$$

where $\Omega(k) \triangleq \lambda R^{-1} + \Psi^{\mathrm{T}}(k)P(k)\Psi(k)$, $P(0) = Q^{-1}$, $\Theta(0) \in \mathbb{R}^{n_c l_u (l_y+l_u)}$, and the *retrospective performance measurement* $z_{\mathrm{R}}(k) \triangleq \hat{z}(\Theta(k), k)$. Note that the retrospective performance measurement is computable from (7) using measured signals z, y, u, θ, and the matrix coefficients $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$. The cumulative retrospective cost adaptive control law is thus given by (9), (10), and

$$u(k) = \theta(k)\phi(k) = \mathrm{vec}^{-1}(\Theta(k))\phi(k). \tag{11}$$

The key feature of the adaptive control algorithm is the use of the retrospective performance (7), which modifies the performance variable $z(k)$ based on the difference between the actual past control inputs $u(k-d), \ldots, u(k-\nu)$ and the recomputed past control inputs $\hat{u}(\hat{\Theta}, k-d) \triangleq \mathrm{vec}^{-1}(\hat{\Theta})\phi(k-d), \ldots, \hat{u}(\hat{\Theta}, k-\nu) \triangleq \mathrm{vec}^{-1}(\hat{\Theta})\phi(k-\nu)$, assuming that the current controller $\hat{\Theta}$ had been used in the past. In next two sections, we discuss how to select $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$.

IV. $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ for Minimum-Phase Systems

Consider the case where the transfer function from u to z is minimum phase, that is, the invariant zeros of (A, B, E_1, E_2) are contained inside of the unit circle. In this case, it is shown in [12] that the controller requires only a single Markov parameter, namely, H_d. More specifically, we let $\nu = d$ and $\bar{\beta}_d = H_d$. Under the minimum-phase assumption, [12] proves asymptotic convergence of z to zero.

V. $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ for Nonminimum-Phase Systems

Consider the case where the transfer function from u to z is nonminimum phase, that is, the invariant zeros of (A, B, E_1, E_2) are not all contained inside of the unit circle. For nonminimum-phase systems, we present three constructions for the parameters $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$.

A. Controller Construction Using Numerator Coefficients

First, consider the case where $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ are the coefficients of the numerator polynomial matrix of the transfer function from u to z, that is, $\nu = n$ and, for $i = d, \ldots, n$, $\bar{\beta}_i = \beta_i$.

B. Controller Construction Using Nonminimum-Phase Transmission Zeros

The results of [12] for the minimum-phase case suggests that we require knowledge of only the first nonzero Markov parameter and the nonminimum-phase transmission zeros of the transfer function from u to z. In this section, we choose $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ to capture this information. Consider the matrix transfer function from u to z given by $G_{zu}(\mathbf{z}) \stackrel{\triangle}{=} \frac{1}{\alpha(\mathbf{z})}\beta(\mathbf{z})$, where $\alpha(\mathbf{z}) \stackrel{\triangle}{=} \mathbf{z}^n + \alpha_1\mathbf{z}^{n-1} + \cdots + \alpha_{n-1}\mathbf{z} + \alpha_n$ and $\beta(\mathbf{z}) \stackrel{\triangle}{=} \mathbf{z}^{n-d}\beta_d + \mathbf{z}^{n-d-1}\beta_{d+1} + \cdots + \mathbf{z}\beta_{n-1} + \beta_n$. Next, let $\beta(\mathbf{z})$ have the polynomial matrix factorization $\beta(\mathbf{z}) = \beta_{\rm U}(\mathbf{z})\beta_{\rm S}(\mathbf{z})$, where $\beta_{\rm U}(\mathbf{z})$ is a polynomial matrix of degree $n_{\rm U} \ge 0$ whose leading matrix coefficient is β_d, $\beta_{\rm S}(\mathbf{z})$ is a monic polynomial matrix of degree $n - n_{\rm U} - d$, and each Smith zero of $\beta(\mathbf{z})$ counting multiplicity that lies on or outside the unit circle is a Smith zero of $\beta_{\rm U}(\mathbf{z})$. More precisely, if $\lambda \in \mathbb{C}$, $|\lambda| \ge 1$, and rank $\beta(\lambda) <$ normal rank $\beta(\mathbf{z})$, then rank $\beta_{\rm U}(\lambda) <$ normal rank $\beta_{\rm U}(\mathbf{z})$ and rank $\beta_{\rm S}(\lambda) =$ normal rank $\beta_{\rm S}(\mathbf{z})$. Furthermore, we can write $\beta_{\rm U}(\mathbf{z}) = \beta_{{\rm U},0}\mathbf{z}^{n_{\rm U}} + \beta_{{\rm U},1}z^{n_{\rm U}-1} + \cdots + \beta_{{\rm U},n_{\rm U}-1}z + \beta_{{\rm U},n_{\rm U}}$, where $\beta_{{\rm U},0} \stackrel{\triangle}{=} \beta_d$. In this case, we let $\nu = n_{\rm U} + d$ and for $i = d, \ldots, n_{\rm U} + d$, $\bar{\beta}_i = \beta_{{\rm U},i-d}$.

