

Inertia-Free Spacecraft Attitude Control with Control Moment Gyroscope Actuation

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We extend the inertia-free continuous control law for spacecraft attitude tracking derived in prior work to the case of three single-axis control moment gyroscopes with spherical gyro wheels. These CMGs are assumed to be mounted in a known and linearly independent configuration with an arbitrary and unknown orientation relative to the spacecraft principal axes. We demonstrate the performance of the modified control laws for rest-to-rest, motion-to-rest and spin maneuvers without the need for a separate steering algorithm.

I. Introduction

In spacecraft applications it is often expensive to determine the mass properties with a high degree of accuracy. To alleviate this requirement, the control algorithms given in refs.¹⁻³ require no prior modeling of the mass distribution. These algorithms incorporate internal states that can be viewed as estimates of the moments and products of inertia; however, these estimates need not converge to the true values, and in fact do not converge to the true values except in cases of sufficiently persistent motion.

The results of ref.¹ are based on rotation matrices⁴ as an alternative to quaternions as used in refs.^{3,5,6} Quaternions provide a double cover of the rotation group $SO(3)$, and thus, when used as the basis of a *continuous* control algorithm, cause unwinding, that is, unnecessary rotation away from and then back to the desired physical attitude.⁷ To avoid unwinding while using quaternions, it is thus necessary to resort to *discontinuous* control algorithms, which introduces the possibility of chatter due to noise as well as mathematical complications.⁸⁻¹⁰ On the other hand, rotation matrices allow for continuous control laws but introduce multiple equilibria. However, the equilibria that do not represent the desired equilibrium are saddle points of the closed-loop system, and thus the attitude of the spacecraft converges almost globally to the desired equilibrium.

The inertia-free control law developed in ref.¹ was extended in ref.² to the case of reaction wheel actuation. The goal of the present paper is to further extend the control law to include control moment gyroscope (CMG) actuation. We derive coordinate-free equations of motion for a satellite with three single-axis spherical CMGs and specialize them to the case where the gyro wheels rotate with a fixed angular velocity and the gimbals are aligned orthogonally with the spacecraft bus. We derive the inertia-free control law for such actuation, and we demonstrate the performance of the control law for rest-to-rest, motion-to-rest and spin maneuvers.

The traditional approach to controlling spacecraft with CMG's is to implement a controller that specifies a torque input, and then command the gimbal angular velocities in order to synthesize the requested torque using a suitable steering algorithm.¹¹ However, the gimbal commands must be chosen to avoid singularities that can arise from the alignment of the gimbals.¹²⁻¹⁶ An experiment devoted to singularity avoidance is described in ref.¹⁷ The singularity problem can be avoided by implementing CMG's with variable gyro wheel speed.¹⁸

In the present paper we consider fixed speed CMG's, where the adaptive controller directly commands the gimbal angular velocities. This approach avoids the need to separately synthesize gimbal angular velocities

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from a requested torque. We demonstrate the adaptive controller for rest to rest, motion to rest and spin maneuvers, where singularity avoidance is also demonstrated.

II. Spacecraft Model with CMGs

In this section we derive the equations of motion for a spacecraft with CMGs, while highlighting the underlying assumptions on CMG geometry, inertia, and attachment to the bus. Throughout the paper $\vec{r}_{q/p}$ denotes the position of point q relative to point p , $\vec{v}_{q/p/X}$ denotes the velocity of point q relative to point p with respect to frame F_X , and $\vec{\omega}_{Y/X}$ denotes the angular velocity of frame F_Y relative to frame F_X . All frames are orthogonal and right handed.

Def. 1. Let F_X be a frame, let \mathcal{B} be a collection of rigid bodies $\mathcal{B}_1, \dots, \mathcal{B}_l$, and let p be a point. Then, the angular momentum of \mathcal{B} relative to p with respect to F_X is defined by

$$\vec{H}_{\mathcal{B}/p/X} \triangleq \sum_{i=1}^l \vec{H}_{\mathcal{B}_i/p/X}, \quad (1)$$

where the angular momentum $\vec{H}_{\mathcal{B}_i/p/X}$ of \mathcal{B}_i relative to z with respect to F_X is defined by

$$\vec{H}_{\mathcal{B}_i/p/X} \triangleq \int_{\mathcal{B}_i} \vec{r}_{dm/p} \times \vec{v}_{dm/p/X} dm. \quad (2)$$

Lemma 1. Let \mathcal{B} be a rigid body, let F_X and F_Y be frames, and let p be a point. Then,

$$\vec{H}_{\mathcal{B}/p/X} = \vec{I}_{\mathcal{B}/p} \vec{\omega}_{Y/X} + \vec{H}_{\mathcal{B}/p/Y}, \quad (3)$$

where the physical inertia matrix $\vec{I}_{\mathcal{B}/p}$ is defined by

$$\vec{I}_{\mathcal{B}/p} \triangleq \int_{\mathcal{B}} |\vec{r}_{dm/p}|^2 \vec{U} - \vec{r}_{dm/p} \vec{r}'_{dm/p} dm, \quad (4)$$

where \vec{U} is the second-order identity tensor.

Lemma 2. Let \mathcal{B} be a rigid body, let F_X and F_Y be frames, let F_Y be a body-fixed frame, and let p be a point that is fixed in \mathcal{B} . Then,

$$\vec{H}_{\mathcal{B}/p/Y} = 0 \quad (5)$$

and

$$\vec{H}_{\mathcal{B}/p/X} = \vec{I}_{\mathcal{B}/p} \vec{\omega}_{Y/X}. \quad (6)$$

Lemma 3. Let F_X be a frame, let p be a point, let \mathcal{B} be a rigid body with mass $m_{\mathcal{B}}$, and let c be the center of mass of \mathcal{B} . Then,

$$\vec{H}_{\mathcal{B}/p/X} = \vec{H}_{\mathcal{B}/c/X} + \vec{r}_{c/p} \times m_{\mathcal{B}} \vec{v}_{c/p/X}. \quad (7)$$

We consider a spacecraft consisting of three single-axis CMGs with spherical gyro wheels attached to a rigid bus. Each CMG is mounted so that its gimbal is free to rotate about an axis passing through the center of mass of the gyro wheel. For simplicity, the gimbals are assumed to be massless. However, this paper does not assume that each gimbal's axis of rotation passes through the center of mass of the bus, nor does it assume that the CMGs are balanced with respect to the bus in order to preserve the location of its center of mass. Thus the center of mass of the spacecraft and the center of mass of the bus may be distinct points.

Let the spacecraft be denoted by sc , and let c denote its center of mass. Although the spacecraft is not a rigid body, the spherical symmetry of the gyro wheels implies that c is fixed in both the bus and the spacecraft. Let c_i denote the center of mass of the i th gyro wheel. We assume a bus-fixed frame F_B , three gimbal-fixed frames F_{G_i} whose y -axes are aligned with the rotation axes of their respective gimbals, three

gyro wheel-fixed frames F_{W_i} whose x -axes are aligned with the rotation axes of their respective gyro wheels, and an Earth-centered inertial frame F_E . The angular momentum of the spacecraft relative to its center of mass with respect to the inertial frame is given by

$$\vec{H}_{sc/c/E} = \vec{H}_{b/c/E} + \sum_{i=1}^3 \vec{H}_{w_i/c/E}, \quad (\text{Def. 1}) \quad (8)$$

where the angular momentum $\vec{H}_{b/c/E}$ of the bus relative to c with respect to F_E is given by

$$\vec{H}_{b/c/E} = \vec{I}_{b/c} \vec{\omega}_{B/E}, \quad (\text{Lemma 2}) \quad (9)$$

where $\vec{I}_{b/c}$ is the positive-definite inertia tensor of the bus relative to the center of mass of the spacecraft, and $\vec{\omega}_{B/E}$ is the angular velocity of F_B with respect to F_E . The angular momentum $\vec{H}_{w_i/c/E}$ of gyro wheel i relative to the center of mass of the spacecraft with respect to the inertial frame is given by

$$\begin{aligned} \vec{H}_{w_i/c/E} &= \vec{I}_{w_i/c} \vec{\omega}_{B/E} + \vec{H}_{w_i/c/B} && (\text{Lemma 1}) \\ &= \vec{I}_{w_i/c} \vec{\omega}_{B/E} + \vec{H}_{w_i/c_i/B} + \vec{r}_{c_i/c} \times m_{w_i} \vec{v}_{c_i/c/B} && (\text{Lemma 3}) \\ &= \vec{I}_{w_i/c} \vec{\omega}_{B/E} + \vec{I}_{w_i/c_i} \vec{\omega}_{W_i/B}, && (\text{Lemma 2}) \end{aligned} \quad (10)$$

where $\vec{I}_{w_i/c}$ is the positive-definite inertia tensor of gyro wheel i relative to the spacecraft's center of mass, \vec{I}_{w_i/c_i} is the positive-definite inertia tensor of gyro wheel i relative to its own center of mass c_i , and $\vec{\omega}_{W_i/B}$ is the angular velocity of gyro wheel i relative to the bus, and is given by

$$\vec{\omega}_{W_i/B} = \vec{\omega}_{W_i/G_i} + \vec{\omega}_{G_i/B}, \quad (11)$$

where $\vec{\omega}_{W_i/G_i}$ is the angular velocity of gyro wheel i relative to the gimbal i , and $\vec{\omega}_{G_i/B}$ is the angular velocity of gimbal i relative to the bus. Thus (8) is given by

$$\vec{H}_{sc/c/E} = \left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \vec{I}_{w_i/c_i} \vec{\omega}_{W_i/B} \quad (12)$$

Note that $\vec{\omega}_{W_i/B}$ has only two degrees of freedom relative to the bus, which can be seen when resolving $\vec{\omega}_{W_i/G_i}$ and $\vec{\omega}_{G_i/B}$ in F_{W_i} and F_{G_i} , respectively, that is,

$$\vec{\omega}_{W_i/G_i} \Big|_{W_i} = \begin{bmatrix} q_i \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\omega}_{G_i/B} \Big|_{G_i} = \begin{bmatrix} 0 \\ p_i \\ 0 \end{bmatrix}. \quad (13)$$

Since F_{W_i} is aligned with the principal axes of gyro wheel i , we have

$$\vec{I}_{w_i/c_i} \Big|_{W_i} = \begin{bmatrix} \beta_i & 0 & 0 \\ 0 & \beta_i & 0 \\ 0 & 0 & \beta_i \end{bmatrix}. \quad (14)$$

The repeated entries are due to the spherical nature of the gyro wheels. F_{W_i} rotates relative to F_{G_i} about its x -axis. Consequently,

$$\vec{I}_{w_i/c_i} \Big|_{G_i} = \vec{I}_{w_i/c_i} \Big|_{W_i}. \quad (15)$$

Therefore,

$$\overbrace{\vec{I}_{w_i/c_i}}^{G_i \bullet} = 0. \quad (16)$$

Note that $\vec{\omega}_{W_i/G_i}$ is an eigenvector of \vec{I}_{w_i/c_i} with eigenvalue β_i , and $\vec{\omega}_{G_i/B}$ is an eigenvector of \vec{I}_{w_i/c_i} with eigenvalue β_i . That is, $\vec{I}_{w_i/c_i} \vec{\omega}_{W_i/G_i} = \beta_i \vec{\omega}_{W_i/G_i}$ and $\vec{I}_{w_i/c_i} \vec{\omega}_{G_i/B} = \beta_i \vec{\omega}_{G_i/B}$.

III. Spacecraft Equations of Motion

We now derive the equations of motion for a spacecraft actuated with CMGs as described above. It follows from Newton's second law for rotation, that

$$\begin{aligned} \vec{M}_{sc/c} &= \overbrace{\vec{H}_{sc/c/E}}^{E \bullet} \\ &= \overbrace{\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right)}^{E \bullet} \vec{\omega}_{B/E} + \overbrace{\sum_{i=1}^3 \vec{I}_{w_i/c_i} \vec{\omega}_{W_i/B}}^{E \bullet} \\ &= \overbrace{\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right)}^{B \bullet} \vec{\omega}_{B/E} + \vec{\omega}_{B/E} \times \left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} \\ &\quad + \overbrace{\sum_{i=1}^3 \vec{I}_{w_i/c_i} \vec{\omega}_{W_i/B}}^{B \bullet} + \vec{\omega}_{B/E} \times \sum_{i=1}^3 \vec{I}_{w_i/c_i} \vec{\omega}_{W_i/B} \\ &= \left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \overbrace{\vec{\omega}_{B/E}}^{B \bullet} + \sum_{i=1}^3 \beta_i \overbrace{\vec{\omega}_{W_i/G_i}}^{B \bullet} + \sum_{i=1}^3 \beta_i \overbrace{\vec{\omega}_{G_i/B}}^{B \bullet} \\ &\quad + \vec{\omega}_{B/E} \times \left(\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \beta_i \left(\vec{\omega}_{W_i/G_i} + \vec{\omega}_{G_i/B} \right) \right). \end{aligned} \quad (17)$$

Rearranging (17) yields

$$\begin{aligned} \left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \overbrace{\vec{\omega}_{B/E}}^{B \bullet} &= \left(\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \beta_i \left(\vec{\omega}_{W_i/G_i} + \vec{\omega}_{G_i/B} \right) \right) \times \vec{\omega}_{B/E} \\ &\quad - \sum_{i=1}^3 \beta_i \overbrace{\vec{\omega}_{W_i/G_i}}^{B \bullet} - \sum_{i=1}^3 \beta_i \overbrace{\vec{\omega}_{G_i/B}}^{B \bullet} + \vec{M}_{sc/c}. \end{aligned} \quad (18)$$

To resolve (18) in F_B , the following notation is used:

$$\begin{aligned}
J_b &\triangleq \vec{I}_{b/c} \Big|_B, & J_{w_i/c} &\triangleq \vec{I}_{w_i/c} \Big|_B, \\
J_{w_i/c_i} &\triangleq \vec{I}_{w_i/c_i} \Big|_B, & J_{sc} &\triangleq J_b + \sum_{i=1}^3 J_{w_i/c}, \\
\omega &\triangleq \vec{\omega}_{B/E} \Big|_B, & \dot{\omega} &\triangleq \overset{B\bullet}{\vec{\omega}}_{B/E} \Big|_B, \\
\nu_i &\triangleq \vec{\omega}_{G_i/B} \Big|_B, & \dot{\nu}_i &\triangleq \overset{B\bullet}{\vec{\omega}}_{G_i/B} \Big|_B, \\
\omega_{W_i} &\triangleq \vec{\omega}_{W_i/G_i} \Big|_B, & \dot{\omega}_{W_i} &\triangleq \overset{B\bullet}{\vec{\omega}}_{W_i/G_i} \Big|_B, \\
\tau_{\text{dist}} &\triangleq \vec{M}_{sc/c} \Big|_B.
\end{aligned}$$

We let the vector τ_{dist} represent disturbance torques, that is, all internal and external torques applied to the spacecraft aside from control torques. Disturbance torques may be due to onboard components, gravity gradients, solar pressure, atmospheric drag, or the ambient magnetic field.

Resolving (18) in F_B yields the equation of motion for a spacecraft with CMGs, which has the form

$$J_{sc} \dot{\omega} = \left(J_{sc} \omega + \sum_{i=1}^3 \beta_i (\nu_i + \omega_{W_i}) \right) \times \omega - \sum_{i=1}^3 \beta_i \dot{\nu}_i - \sum_{i=1}^3 \beta_i \dot{\omega}_{W_i} + \tau_{\text{dist}}. \quad (19)$$

A. Special Case: gyro wheels rotate with constant angular velocity

We now specialize the equation of motion (19) by assuming that the gyro wheels rotate with a constant angular velocity. We thus have,

$$\overset{W_i\bullet}{\vec{\omega}}_{W_i/G_i} = 0. \quad (20)$$

Then,

$$\begin{aligned}
\overset{B\bullet}{\vec{\omega}}_{W_i/G_i} &= \overset{G_i\bullet}{\vec{\omega}}_{W_i/G_i} + \vec{\omega}_{G_i/B} \times \vec{\omega}_{W_i/G_i} \\
&= \overset{W_i\bullet}{\vec{\omega}}_{W_i/G_i} + \vec{\omega}_{G_i/B} \times \vec{\omega}_{W_i/G_i} \\
&= \vec{\omega}_{G_i/B} \times \vec{\omega}_{W_i/G_i}.
\end{aligned} \quad (21)$$

Substituting (21) into (18) and resolving in F_B yields

$$J_{sc}\dot{\omega} = \left(J_{sc}\omega + \sum_{i=1}^3 \beta_i (\nu_i + \omega_{W_i}) \right) \times \omega - \sum_{i=1}^3 \beta_i \dot{\nu}_i - \sum_{i=1}^3 \beta_i (\nu_i \times \omega_{W_i}) + \tau_{dist}. \quad (22)$$

B. Special case: orthogonal gimbals aligned with F_B

We further specialize the equations of motion by assuming that the gimbals are mounted orthogonally to each other, but their rotation axes need not intersect. For further simplification, we align F_B with the rotational axes of the three orthogonal gimbals. However, we do not assume that F_B is aligned with the principal axes of the bus. In this configuration

$$\mathcal{O}_{B/G_1} e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1, \quad \mathcal{O}_{B/G_2} e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = e_2, \quad \mathcal{O}_{B/G_3} e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = e_3, \quad (23)$$

where $\mathcal{O}_{B/G_i} \in \mathbb{R}^{3 \times 3}$ is the proper orthogonal matrix (that is, the orientation matrix) that transforms the components of a vector resolved in F_{G_i} into the components of the same vector resolved in F_B .

Additionally, as we will be directly commanding the gimbal angular rates, we rearrange (18) in order to group all terms involving $\vec{\omega}_{G_i/B}$, which we now write as u_i . We thus have

$$\begin{aligned} \left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E}^{\bullet} &= \left(\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \beta_i \left(\vec{\omega}_{W_i/G_i} + \vec{\omega}_{G_i/B} \right) \right) \times \vec{\omega}_{B/E} \\ &\quad - \sum_{i=1}^3 \beta_i \left(\vec{\omega}_{G_i/B} \times \vec{\omega}_{W_i/G_i} \right) - \sum_{i=1}^3 \beta_i \vec{\omega}_{G_i/B}^{\bullet} + \vec{M}_{sc/c} \\ &= \left(\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \beta_i \vec{\omega}_{W_i/G_i} \right) \times \vec{\omega}_{B/E} - \sum_{i=1}^3 \beta_i \vec{\omega}_{B/E}^{\times} \vec{\omega}_{G_i/B} \\ &\quad + \sum_{i=1}^3 \beta_i \vec{\omega}_{W_i/G_i}^{\times} \vec{\omega}_{G_i/B} - \sum_{i=1}^3 \beta_i \vec{\omega}_{G_i/B}^{\bullet} + \vec{M}_{sc/c} \\ &= \left(\left(\vec{I}_{b/c} + \sum_{i=1}^3 \vec{I}_{w_i/c} \right) \vec{\omega}_{B/E} + \sum_{i=1}^3 \beta_i \vec{\omega}_{W_i/G_i} \right) \times \vec{\omega}_{B/E} \\ &\quad + \sum_{i=1}^3 \beta_i \left(\vec{\omega}_{W_i/G_i}^{\times} - \vec{\omega}_{B/E}^{\times} \right) \mathcal{O}_{B/G_i} e_2 u_i - \sum_{i=1}^3 \beta_i \vec{\omega}_{G_i/B}^{\bullet} + \vec{M}_{sc/c}. \end{aligned} \quad (24)$$

Expanding the control term in (24) and plugging in (23) yields,

$$\begin{aligned} &\sum_{i=1}^3 \beta_i \left(\vec{\omega}_{W_i/G_i}^{\times} - \vec{\omega}_{B/E}^{\times} \right) \mathcal{O}_{B/G_i} e_2 u_i \\ &= \beta_1 \left(\vec{\omega}_{W_1/G_1}^{\times} - \vec{\omega}_{B/E}^{\times} \right) e_1 u_1 + \beta_2 \left(\vec{\omega}_{W_2/G_2}^{\times} - \vec{\omega}_{B/E}^{\times} \right) e_2 u_2 + \beta_3 \left(\vec{\omega}_{W_3/G_3}^{\times} - \vec{\omega}_{B/E}^{\times} \right) e_3 u_3 \\ &= Y u, \end{aligned} \quad (25)$$

where

$$Y \triangleq \begin{bmatrix} \beta_1 \left(\bar{\omega}_{W_1/G_1}^\times - \bar{\omega}_{B/E}^\times \right) e_1 & \beta_2 \left(\bar{\omega}_{W_2/G_2}^\times - \bar{\omega}_{B/E}^\times \right) e_2 & \beta_3 \left(\bar{\omega}_{W_3/G_3}^\times - \bar{\omega}_{B/E}^\times \right) e_3 \end{bmatrix}, \quad u \triangleq \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Finally, resolving the equation of motion (24) in F_B yields

$$J_{sc}\dot{\omega} = \left(J_{sc}\omega + \sum_{i=1}^3 \beta_i \omega_{W_i} \right) \times \omega + Yu - J_\beta \dot{u} + \tau_{\text{dist}}, \quad (26)$$

where,

$$J_\beta = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}. \quad (27)$$

IV. Spacecraft Model, Assumptions, and Objectives for Control Design

The kinematics of the spacecraft model are given by Poisson's equation

$$\dot{R} = R\omega^\times, \quad (28)$$

which complements (26). In (28), ω^\times denotes the skew-symmetric matrix of ω , and $R \triangleq \mathcal{O}_{E/B} \in \mathbb{R}^{3 \times 3}$.

Compared to the case of reaction wheels treated in ref.,² control moment gyroscope actuation complicates the dynamic equations due to the term $J_\beta \dot{u}$ in (26), as well as the state-dependant, time-varying, and potentially singular input matrix, Y . The kinematic relation (28) remains unchanged. The torque inputs applied to each gyro wheel are constrained by current limitations on the electric motors and amplifiers as well as angular-velocity constraints on the gimbals. However, these constraints are not addressed explicitly.

Both rate (inertial) and attitude (noninertial) measurements are assumed to be available. Gyro measurements $y_{\text{rate}} \in \mathbb{R}^3$ are assumed to provide measurements of the angular velocity resolved in the spacecraft frame, that is,

$$y_{\text{rate}} = \omega + v_{\text{rate}}, \quad (29)$$

where $v_{\text{rate}} \in \mathbb{R}^3$ represents the presence of noise in the gyro measurements. Attitude is measured indirectly using sensors such as magnetometers or star trackers. The attitude is determined to be

$$y_{\text{attitude}} = R. \quad (30)$$

When attitude measurements are given in terms of an alternative attitude representation, such as quaternions, Rodrigues's formula can be used to determine the corresponding rotation matrix. Attitude estimation on $SO(3)$ is considered in ref.²⁰

The objective of the attitude control problem is to determine control inputs such that the spacecraft attitude given by R follows a commanded attitude trajectory given by a possibly time-varying C^1 rotation matrix $R_d(t)$. For $t \geq 0$, $R_d(t)$ is given by

$$\dot{R}_d(t) = R_d(t)\omega_d(t)^\times, \quad (31)$$

$$R_d(0) = R_{d0}, \quad (32)$$

where ω_d is the desired, possibly time-varying angular velocity. The error between $R(t)$ and $R_d(t)$ is given in terms of the attitude-error rotation matrix

$$\tilde{R} \triangleq R_d^T R,$$

which satisfies the differential equation

$$\dot{\tilde{R}} = \tilde{R} \tilde{\omega}^\times, \quad (33)$$

where the angular velocity error $\tilde{\omega}$ is defined by

$$\tilde{\omega} \triangleq \omega - \tilde{R}^T \omega_d.$$

We rewrite (26) in terms of the angular-velocity error as

$$\begin{aligned} J_{sc} \dot{\tilde{\omega}} = & \left(J_{sc}(\tilde{\omega} + \tilde{R}^T \omega_d) + \sum_{i=1}^3 \beta_i \omega_{W_i} \right) \times (\tilde{\omega} + \tilde{R}^T \omega_d) + J_{sc}(\tilde{\omega} \times \tilde{R}^T \omega_d - \tilde{R}^T \dot{\omega}_d) \\ & + Y u - J_\beta \dot{u} + \tau_{dist}. \end{aligned} \quad (34)$$

A scalar measure of attitude error is given by the rotation angle $\theta(t)$ about an eigenaxis needed to rotate the spacecraft from its attitude $R(t)$ to the desired attitude $R_d(t)$. This angle is given by¹⁹

$$\theta(t) = \cos^{-1}(\frac{1}{2}[\text{tr } \tilde{R}(t) - 1]), \quad (35)$$

and referred to as the eigenaxis attitude error.