C. Controller Construction Using Markov Parameters

Consider the μ-MARKOV model of (4) obtained from μ successive back-substitutions of (4) into itself, and given by

$$\begin{aligned} z(k) = &-\sum_{i=1}^{n}\alpha_{\mu,i}z(k-\mu-i) + \sum_{i=d}^{\mu}H_{zu,i}u(k-i) \\ &+\sum_{i=1}^{n}\beta_{\mu,i}u(k-\mu-i) + \sum_{i=0}^{\mu}H_{zw,i}w(k-i) \\ &+\sum_{i=1}^{n}\gamma_{\mu,i}w(k-\mu-i), \end{aligned} \tag{12}$$

where $\alpha_{\mu,i} \in \mathbb{R}$, $\beta_{\mu,i} \in \mathbb{R}^{l_z \times l_u}$, $\gamma_{\mu,i} \in \mathbb{R}^{l_z \times l_w}$, $H_{zu,i} \in \mathbb{R}^{l_z \times l_u}$, $H_{zw,i} \in \mathbb{R}^{l_z \times l_w}$, and $\mu \ge d$. Thus, the μ-MARKOV transfer function from u to z is given by

$$\begin{aligned} G_{zu,\mu}(\mathbf{z}) =& \frac{1}{p_\mu(\mathbf{z})}\left(H_{zu,d}\mathbf{z}^{\mu+n-d} + \cdots + H_{zu,\mu}\mathbf{z}^n\right) \\ &+ \frac{1}{p_\mu(\mathbf{z})}\left(\beta_{\mu,1}\mathbf{z}^{n-1} + \cdots + \beta_{\mu,n}\right), \end{aligned} \tag{13}$$

where $p_\mu(\mathbf{z}) \stackrel{\triangle}{=} \mathbf{z}^{\mu+n} + \alpha_{\mu,1}\mathbf{z}^{n-1} + \cdots + \alpha_{\mu,n}$.

The Laurent series expansion of $G_{zu}(\mathbf{z})$ about $\mathbf{z} = \infty$ is $G_{zu}(\mathbf{z}) = \sum_{i=d}^{\infty}\mathbf{z}^{-i}H_{zu,i}$. Truncating the numerator and denominator of (13) is equivalent to the truncated Laurent series expansion of $G_{zu}(\mathbf{z})$ about $\mathbf{z} = \infty$. Thus, the truncated Laurent series expansion of $G_{zu}(\mathbf{z})$ is $\bar{G}_{zu,\mu}(\mathbf{z}) \stackrel{\triangle}{=} \sum_{i=d}^{\mu}\mathbf{z}^{-i}H_{zu,i}$.

Note that, for a single-input, single-output system, a subset of the roots of the polynomial $H(\mathbf{z}) \stackrel{\triangle}{=} \mathbf{z}^{\mu-d}H_{zu,d} + \mathbf{z}^{\mu-d-1}H_{zu,d+1} + \cdots + \mathbf{z}H_{zu,\mu-1} + H_{zu,\mu}$ can be shown to approximate the nonminimum-phase zeros from u to z that lie outside of a circle in the complex plane centered at the origin with radius equal to the spectral radius of A. Thus, knowledge of $H_{zu,d}, \ldots, H_{zu,\mu}$ encompasses knowledge of the nonminimum-phase zeros from u to z that lie outside of the spectral radius of A.

Therefore, we present a variation of the cumulative retrospective cost adaptive controller (9)-(11) that uses only the Markov parameters $H_{zu,d}, \ldots, H_{zu,\mu}$. In this case, we let $\nu = \mu$ and for $i = d, \ldots, \mu$, $\bar{\beta}_i = H_{zu,i}$. This choice of $\bar{\beta}_d, \ldots, \bar{\beta}_\nu$ works well provided that $\mu \ge d$ is chosen large enough so that roots of $H(\mathbf{z})$ approximate the nonminimum-phase zeros from u to z.

VI. Simulation Results

In this section, we present numerical examples to demonstrate the cumulative retrospective cost adaptive controller. In all simulations, we initialize the adaptive controller to zero, that is, $\theta(0) = 0$. Unless otherwise stated, the numerical examples in this section are constructed as follows.

(i) We assume that the performance equals the output measurement, that is, $z = y$.

(ii) We do not use a forgetting factor, that is, $\lambda = 1$.

(iii) The exogenous command and disturbance signal $w(k) \stackrel{\triangle}{=} [\, w_1(k) \quad w_2(k) \quad w_3(k) \,]^{\rm T}$, where, for $i = 1, 2, 3$, $w_i(k) \stackrel{\triangle}{=} A_i \sin(2\pi\omega_i T_{\rm s} k) + b_i$, where $A_1 = 6$, $A_2 = 8$, and $A_3 = 10$; $\omega_1 = 5$ Hz, $\omega_2 = 10$ Hz, and $\omega_3 = 15$ Hz; $b_1 = 0$, $b_2 = 0$, and $b_3 = 20$; and $T_{\rm s} = 0.002$ seconds.

(iv) All transfer functions from u to z are realized in controllable canonical form, where

$$D_1 = \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline \multicolumn{3}{c}{0_{(n-3)\times 3}} \end{array}\right],$$

$$E_0 = \left[\begin{array}{ccc} 0 & 0 & -1 \\ \hline \multicolumn{3}{c}{0_{(l_z-1)\times 3}} \end{array}\right],$$

and $E_2 = 0$. Therefore, the control objective is to reject the disturbances w_1 and w_2 while having the first component of $E_1 x$ follow the command w_3. The control effort is not weighted.

A. Stabilization for an unstable, SISO, minimum-phase plant

Consider the unstable, SISO, minimum-phase transfer function from u to z, given by

$$G_{zu}(\mathbf{z}) = \beta_1 \frac{(\mathbf{z}+0.5)(\mathbf{z}-0.8)}{(\mathbf{z}+1.1)(\mathbf{z}-1.2)(\mathbf{z}+0.3)},$$

where $\beta_1 = -3$. To represent the stabilization problem, let D_1, E_2, and E_0 be zero. Since G_{zu} is minimum phase, the adaptive controller (9)-(11) requires knowledge of only the first nonzero Markov parameter. More specifically, we let $\nu = d = 1$ and $\bar{\beta}_1 = \beta_1 = -3$. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 3$ and $P(0) = 0.1 I_6$. The plant has the initial condition $x(0) = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}^{\mathrm{T}}$. Figure 1 shows the time history of the closed-loop performance z and control u. The adaptive controller is turned on at $k = 0$, and the closed-loop performance approaches zero after approximately 50 time steps.