V. Controller Design

Let I denote the identity matrix, whose dimensions are determined by context, and let M_{ij} denote the i, j entry of the matrix M . The following result is given in ref.¹

Lemma 4. Let $A \in \mathbb{R}^{3 \times 3}$ be a diagonal positive-definite matrix. Then the following statements hold for a proper orthogonal matrix R :

- i) For all $i, j = 1, 2, 3$, $R_{ij} \in [-1, 1]$.
- ii) $\text{tr}(A - AR) \geq 0$.
- iii) $\text{tr}(A - AR) = 0$ if and only if $R = I$.

For convenience we note that, if R is a rotation matrix and $x, y \in \mathbb{R}^3$, then

$$(Rx)^\times = Rx^\times R^T,$$

and, therefore,

$$R(x \times y) = (Rx) \times Ry.$$

Next we introduce the notation

$$J_{sc} \omega = L(\omega) \gamma,$$

where $\gamma \in \mathbb{R}^6$ is defined by

$$\gamma \triangleq \begin{bmatrix} J_{11} & J_{22} & J_{33} & J_{23} & J_{13} & J_{12} \end{bmatrix}^T$$

and

$$L(\omega) \triangleq \begin{bmatrix} \omega_1 & 0 & 0 & 0 & \omega_3 & \omega_2 \\ 0 & \omega_2 & 0 & \omega_3 & 0 & \omega_1 \\ 0 & 0 & \omega_3 & \omega_2 & \omega_1 & 0 \end{bmatrix}.$$

Next, let $\hat{J}_{\text{sc}} \in \mathbb{R}^{3 \times 3}$ denote an estimate of J_{sc} , and define the inertia-estimation error

$$\tilde{J}_{\text{sc}} \triangleq J_{\text{sc}} - \hat{J}_{\text{sc}}.$$

Letting $\hat{\gamma}, \tilde{\gamma} \in \mathbb{R}^6$ represent $\hat{J}_{\text{sc}}, \tilde{J}_{\text{sc}}$, respectively, it follows that

$$\tilde{\gamma} = \gamma - \hat{\gamma}.$$

Likewise, let $\hat{\tau}_{\text{dist}} \in \mathbb{R}^3$ denote an estimate of τ_{dist} , and define the disturbance-estimation error

$$\tilde{\tau}_{\text{dist}} \triangleq \tau_{\text{dist}} - \hat{\tau}_{\text{dist}}.$$

The assumptions upon which the following development is based are now stated.

Assumption 1. J_{sc} is constant but unknown.

Assumption 2. J_{β} is constant, nonsingular, and known. That is, we have three orthogonal CMGs with symmetric gyro wheels, and we know the moments of inertia about their spin axes.

Assumption 3. \dot{u} , the gimbale accelerations, are negligible and can be ignored. We will ignore the effect of \dot{u} in our derivation of the control law. This treatment of \dot{u} is consistent with prior literature.^{21–23} We will however, consider it in the actual plant dynamics for simulations and show through numerical examples that this is a reasonable assumption. Removing \dot{u} from (26) yields

$$J_{\text{sc}}\dot{\omega} = \left(J_{\text{sc}}\omega + \sum_{i=1}^3 \beta_i \omega_{\text{W}_i} \right) \times \omega + Yu + \tau_{\text{dist}}. \quad (36)$$

Two controllers are presented in ref.¹ They are now modified, as necessary, for control moment gyroscope actuation.

A. Control Law for Slew Maneuvers

When no disturbances are present, the inertia-free control law given by (38) of ref.¹ achieves almost global stabilization of a constant desired attitude R_d , that is, a slew maneuver that brings the spacecraft to rest. The initial conditions of the slew maneuver may be arbitrary, that is, the spacecraft may have nonzero initial velocity.

Given $a = [a_1 \ a_2 \ a_3]^T \in \mathbb{R}^3$ such that a_1, a_2 and a_3 are positive and distinct, define the vector measure of attitude error

$$S \triangleq \sum_{i=1}^3 a_i (\tilde{R}^T e_i) \times e_i, \quad (37)$$

where, for $i = 1, 2, 3$, e_i denotes the i th column of the 3×3 identity matrix.

Theorem 1. Let K_p be a positive number and let $A = \text{diag}(a)$ be a diagonal positive-definite matrix. Then the function

$$V(\omega, \tilde{R}) \triangleq \frac{1}{2} \omega^T J_{\text{sc}} \omega + K_p \text{tr}(A - A\tilde{R}), \quad (38)$$

is positive definite, that is, V is nonnegative, and $V = 0$ if and only if $\omega = 0$ and $\tilde{R} = I$.

Proof. It follows from statement 2 of Lemma 4 that $\text{tr}(A - A\tilde{R})$ is nonnegative. Hence V is nonnegative. Now suppose that $V = 0$. Then, $\omega = 0$, and it follows from statement 3 of Lemma 4 that $\tilde{R} = I$. \square

Note that (38), which we intend to use as a Lyapunov function, is not positive in all the states, namely, it makes no use of the relative angular velocity of the gyro wheels or the gimbals since gyro wheel-speed regulation and gimbal-speed regulation are not goals of the control objective.

Theorem 2. Let K_p be a positive number, let $K_v \in \mathbb{R}^{3 \times 3}$ be a positive-definite matrix, let $A = \text{diag}(a_1, a_2, a_3)$ be a diagonal positive-definite matrix, define S as in (37), and define V as in Theorem 1. Furthermore, consider the control law

$$u = -Y^{-1}(K_p S + K_v \omega), \quad (39)$$

and assume that Y is nonsingular. Then,

$$\dot{V}(\omega, \tilde{R}) = -\omega^T K_v \omega, \quad (40)$$

along the trajectories of (36), is negative semidefinite.

Proof. Noting that

$$\begin{aligned} \frac{d}{dt} \text{tr}(A - A\tilde{R}) &= -\text{tr} A \dot{\tilde{R}} \\ &= -\text{tr} A(\tilde{R}\omega^\times - \omega_d^\times \tilde{R}) \\ &= -\sum_{i=1}^3 a_i e_i^T (\tilde{R}\omega^\times - \omega_d^\times \tilde{R}) e_i \\ &= -\sum_{i=1}^3 a_i e_i^T \tilde{R}(\omega^\times - \tilde{R}^T \omega_d^\times \tilde{R}) e_i \\ &= -\sum_{i=1}^3 a_i e_i^T \tilde{R}(\omega - \tilde{R}^T \omega_d)^\times e_i \\ &= \sum_{i=1}^3 a_i e_i^T \tilde{R} e_i^\times \tilde{\omega} \\ &= \left[-\sum_{i=1}^3 a_i e_i \times \tilde{R}^T e_i \right]^T \tilde{\omega} \\ &= \left[\sum_{i=1}^3 a_i (\tilde{R}^T e_i) \times e_i \right]^T \tilde{\omega} \\ &= \tilde{\omega}^T S, \end{aligned}$$

then

$$\begin{aligned} \dot{V}(\omega, \tilde{R}) &= \omega^T J_{sc} \dot{\omega} + K_p \omega^T S \\ &= \omega^T \left[\left(J_{sc} \omega + \sum_{i=1}^3 \beta_i \omega_{W_i} \right) \times \omega + Y u \right] + K_p \omega^T S \\ &= \omega^T (-K_p S - K_v \omega) + K_p \omega^T S \\ &= -\omega^T K_v \omega. \quad \square \end{aligned}$$

Note that $-Y$ is substituted for the input matrix B used in the inertia-free control law (38) of ref.¹ but otherwise the controller requires no modification for the case of control moment gyroscope actuation.

B. Control Law for Attitude Tracking

A control law that tracks a desired attitude trajectory in the presence of disturbances is given by (21) of ref.¹ This controller is based on an additional assumption.

Assumption 4. Each component of τ_{dist} is a linear combination of constant and harmonic signals, whose frequencies are known but whose amplitudes and phases are unknown.

Assumption 4 implies that τ_{dist} can be modeled as the output of an autonomous system of the form

$$\dot{d} = A_d d, \quad (41)$$

$$\tau_{\text{dist}} = C_d d, \quad (42)$$

where $A_d \in \mathbb{R}^{n_d \times n_d}$ and $C_d \in \mathbb{R}^{3 \times n_d}$ are known matrices and A_d is a Lyapunov-stable matrix. In this model, $d(0)$ is unknown, which is equivalent to the assumption that the amplitude and phase of each harmonic component of the disturbance is unknown. The matrix A_d is chosen to include eigenvalues of all frequency components that may be present in the disturbance signal, where the zero eigenvalue corresponds to a constant disturbance. In effect, the controller provides infinite gain at the disturbance frequency, which results in asymptotic rejection of harmonic disturbance components. In particular, an integral controller provides infinite gain at DC in order to reject constant disturbances. In the case of orbit-dependent disturbances, the frequencies can be estimated from the orbital parameters. Likewise, in the case of disturbances originating from on-board devices, the spectral content of the disturbances may be known. In other cases, it may be possible to estimate the spectrum of the disturbances through signal processing. Assumption 4 implies that A_d can be chosen to be skew symmetric, which we do henceforth. Let $\hat{d} \in \mathbb{R}^{n_d}$ denote an estimate of d , and define the disturbance-state estimation error

$$\tilde{d} \triangleq d - \hat{d}.$$

The attitude tracking controller in the presence of disturbances is modified for CMG actuators below.

Theorem 3. *Let K_p be a positive number, let $K_1 \in \mathbb{R}^{3 \times 3}$, let $Q \in \mathbb{R}^{6 \times 6}$ and $D \in \mathbb{R}^{n_d \times n_d}$ be positive definite matrices, let $A = \text{diag}(a_1, a_2, a_3)$ be a diagonal positive-definite matrix, and define S as before. Then the function*

$$V(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{d}) \triangleq \frac{1}{2}(\tilde{\omega} + K_1 S)^T J_{\text{sc}}(\tilde{\omega} + K_1 S) + K_p \text{tr}(A - A\tilde{R}) + \frac{1}{2}\tilde{\gamma}^T Q \tilde{\gamma} + \frac{1}{2}\tilde{d}^T D \tilde{d} \quad (43)$$

is positive definite, that is, V is nonnegative, and $V = 0$ if and only if $\tilde{\omega} = 0$, $\tilde{R} = I$, $\tilde{\gamma} = 0$, and $\tilde{d} = 0$.

Proof. It follows from statement 2 of Lemma 4 that $\text{tr}(A - A\tilde{R})$ is nonnegative. Hence V is nonnegative. Now suppose that $V = 0$. Then, $\tilde{\omega} + K_1 S = 0$, $\tilde{\gamma} = 0$, and $\tilde{d} = 0$, and it follows from statement 3 of Lemma 4 that $\tilde{R} = I$, and thus $S = 0$. Therefore, $\tilde{\omega} = 0$. \square

Note that (43), which we intend to use as a Lyapunov function, is not positive in all the states, namely, it makes no use of the relative angular velocity of the gyro wheels or the gimbals since gyro wheel-speed regulation and gimbal-speed regulation are not goals of the control objective.

Theorem 4. *Let K_p be a positive number, let $K_v \in \mathbb{R}^{3 \times 3}$, $K_1 \in \mathbb{R}^{3 \times 3}$, $Q \in \mathbb{R}^{6 \times 6}$, and $D \in \mathbb{R}^{n_d \times n_d}$ be positive definite matrices, assume that $A_d^T D + D A_d$ is negative semidefinite, let $A = \text{diag}(a_1, a_2, a_3)$ be a diagonal positive-definite matrix, define S and V as in Theorem 1, and let $\hat{\gamma}$ and \hat{d} satisfy*

$$\dot{\hat{\gamma}} = Q^{-1}[L^T(\omega)\omega^\times + L^T(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)](\tilde{\omega} + K_1 S), \quad (44)$$

where

$$\dot{S} = \sum_{i=1}^3 a_i [(\tilde{R}^T e_i) \times \tilde{\omega}] \times e_i, \quad (45)$$

and

$$\dot{\hat{d}} = A_d \hat{d} + D^{-1} C_d^T (\tilde{\omega} + K_1 S), \quad (46)$$

$$\hat{\tau}_{\text{dist}} = C_d \hat{d}, \quad (47)$$

so that $\hat{\tau}_{\text{dist}}$ is the disturbance torque estimator. Furthermore, consider the control law

$$u = Y^{-1}(v_1 + v_2 + v_3), \quad (48)$$

where

$$v_1 \triangleq -(\hat{J}_{\text{sc}} \omega + \sum_{i=1}^3 \beta_i \omega_{W_i}) \times \omega - \hat{J}_{\text{sc}}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d), \quad (49)$$

$$v_2 \triangleq -\hat{\tau}_{\text{dist}}, \quad (50)$$

and

$$v_3 \triangleq -K_v(\tilde{\omega} + K_1 S) - K_p S. \quad (51)$$

Assume that Y is nonsingular. Then,

$$\dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{d}) = -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_1 S + \frac{1}{2} \tilde{d}^T (A_d^T D + D A_d) \tilde{d}, \quad (52)$$

along the trajectories of (36), is negative semidefinite.

Proof.

$$\begin{aligned} \dot{V}(\tilde{\omega}, \tilde{R}, \tilde{\gamma}, \tilde{d}) &= (\tilde{\omega} + K_1 S)^T (J_{\text{sc}} \dot{\tilde{\omega}} + J_{\text{sc}} K_1 \dot{S}) - K_p \text{tr} A \dot{\tilde{R}} - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} + \tilde{d}^T D \dot{\tilde{d}} \\ &= (\tilde{\omega} + K_1 S)^T [(J_{\text{sc}} \omega + \sum_{i=1}^3 \beta_i \omega_{W_i}) \times \omega + J_{\text{sc}}(\tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d) + Y u + \tau_{\text{dist}} + J_{\text{sc}} K_1 \dot{S}] \\ &\quad + K_p \tilde{\omega}^T S - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} + \tilde{d}^T D \dot{\tilde{d}} \\ &= (\tilde{\omega} + K_1 S)^T [(J_{\text{sc}} \omega + \sum_{i=1}^3 \beta_i \omega_{W_i}) \times \omega + J_{\text{sc}}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d) + v_1 + v_2 + v_3 + \tau_{\text{dist}}] \\ &\quad + K_p \tilde{\omega}^T S - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} + \tilde{d}^T D \dot{\tilde{d}} \\ &= (\tilde{\omega} + K_1 S)^T [(\tilde{J}_{\text{sc}} \omega) \times \omega + \tilde{J}_{\text{sc}}(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)] \\ &\quad + (\tilde{\omega} + K_1 S)^T \tilde{\tau}_{\text{dist}} - (\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p(\tilde{\omega} + K_1 S)^T S \\ &\quad + K_p \tilde{\omega}^T S - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} + \tilde{d}^T D \dot{\tilde{d}} \\ &= (\tilde{\omega} + K_1 S)^T [L(\omega) \tilde{\gamma} \times \omega + L(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d) \tilde{\gamma}] \\ &\quad - (\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_1 S - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} \\ &\quad + \tilde{d}^T C_d^T (\tilde{\omega} + K_1 S) + \tilde{d}^T D [A_d \tilde{d} - D^{-1} C_d^T (\tilde{\omega} + K_1 S)] \\ &= -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_1 S - \tilde{\gamma}^T Q \dot{\tilde{\gamma}} \\ &\quad + (\tilde{\omega} + K_1 S)^T [-\omega^\times L(\omega) + L(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d)] \tilde{\gamma} \\ &\quad + \frac{1}{2} \tilde{d}^T (A_d^T D + D A_d) \tilde{d} \\ &= -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_1 S \\ &\quad + \tilde{\gamma}^T [-Q \dot{\tilde{\gamma}} + (L^T(\omega) \omega^\times + L^T(K_1 \dot{S} + \tilde{\omega} \times \omega - \tilde{R}^T \dot{\omega}_d))(\tilde{\omega} + K_1 S)] \\ &\quad + \frac{1}{2} \tilde{d}^T (A_d^T D + D A_d) \tilde{d} \\ &= -(\tilde{\omega} + K_1 S)^T K_v(\tilde{\omega} + K_1 S) - K_p S^T K_1 S + \frac{1}{2} \tilde{d}^T (A_d^T D + D A_d) \tilde{d}. \quad \square \end{aligned}$$

Future work will complete the proof for almost global stabilization (that is, Lyapunov stability with almost global convergence) of spacecraft tracking using CMGs. The proof will rely on partial stability theory and invariance theorems.

VI. Modification to avoid singularities

A difficulty encountered with CMGs is that the torque they are able to generate may sometimes lie in a plane perpendicular to the demanded torque. When this condition takes place they are considered to be in a singular state, and gimbal angular velocities that synthesize the requested torque do not exist. Much of the work on CMGs has thus been to develop steering laws, that is, laws that modify the controller requested torque to either avoid these singular states, or steer the controller through them

While Theorems 2 and 4 do not use an explicit steering law to synthesize a desired torque, the matrix Y , which is assumed to be invertible in these theorems, plays a similar role and is sometimes not invertible.

Borrowing ideas from the steering law literature, we therefore examine two practical, albeit approximate, methods for inverting Y . Section VI.A discusses a modified version of the pseudoinverse, while section VI.B discusses the singularity-robust (SR) inverse.²⁴ Other methods such as the singular-direction avoidance (SDA) inverse²² are left for future study.

A. Saturated pseudoinverse method

We consider taking the saturated pseudoinverse of Y . We define the saturated pseudoinverse as

$$Y^{+sat} = V\Sigma^{-1sat}U^T, \quad (53)$$

where $U \in \mathbb{R}^{3 \times 3}$ and $V \in \mathbb{R}^{3 \times 3}$ are orthogonal matrices obtained from the singular value decomposition of Y , which is given by

$$Y = U\Sigma V^T, \quad (54)$$

where $\Sigma \in \mathbb{R}^{3 \times 3}$ is a diagonal matrix with singular values of Y on the diagonal,

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}, \quad (55)$$

and Σ^{-1sat} is defined as

$$\Sigma^{-1sat} = \begin{bmatrix} \min(\sigma_1^{-1}, c_1^{-1}) & 0 & 0 \\ 0 & \min(\sigma_2^{-1}, c_1^{-1}) & 0 \\ 0 & 0 & \min(\sigma_3^{-1}, c_1^{-1}) \end{bmatrix}, \quad (56)$$

where $c_1 > 0$ is a saturation limit. Note that unlike the standard pseudoinverse of a matrix approaching rank deficiency, whose matrix norm grows quite large until the matrix is within some tolerance of singularity and then becomes singular, the saturated pseudoinverse is always full rank and is saturated at c_1^{-1} .

Using the saturated pseudoinverse, control laws (39) and (48) become

$$u = -Y^{+sat}(K_p S + K_v \omega). \quad (57)$$

and

$$u = Y^{+sat}(v_1 + v_2 + v_3). \quad (58)$$

Additionally, if the saturated pseudoinverse based controllers (57), (58) specify inputs with magnitudes larger than u_{\max} , let

$$u_{\text{sat}} = u_{\max} \frac{u}{\|u\|}. \quad (59)$$

We use (59) only when $\Sigma^{-1} \neq \Sigma^{-1sat}$ in order to prevent \dot{u} from becoming arbitrarily large. Note that (57), (58) reduce to (39), (48) when Y is nonsingular.

B. Singularity Robust inverse method

One technique used in steering laws is the singularity robust (SR) inverse.²⁵ The idea behind the SR inverse is that it trades off between introducing torque errors in the vicinity of a singularity and the feasibility of the solution - feasibility indicating that the gimbal angular velocities remain bounded, unlike the case of the Moore-Penrose inverse.

The SR inverse is derived from the following optimization problem,

$$\text{minimize } e^T W e \quad (60)$$

where $e = \begin{bmatrix} \tau - Yu & u \end{bmatrix}^T$, τ is the desired torque in a steering law formulation of the CMG problem, and $W = \text{diag}(W_1, W_2)$ is a block diagonal weight matrix.

The SR inverse is thus given by

$$Y^\# = W_2^{-1}Y^T (YW_2Y^T + W_1^{-1}). \quad (61)$$

Note that different values of W_1 and W_2 yield different SR inverses and that by selecting $W_1 = 0$ and $W_2 = I$ one recovers the Moore-Penrose inverse.

We consider the SR inverse for the inversion of Y in (39) and (48). Since the SR inverse introduces error into the inversion, we test both control laws ability to compensate for this disturbance. The control laws thus become

$$u = -Y^\#(K_p S + K_v \omega). \quad (62)$$

and

$$u = Y^\#(v_1 + v_2 + v_3). \quad (63)$$

In the simulations below we do not modify the weight matrices W_1 and W_2 based on the distance of Y from singularity.

VII. Simulation

Simulation results are used to illustrate the effectiveness of controllers (57), (58), (62) and (63) in controlling the spacecraft attitude and angular velocity using CMG actuators. To simulate the rest-to-rest, motion-to-rest and spin maneuvers, the following spacecraft parameters are assumed. The bus inertia matrix J_b is given by

$$J_b = \begin{bmatrix} 5 & -0.1 & -0.5 \\ -0.1 & 2 & 1 \\ -0.5 & 1 & 3.5 \end{bmatrix} \text{kg-m}^2, \quad (64)$$

with principal moments of inertia 1.4947, 3.7997, and 5.2056 kg-m². J_b is unknown to the controller. The axes of rotation of the three CMGs are aligned with the spacecraft body-fixed frame unit vectors, and we assume that the gyro wheel inertias in their home position with respect to their own centers are given by $J_{w_1} = \text{diag}(0.3, 0.3, 0.3)$ kg-m², $J_{w_2} = \text{diag}(0.3, 0.3, 0.3)$ kg-m², and $J_{w_3} = \text{diag}(0.3, 0.3, 0.3)$ kg-m². The gyro wheels are assumed to be running at a constant speed of 20 rad/sec. Let $K_v = I_3$, and let K_p be given by

$$K_p = \frac{\alpha}{\text{tr} A}. \quad (65)$$

Illustrative rest-to-rest, motion-to-rest and spin maneuvers are considered below.

A. Rest-to-Rest Maneuver using Control Laws (57) and (62)

We use controller (57) for a rest-to-rest maneuver, where the objective is to bring the spacecraft from the initial attitude $R_0 = I_3$ to the desired final orientation

$$R_d = \begin{bmatrix} 0.4536 & 0.8912 & 0 \\ -0.8912 & 0.4536 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}, \quad (66)$$

which represents a rotation of 1.1 radians about the z -axis. We assume that the gimbals are initially at rest relative to the spacecraft, that is,

$$u(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \text{rad/sec.}$$

The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.419$ rad, $\phi_3 = 3.14$ rad. Let $c_1 = 2$ and $u_{\max} = 3$.