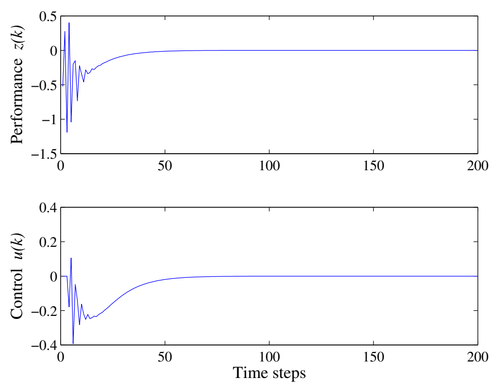

Fig. 1. *Stabilization for an unstable, SISO, minimum-phase plant:* The adaptive control (9)-(11) with $\bar{\beta}_1 = \beta_1$, $n_c = 3$, and $P(0) = 0.1 I_6$ is turned on at $k = 0$ and drives z to zero.

B. Command following and disturbance rejection for a stable, SISO, minimum-phase plant

Consider the stable, SISO, minimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \beta_2 \frac{\mathbf{z}-0.3}{(\mathbf{z}-0.4)(\mathbf{z}+0.6)(\mathbf{z}-0.8)},$$

where $\beta_2 = 2$. We let $\nu = d = 2$ and $\bar{\beta}_2 = \beta_2 = 2$. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 20$ and $P(0) = 100 I_{40}$. The plant has the initial condition $x(0) = \begin{bmatrix} -2 & 2 & 0 \end{bmatrix}^{\mathrm{T}}$. Figure 1 shows the time history of the closed-loop performance z and control u. The system is allowed to run open loop for 100 time steps, and the adaptive controller is turned on at $k = 100$. The closed-loop performance approaches zero after approximately 70 time steps.

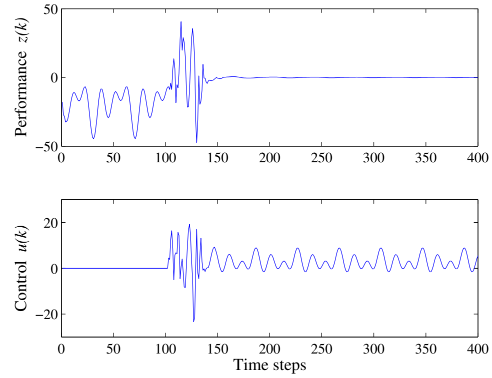

Fig. 2. *Command following and disturbance rejection for a stable, SISO, minimum-phase plant:* The adaptive control (9)-(11) with $\bar{\beta}_2 = \beta_2$, $n_c = 20$, and $P(0) = 100 I_{40}$ is turned on at $k = 100$ and drives z to zero.

C. Command following and disturbance rejection for an unstable, SISO, minimum-phase plant

Consider the unstable, SISO, minimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \beta_1 \frac{(\mathbf{z}-0.7)(\mathbf{z}-0.8)(\mathbf{z}-0.9)}{(\mathbf{z}-1)^2(\mathbf{z}+0.3+\jmath 0.4)(\mathbf{z}+0.3-\jmath 0.4)},$$

where $\beta_1 = -1$. We let $\nu = d = 1$ and $\bar{\beta}_1 = \beta_1 = -1$. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 20$ and $P(0) = I_{40}$. The plant has the initial condition $x(0) = 0$. Figure 3 shows the time history of the closed-loop performance z and control u. The adaptive controller is turned on at $k = 0$, and the closed-loop performance approaches zero.

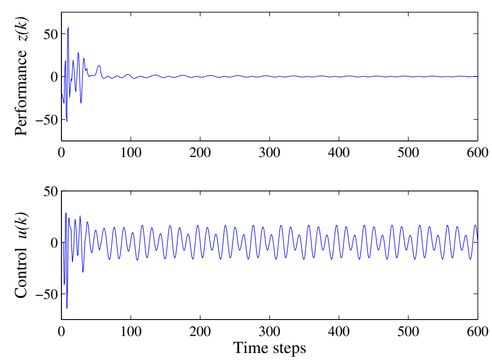

Fig. 3. *Command following and disturbance rejection for an unstable, SISO, minimum-phase plant:* The adaptive control (9)-(11) with $\bar{\beta}_1 = \beta_1$, $n_c = 20$, and $P(0) = I_{40}$ is turned on at $k = 0$ and drives z to zero.