Figures 1(a)-(f) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}) and singular values of Y . The spacecraft attitude and angular velocity components are brought close to the desired values in about 10 sec. It can be seen from Figure 1(e) that the control input derivatives decrease to zero in about 0.5 sec. Figure 1(f) confirms that the controller is successfully able to avoid singularities during this maneuver.

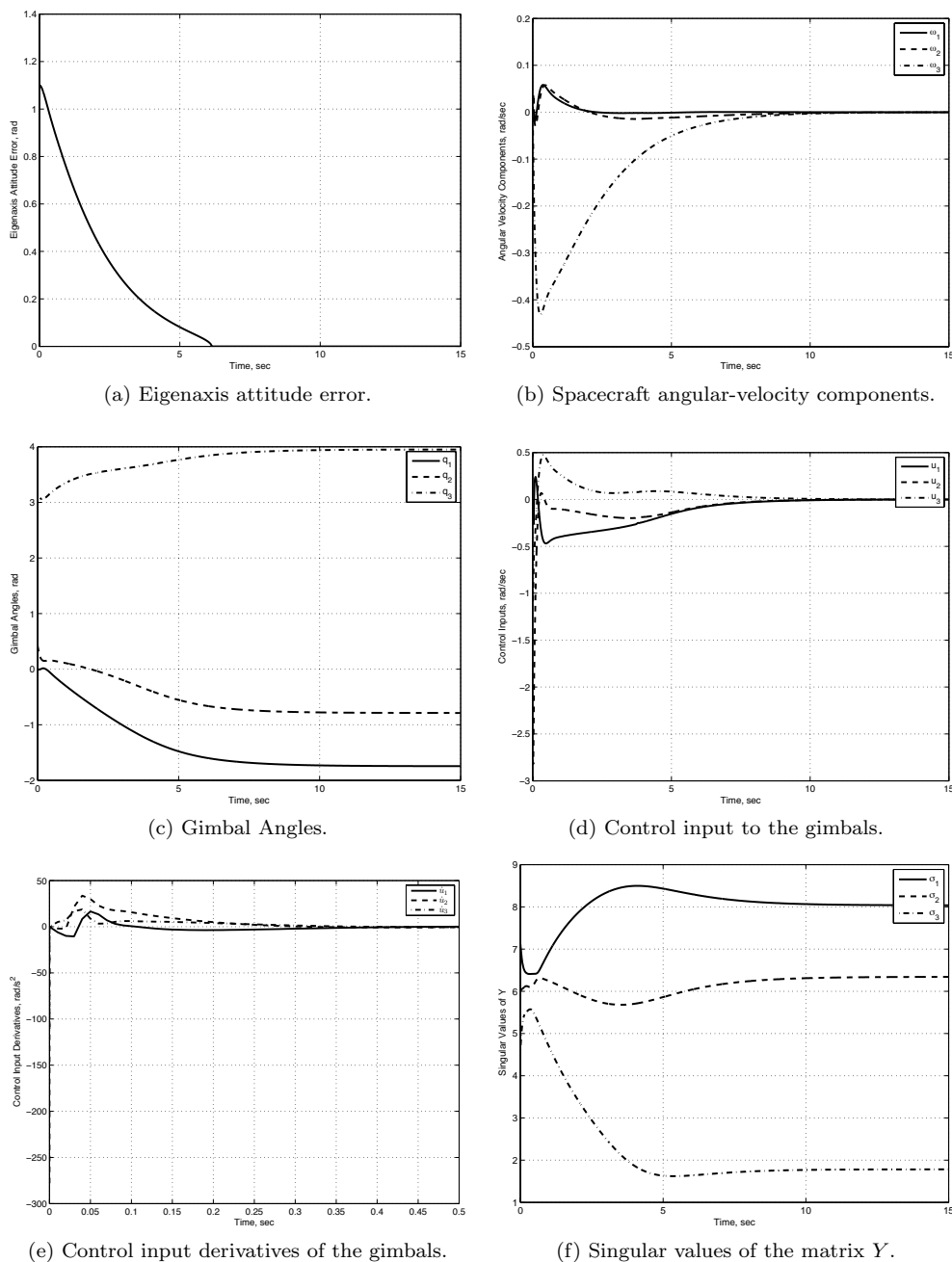


Figure 1: Rest-to-Rest maneuver with CMGs using control law (57).

We now use controller (62) for the same maneuver. Let $W_1 = \text{diag}(7, 8, 9)$ and $W_2 = \text{diag}(14, 15, 6)$. The initial gimbal angles in this case are $\phi_1 = 0$ rad, $\phi_2 = 0.897$ rad, $\phi_3 = 1.744$ rad. All other initial conditions are the same as in the previous maneuver.

Figures 2(a)-(f) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}) and singular values of Y . The spacecraft

attitude and angular velocity components are brought close to the desired values in about 10 sec. It can be seen from Figure 2(e) that the control input derivatives decrease to zero in about 1 sec. Figure 2(f) confirms that the controller is successfully able to avoid singularities during this maneuver.

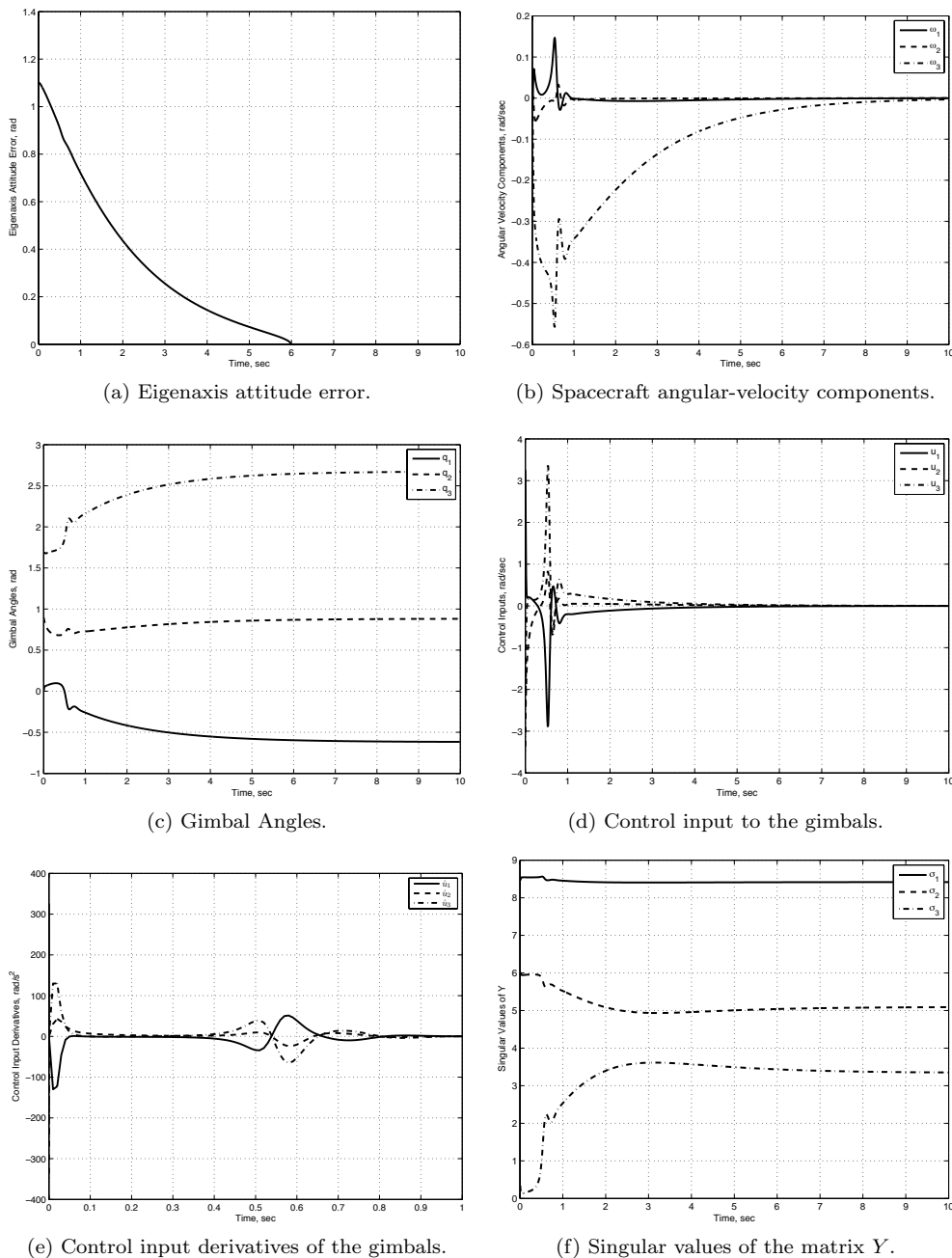


Figure 2: Rest-to-Rest maneuver with CMGs using control law (62).

B. Motion-to-Rest Maneuver using Control Laws (57) and (62)

We now consider a motion-to-rest maneuver, where the spacecraft is initially spinning with an angular velocity

$$\omega_0 = \begin{bmatrix} 0.5 & -0.5 & -0.5 \end{bmatrix}^T \text{ rad/sec,}$$

and initial attitude $R_d(0) = I_3$, where the objective is to bring the spacecraft to rest at the desired final orientation R_d as given above. The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.419$ rad, $\phi_3 = 3.14$ rad. Let $c_1 = 1$ and $u_{\max} = 15$.

Figures 3(a)-(f) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}) and singular values of Y . The spacecraft attitude and angular velocity components are brought close to the desired values in about 5 sec. It can be seen from Figure 3(e) that the control input derivatives decrease to zero in about 0.8 sec. Figure 3(f) confirms that the controller is successfully able to avoid singularities during this maneuver.