D. Stabilization for an unstable, SISO, nonminimum-phase plant

Consider the unstable, SISO, nonminimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \beta_2 \frac{\mathbf{z}-1.1}{\mathbf{z}(\mathbf{z}-1.2)(\mathbf{z}-0.1)},$$

where $\beta_2 = 2$. To represent the stabilization problem, let D_1, E_2, and E_0 be zero. Note that G_{zu} is not strongly stabilizable, that is, an unstable linear controller is required to stabilize G_{zu} [19]. We let $\nu = n = 2$ and let $\bar{\beta}_2, \bar{\beta}_3$ be the coefficients of the numerator polynomial of G_{zu} (as described in Section V-A), that is, $\bar{\beta}_2 = 2$ and $\bar{\beta}_3 = -2.2$. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 3$ and $P(0) = I_6$. The plant has the initial condition $x(0) = \begin{bmatrix} 0.1 & -0.1 & 0.2 \end{bmatrix}^T$. Figure 4 shows the time history of the closed-loop performance z and control u. The adaptive controller is turned on at $k = 0$, and the closed-loop performance approaches zero.

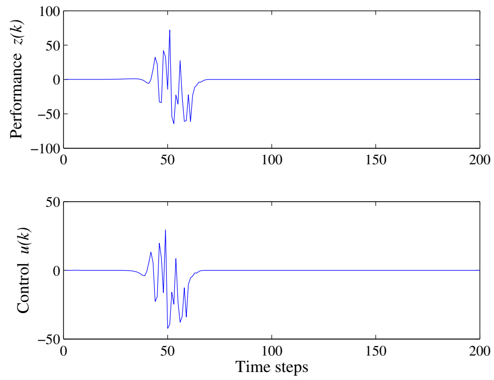

Fig. 4. *Stabilization for an unstable, SISO, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 3$, $P(0) = I_6$, and $\bar{\beta}_2, \bar{\beta}_3$ selected as the numerator coefficients is turned on at $k = 0$ and drives z to zero.

E. Command following and disturbance rejection for a stable, SISO, nonminimum-phase plant

Consider the stable, SISO, nonminimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \beta_2 \frac{(\mathbf{z} - 1.1)(\mathbf{z} + 1.1)}{\mathbf{z}^2(\mathbf{z} + 0.1 + \jmath 0.3)(\mathbf{z} + 0.1 - \jmath 0.3)},$$

where $\beta_2 = 0.5$. We let $\nu = n = 4$ and let $\bar{\beta}_2, \bar{\beta}_3, \bar{\beta}_4$ be the coefficients of the numerator polynomial of G_{zu}. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 40$ and $P(0) = 0.1 I_{80}$. The plant has the initial condition $x(0) = 0$. Figure 5 shows the time history of the closed-loop performance z and control u. The system is allowed to run open loop for 100 time steps, and the adaptive controller is turned on at $k = 100$. The closed-loop performance approaches zero.

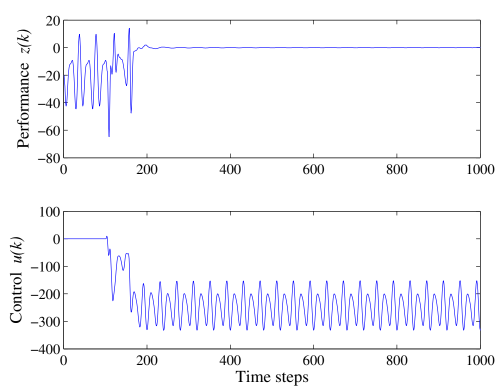

Fig. 5. *Command following and disturbance rejection for a stable, SISO, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 40$, $P(0) = 0.1 I_{80}$, and $\bar{\beta}_2, \ldots, \bar{\beta}_4$ selected as the numerator coefficients is turned on at $k = 100$ and drives z to zero.