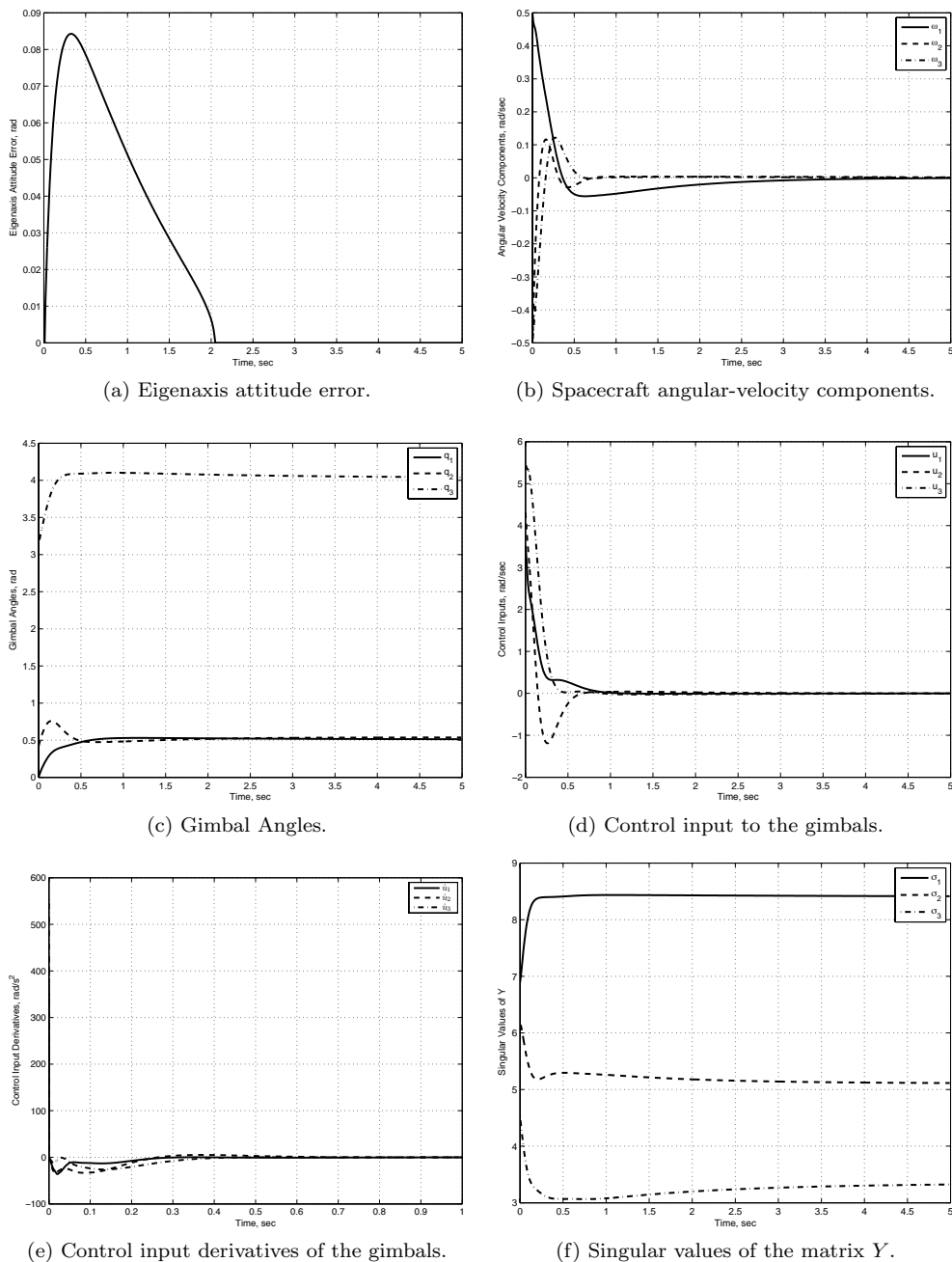


Figure 3: Motion-to-Rest maneuver with CMGs using control law (57).

We now use controller (62) for the same maneuver. Let $W_1 = \text{diag}(7, 8, 9)$ and $W_2 = \text{diag}(14, 15, 6)$. The initial gimbal angles in this case are $\phi_1 = 0$ rad, $\phi_2 = 0.897$ rad, $\phi_3 = 1.744$ rad. All other initial conditions are the same as in the previous maneuver.

Figures 4(a)-(f) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}) and singular values of Y . The spacecraft attitude and angular velocity components are brought close to the desired values in about 2.5 sec. It can be seen from Figure 4(e) that the control input derivatives decrease to zero in about 0.3 sec. Figure 4(f) confirms that the controller is successfully able to avoid singularities during this maneuver.

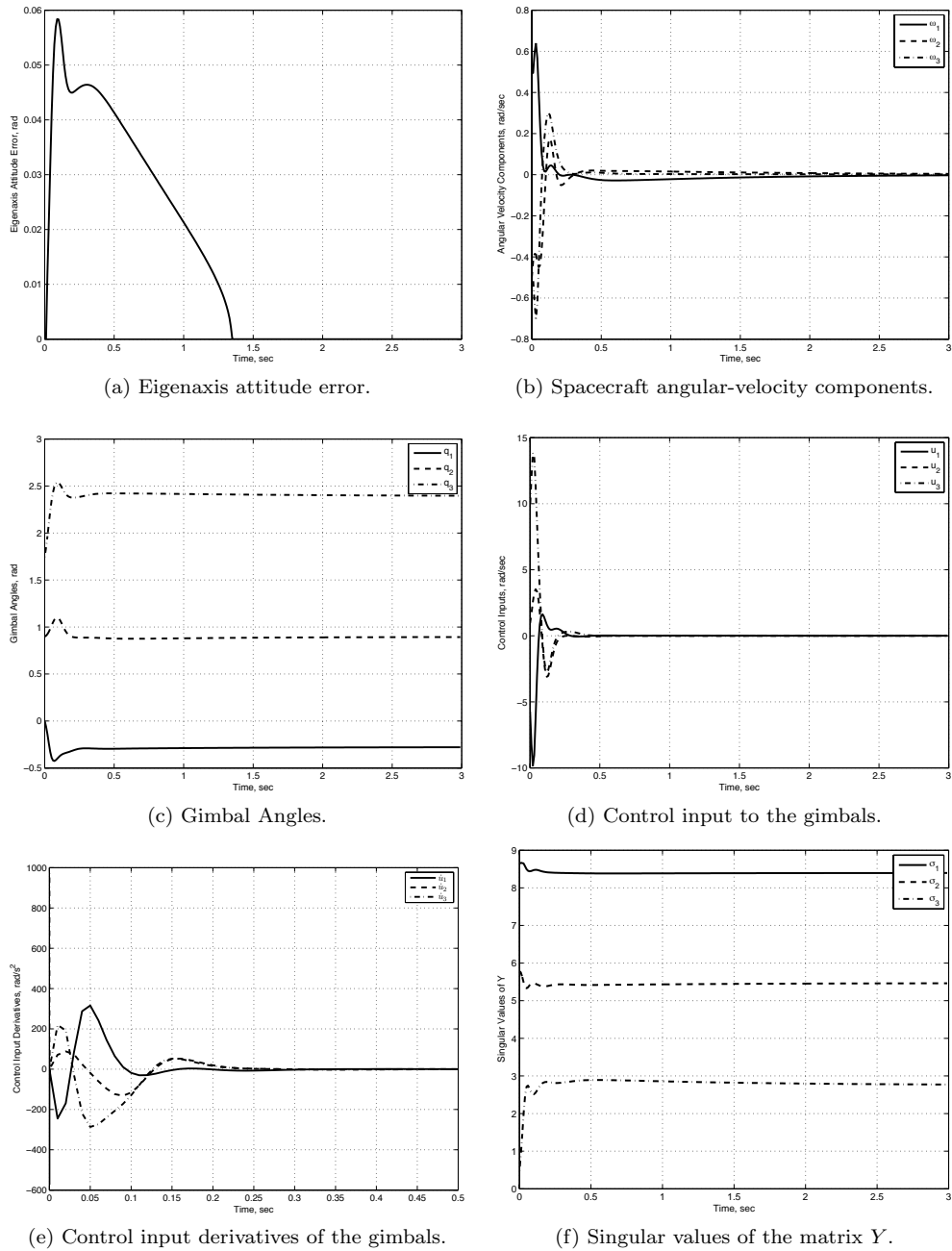


Figure 4: Motion-to-Rest maneuver with CMGs using control law (62).

C. Rest-to-Rest Maneuver using Control Laws (58) and (63)

We use controller (58) for a rest-to-rest maneuver, where the objective is to bring the spacecraft from the initial attitude $R_0 = I_3$ to the desired final orientation

$$R_d = \begin{bmatrix} 0.0707 & 0.9975 & 0 \\ -0.9975 & 0.0707 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix} \quad (67)$$

which represents a rotation of 1.5 radians about the z -axis. We assume that the gimbals are initially at rest relative to the spacecraft, that is,

$$u(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \text{ rad/sec.}$$

The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.419$ rad, $\phi_3 = 3.14$ rad. The parameters of controller (58) are chosen to be $K_1 = I_3$, $\alpha = 1$, $A = \text{diag}(1, 2, 3)$, $D = I_3$, and $Q = I_6$. Let $c_1 = 1$ and $u_{\max} = 1$.

Figures 5(a)-(g) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}), singular values of Y and inertia estimate errors. The spacecraft attitude and angular velocity components are brought close to the desired values in about 15 sec. It can be seen from Figure 5(e) that the control input derivatives decrease to zero in about 6 sec. Figure 5(f) confirms that the controller is successfully able to pass through singularities during this maneuver.

We now use controller (63) for the same maneuver. Let $W_1 = \text{diag}(7, 8, 9)$ and $W_2 = \text{diag}(14, 15, 6)$. The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.897$ rad, $\phi_3 = 1.744$ rad. All other initial conditions and controller parameters are the same as in the previous maneuver.

Figures 6(a)-(g) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}), singular values of Y and inertia estimate errors. The spacecraft attitude and angular velocity components are brought close to the desired values in about 15 sec. It can be seen from Figure 6(e) that the control input derivatives decrease to zero in about 5 sec. Figure 6(f) confirms that the controller is successfully able to pass through singularities during this maneuver.

D. Motion-to-Rest Maneuver using Control Laws (58) and (63)

We now consider a motion-to-rest maneuver, where the spacecraft is initially spinning with an angular velocity

$$\omega_0 = \begin{bmatrix} 0.5 & -1 & -0.5 \end{bmatrix}^T \text{ rad/sec,}$$

and initial attitude $R_d(0) = I_3$, where the objective is to bring the spacecraft to rest at the desired final orientation R_d as given above. Likewise, the controller parameters are as given in the previous section. The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.419$ rad, $\phi_3 = 3.14$ rad. Let $c_1 = 1$ and $u_{\max} = 1$.

Figures 7(a)-(g) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}), singular values of Y and inertia estimate errors. The spacecraft attitude and angular velocity components are brought close to the desired values in about 25 sec. It can be seen from Figure 7(e) that the control input derivatives decrease to zero in about 0.1 sec. Figure 7(f) confirms that the controller is successfully able to pass through singularities during this maneuver.

We now use controller (63) for the same maneuver. Let $W_1 = \text{diag}(7, 8, 9)$ and $W_2 = \text{diag}(140, 150, 60)$. The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.897$ rad, $\phi_3 = 1.744$ rad. All other initial conditions and controller parameters are the same as in the previous maneuver.

Figures 8(a)-(g) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}), singular values of Y and inertia estimate errors. The spacecraft attitude and angular velocity components are brought close to the desired values in about 20 sec. It can be seen from Figure 8(e) that the control input derivatives decrease to zero in about 10 sec. Figure 8(f) confirms that the controller is successfully able to pass through singularities during this maneuver.

E. Spin Maneuver using Control Law (63)

We use controller (63) for a spin maneuver, where the objective is to bring the spacecraft from rest to a spin of

$$\omega_d = \begin{bmatrix} 0.005 & -0.005 & 0.003 \end{bmatrix}^T \text{ rad/sec,}$$

We assume that the gimbals are initially at rest relative to the spacecraft, that is,

$$u(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T \text{ rad/sec.}$$

Let $W_1 = \text{diag}(7, 8, 9)$ and $W_2 = \text{diag}(14, 15, 6)$. The initial gimbal angles are $\phi_1 = 0$ rad, $\phi_2 = 0.897$ rad, $\phi_3 = 1.744$ rad. All other controller parameters are the same as in the previous section.

Figures 9(a)-(g) show, respectively, the attitude error, angular velocity components, gimbal angles, gimbal angular velocity control inputs (u), gimbal angular accelerations (\dot{u}), singular values of Y and inertia estimate errors. The spacecraft attitude and angular velocity components are brought close to the desired values in about 70 sec. It can be seen from Figure 9(e) that the control input derivatives decrease to zero in about 20 sec. Figure 9(f) confirms that the controller is successfully able to avoid singularities during this maneuver.

VIII. Conclusion

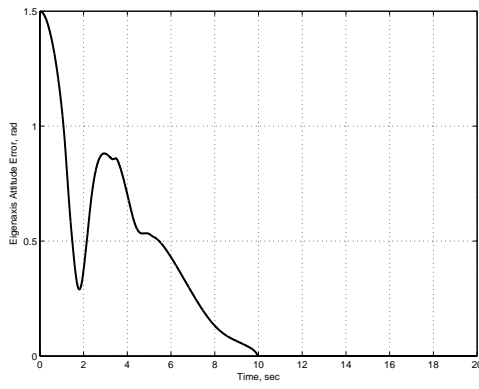
We extended the control laws of ref.¹ to the case of control moment gyroscope actuation with gimbal velocity commands. We carefully derived the equations of motion of a spacecraft with three linearly independent CMG actuators with spherical gyro wheels, whose gimbal axes of rotation are not necessarily aligned with the principal axes of the spacecraft bus, do not necessarily pass through the spacecraft's center of mass, and are not necessarily mass balanced in order to preserve the location of the spacecraft's center of mass. We do not require knowledge of the spacecraft's inertia, only the inertia of the gyro wheels. These results have practical advantages relative to previous controllers that 1) require exact or approximate inertia information or 2) are based on attitude parameterizations such as quaternions that require discontinuous control laws or fail to be physically consistent (that is, specify different control torques for the same physical orientation).