F. Command following and disturbance rejection for an unstable, SISO, nonminimum-phase plant

Consider the unstable, SISO, nonminimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \beta_1 \frac{(\mathbf{z} + 0.7)(\mathbf{z} - 0.9)(\mathbf{z} + 1.5)}{(\mathbf{z} - 1)^2(\mathbf{z} + 0.3 + \jmath 0.4)(\mathbf{z} + 0.3 - \jmath 0.4)},$$

where $\beta_1 = 0.5$. We let $\nu = n_U + d = 2$ and let $\bar{\beta}_1, \bar{\beta}_2$ be the coefficients of the unstable numerator polynomial β_U (as described in Section V-B). More specifically, let $\bar{\beta}_1 = \beta_1 = 0.5$ and $\bar{\beta}_2 = 1.5\beta_1 = 0.75$. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 25$ and $P(0) = 0.01 I_{50}$. The plant has the initial condition $x(0) = 0$. Figure 6 shows the time history of the closed-loop performance z and control u. The adaptive controller is turned on at $k = 0$, and the closed-loop performance approaches zero.

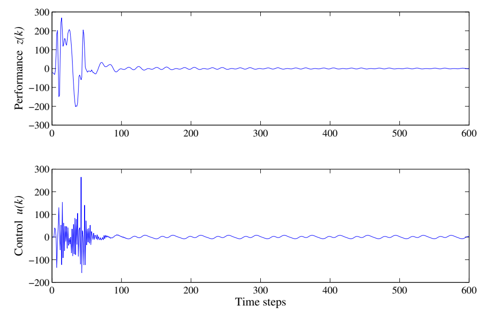

Fig. 6. *Command following and disturbance rejection for an unstable, SISO, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 25$, $P(0) = 0.01 I_{50}$, and $\bar{\beta}_1, \bar{\beta}_2$ selected as the coefficients of the unstable numerator polynomial β_U is turned on at $k = 0$ and drives z to zero.

G. Command following and disturbance rejection for a stable, two-input, two-output, nonminimum-phase plant

Consider the stable, two-input, two-output, nonminimum-phase transfer function

$$G_{zu}(\mathbf{z}) = \begin{bmatrix} \frac{(\mathbf{z}+1.1)(\mathbf{z}-1.1)}{\alpha(\mathbf{z})} & \frac{(\mathbf{z}+1.1)(\mathbf{z}-1.5)}{\alpha(\mathbf{z})} \\ \frac{\mathbf{z}-1.1}{\alpha(\mathbf{z})} & \frac{(\mathbf{z}+1)(\mathbf{z}-1.1)}{\alpha(\mathbf{z})} \end{bmatrix},$$

where $\alpha(\mathbf{z}) \triangleq (\mathbf{z}+0.1)(\mathbf{z}-0.2)(\mathbf{z}-0.1+\jmath 0.3)(\mathbf{z}-0.1-\jmath 0.3)$.

We let $\nu = n = 4$ and let $\bar{\beta}_2, \bar{\beta}_3, \bar{\beta}_4$ be the coefficients of the numerator polynomial matrix of G_{zu} (as described in Section V-A). The adaptive controller (9)-(11) is implemented in feedback with $n_c = 40$ and $P(0) = 0.01 I_{320}$. The plant has the initial condition $x(0) = 0$. Figure 7 is the time histories of the closed-loop performance z. The system is allowed to run open loop for 100 time steps, and the adaptive controller is turned on at $k = 100$. The closed-loop performance approaches zero.

H. White-noise disturbance rejection for a stable, SISO, nonminimum-phase plant

All of the examples thus far have focused on deterministic command and disturbance signals. In this example, we demonstrate that the retrospective cost adaptive controller is able to improve open-loop performance under white-noise

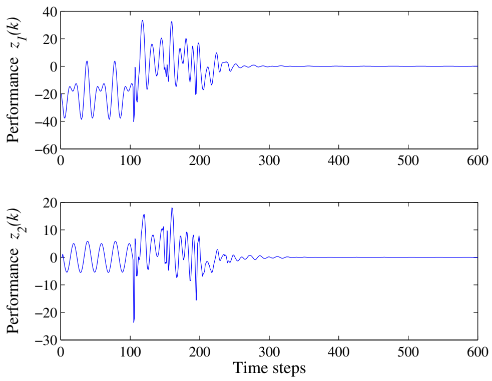

Fig. 7. *Command following and disturbance rejection for a stable, two-input, two-output, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 40$, $P(0) = 0.01 I_{320}$, and $\bar{\beta}_2, \ldots, \bar{\beta}_4$ selected as the numerator coefficients is turned on at $k = 100$ and drives z to zero.

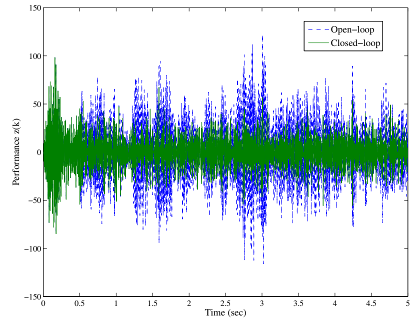

Fig. 8. *White-noise disturbance rejection for a stable, SISO, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 6$, $P(0) = 0.01 I_{12}$, and $\bar{\beta}_4, \ldots, \bar{\beta}_6$ selected as the numerator coefficients is turned on at 0.5 seconds and reduces the magnitude of the performance response to the white-noise disturbance. The open-loop response is shown for comparison.

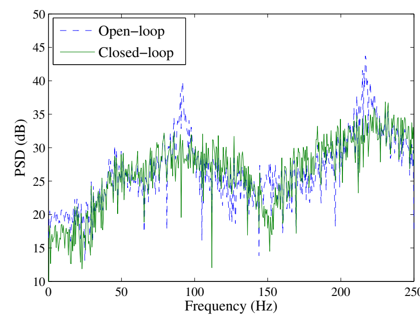

Fig. 9. *White-noise disturbance rejection for a stable, SISO, nonminimum-phase plant:* The adaptive control (9)-(11) with $n_c = 6$, $P(0) = 0.01 I_{12}$, and $\bar{\beta}_4, \ldots, \bar{\beta}_6$ selected as the numerator coefficients yields approximately 10 dB of peak attenuation near the model frequencies of the system.

disturbances. Consider the stable, SISO, nonminimum-phase transfer function from u to z, given by

$$G_{zu}(\mathbf{z}) = \frac{(\mathbf{z} - 1.5)(\mathbf{z} - 2)}{\alpha(\mathbf{z})},$$

where $\alpha(\mathbf{z}) \stackrel{\triangle}{=} (\mathbf{z} - 0.4 + \jmath 0.9)(\mathbf{z} - 0.4 - \jmath 0.9)(\mathbf{z} - 0.8 + \jmath 0.5)(\mathbf{z} - 0.8 - \jmath 0.5)(\mathbf{z} + 0.9 + \jmath 0.4)(\mathbf{z} + 0.9 - \jmath 0.4)$. The transfer function G_{zu} is realized in a controllable canonical form where $D_1 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \end{array}\right]^{\mathrm{T}}$, $E_0 = 0$, $E_2 = 0$, and the initial condition is $x(0) = 0$. Therefore, the control objective is to reject the disturbance w. In this example, w is a white-noise sequence.

We let $\nu = n = 6$ and let $\bar{\beta}_4, \bar{\beta}_5, \bar{\beta}_6$ be the coefficients of the numerator polynomial of G_{zu}. The adaptive controller (9)-(11) is implemented in feedback with $n_c = 6$ and $P(0) = 0.01 I_{12}$. Figure 8 shows the time history of the open-loop and closed-loop performance z. The system is allowed to run open loop for 0.5 seconds, then the adaptive controller is turned on and the controller reduces the magnitude of the response to the white-noise disturbance. Figure 9 shows the power spectral density of the open-loop and closed-loop performance variable. The adaptive controller yields approximately 10 dB of peak attenuation near the modal frequencies of the open-loop system. The open-loop and closed-loop power spectral densities of z are calculated using the final 3 seconds of the time history data presented in Figure 8.

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