We discussed two singularity avoidance methods in this paper, namely, the saturated pseudoinverse and the singularity robust (SR) inverse. We simulated the controllers for rest-to-rest, motion-to-rest, and spin maneuvers that pass through CMG singularities using both methods. We found that the spacecraft achieves the desired orientation and spin.

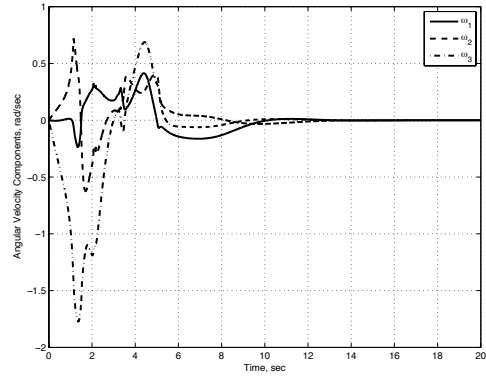
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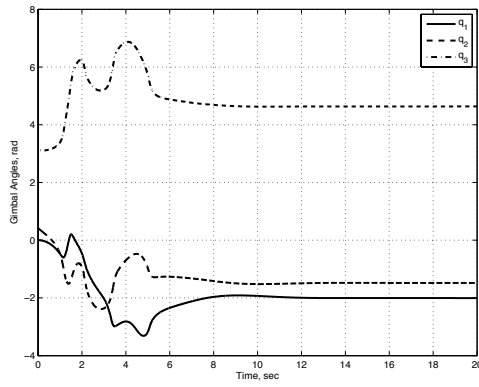
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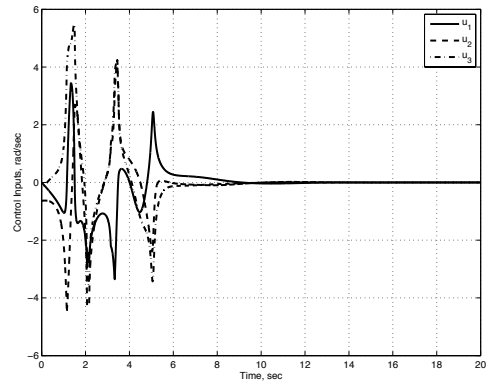
(a) Eigenaxis attitude error.



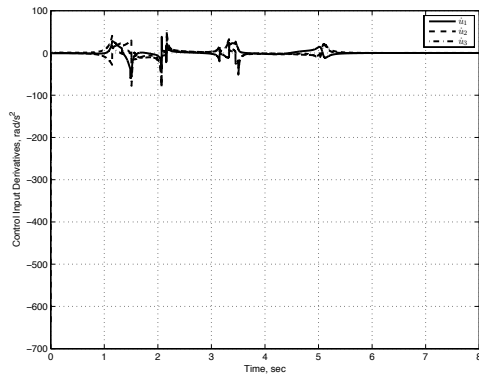
(b) Spacecraft angular-velocity components.



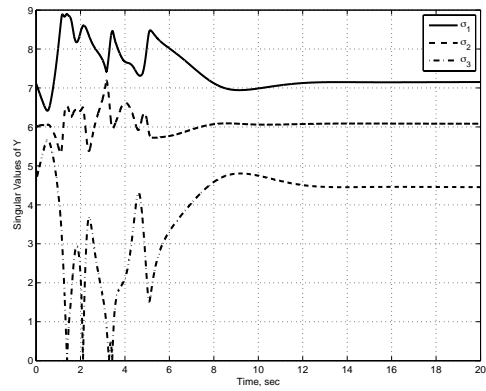
(c) Gimbal Angles.



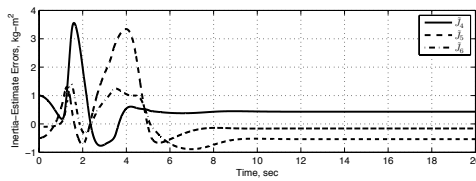
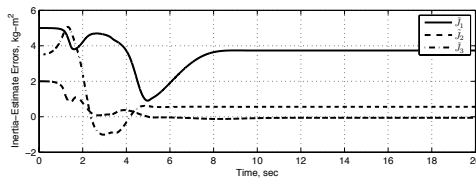
(d) Control input to the gimbals.



(e) Control input derivatives of the gimbals.

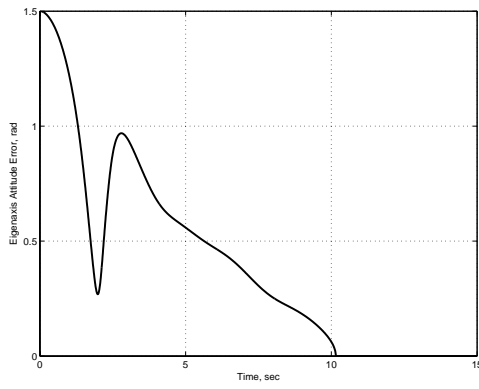


(f) Singular values of the matrix Y .

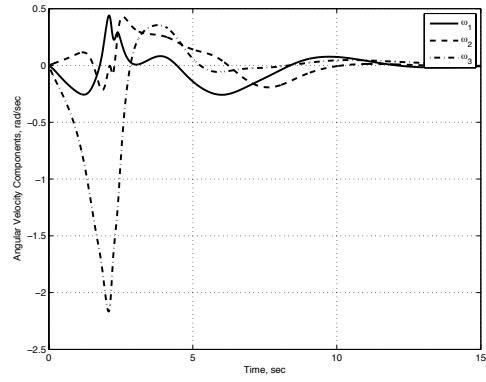


(g) Inertia-estimate errors.

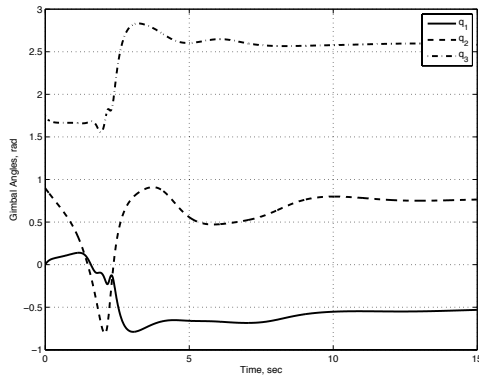
Figure 5: Rest-to-Rest maneuver with CMGs using control law (58).



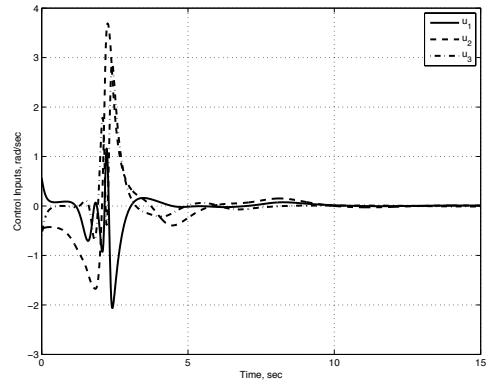
(a) Eigenaxis attitude error.



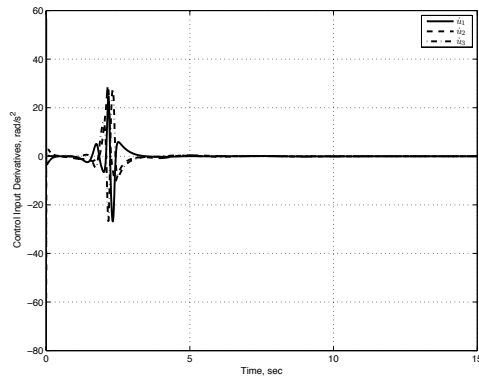
(b) Spacecraft angular-velocity components.



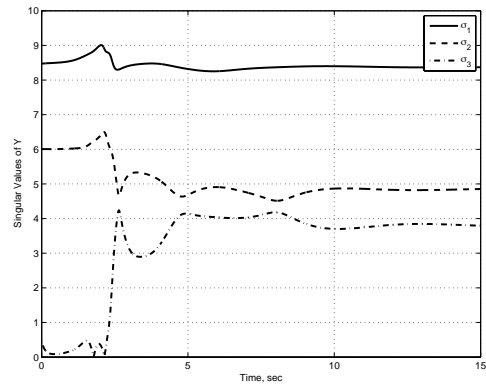
(c) Gimbal Angles.



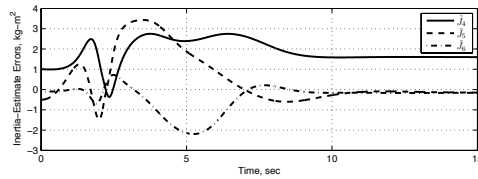
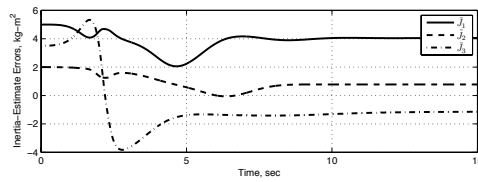
(d) Control input to the gimbals.



(e) Control input derivatives of the gimbals.

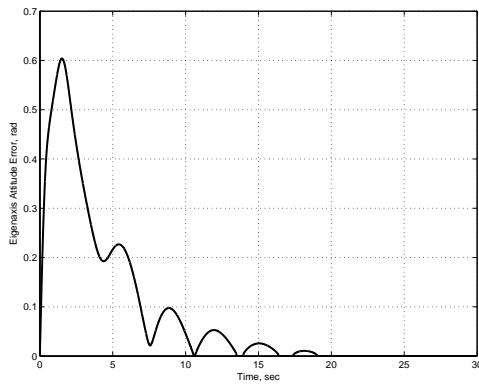


(f) Singular values of the matrix Y .

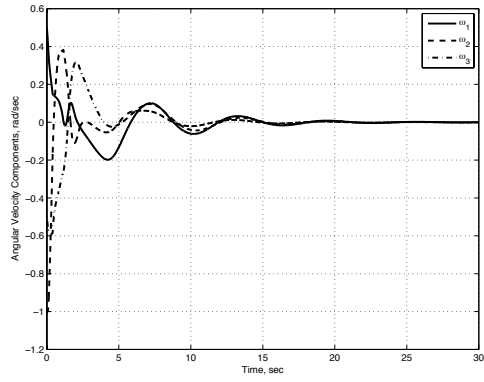


(g) Inertia-estimate errors.

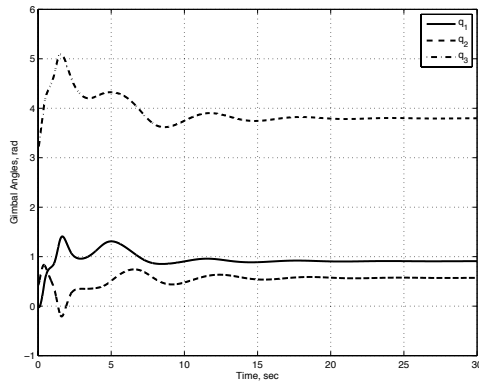
Figure 6: Rest-to-Rest maneuver with CMGs using control law (63).



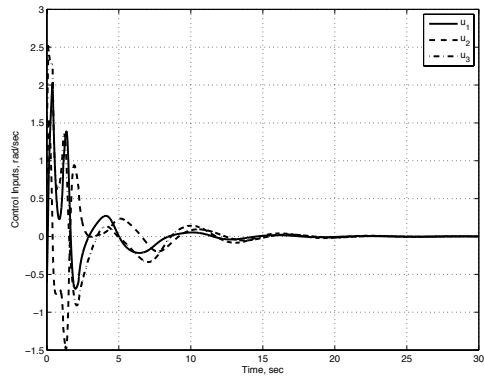
(a) Eigenaxis attitude error.



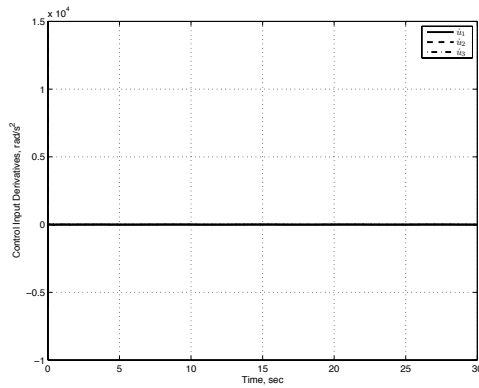
(b) Spacecraft angular-velocity components.



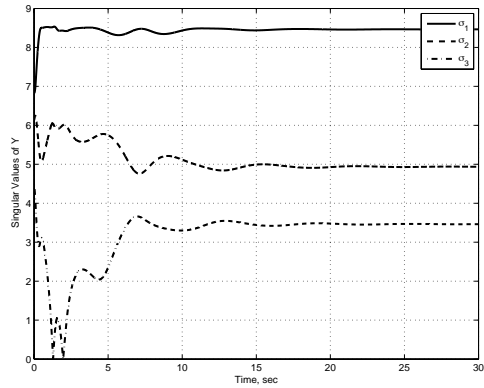
(c) Gimbal Angles.



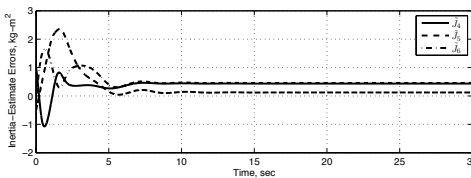
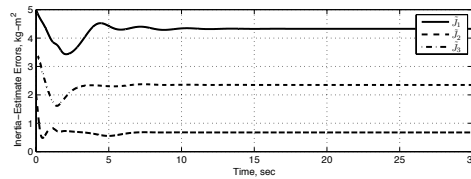
(d) Control input to the gimbals.



(e) Control input derivatives of the gimbals.

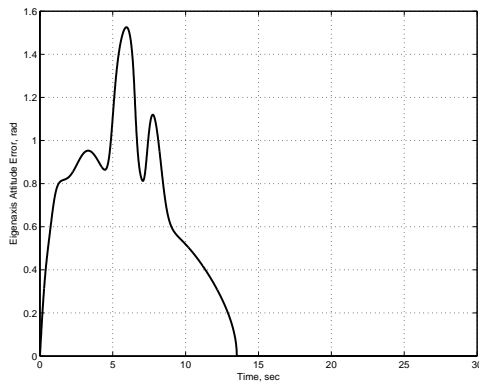


(f) Singular values of the matrix Y .

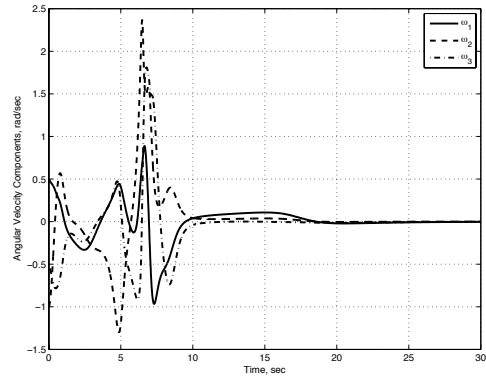


(g) Inertia-estimate errors.

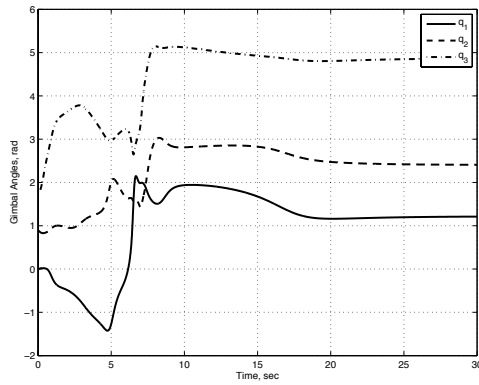
Figure 7: Motion-to-Rest maneuver with CMGs using control law (58).



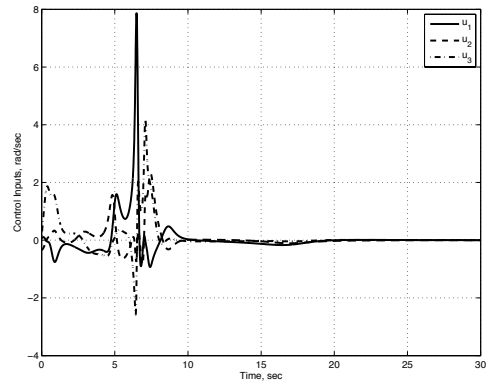
(a) Eigenaxis attitude error.



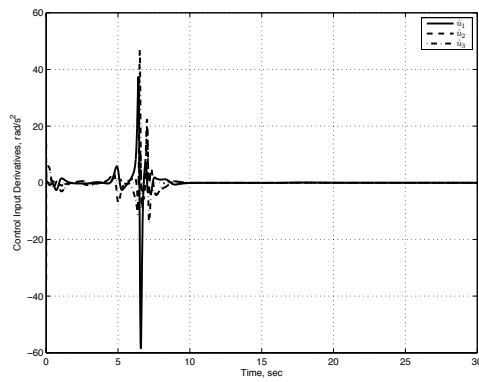
(b) Spacecraft angular-velocity components.



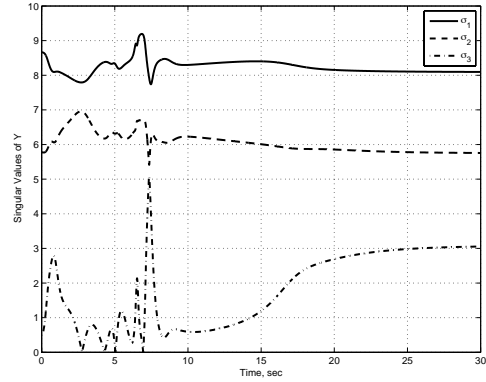
(c) Gimbal Angles.



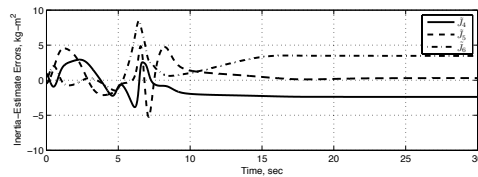
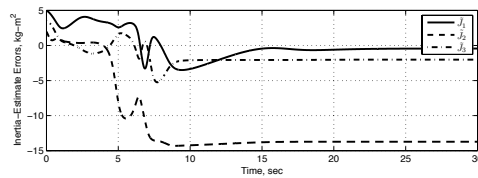
(d) Control input to the gimbals.



(e) Control input derivatives of the gimbals.

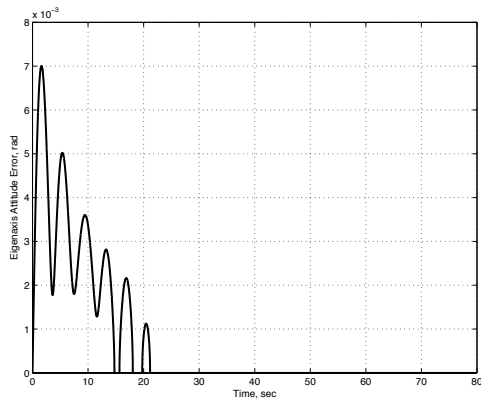


(f) Singular values of the matrix Y .

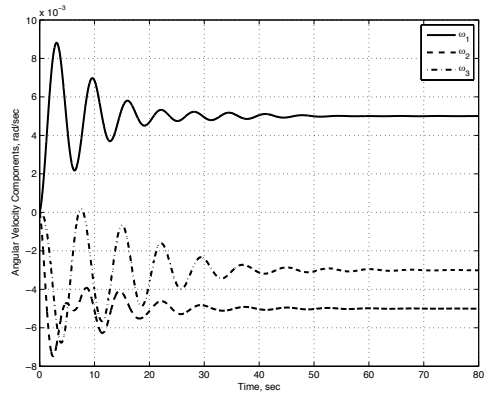


(g) Inertia-estimate errors.

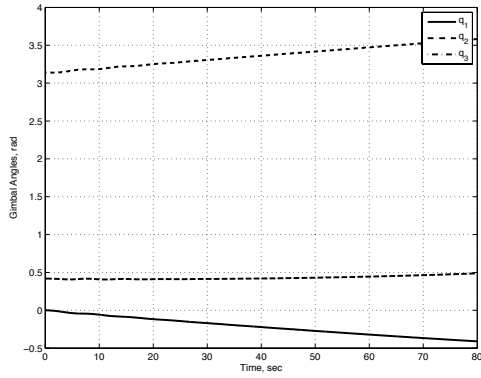
Figure 8: Motion-to-Rest maneuver with CMGs using control law (63).



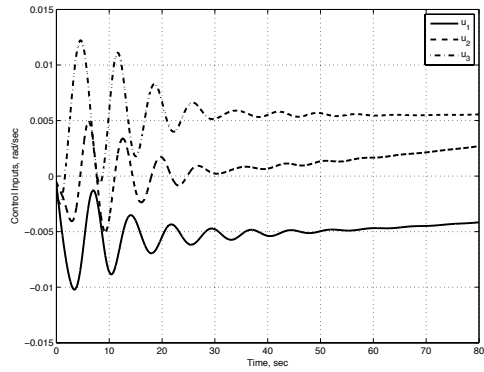
(a) Eigenaxis attitude error.



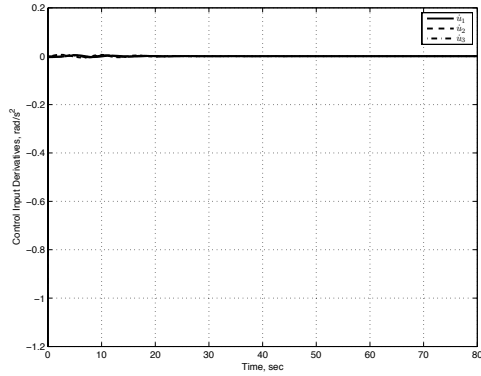
(b) Spacecraft angular-velocity components.



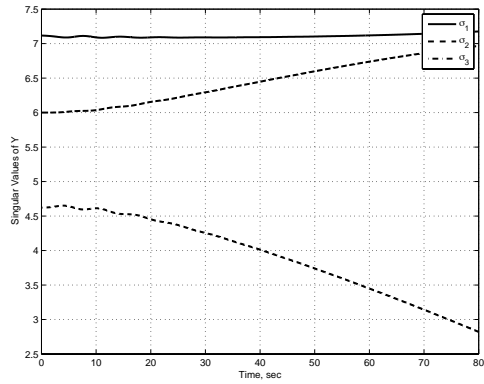
(c) Gimbal Angles.



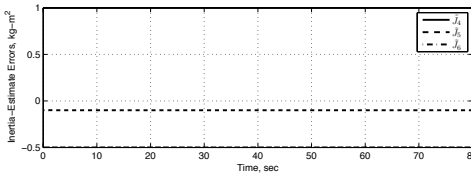
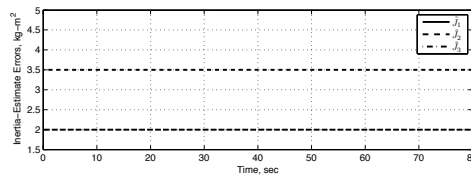
(d) Control input to the gimbals.



(e) Control input derivatives of the gimbals.



(f) Singular values of the matrix Y .



(g) Inertia-estimate errors.

Figure 9: Spin maneuver with CMGs using control law (63).