Identification of Sensor-Only MIMO Pseudo Transfer Functions

Adam J. Brzezinski\textsuperscript{1}, Sunil L. Kukreja\textsuperscript{2}, Jun Ni\textsuperscript{3}, and Dennis S. Bernstein\textsuperscript{4}

Abstract—In many applications of system identification, output measurements are available, but measurements of the input are not available. In these cases, sensor-only identification techniques are needed. In the present paper, we define a discrete-time MIMO pseudo transfer function (PTF) between sensor measurements in a sampled-data system with multiple excitation signals. We show that the MIMO PTF does not depend on the unknown initial state or the unknown excitation. We also characterize the order and relative degree of the MIMO PTF. To consistently estimate the Markov parameters of the MIMO PTF in the presence of sensor noise, we use quadratically constrained least squares identification for MIMO systems. We apply this technique to a three-degree-of-freedom mass-spring-damper system to assess the accuracy of the identified PTFs.

I. INTRODUCTION

In some applications, the excitation may be unknown and thus sensor data may be the only available information for system identification. In this case, it is typically assumed that the excitation is generated by a white random process, and various system identification techniques are used to detect changes in the dynamics of the system [1–4]. Researchers have considered MIMO sensor-to-sensor relationships [5], but correct identification in the presence of multiple excitations is difficult [6]. Authors have noted that parameters identified using only sensor measurements may not correspond to the poles of the system [7], but the connection with system zeros has only recently been established [8].

Pseudo transfer functions (PTFs) are used in [8, 9] to detect system changes under unknown excitation. SISO PTFs for single-input, two-output systems are characterized in [9]. Sampling introduces additional zeros into a discrete-time input-output model if the relative degree of the continuous-time system is greater than 1 [10]. Hence, for a strictly proper continuous-time system, the order of the SISO PTF arising from a sampled-data application is $n - 1$, where $n$ is the order of the underlying system [8, 9]. Since a PTF is essentially a ratio of transfer functions, the information in a PTF consists of information about the zeros of the system from the unknown excitation to each of the sensors.

For applications involving structural dynamics such as operational modal analysis (OMA) [11], PTFs are a generalization of transmissibilities, where PTFs do not require that one of the sensors be colocated with the prescribed displacement [2]. To compare a transmissibility [12] and a PTF, consider the system $G$ in Figure 1. The transmissibility from $y_1$ to $y_2$ in Figure 1(a) assumes $y_1$ is colocated with the displacement excitation $u$ so that the transfer function from $u$ to the displacement $y_1$ is

$$y_1 = u,$$

the transfer function from $u$ to $y_2$ is

$$y_2 = \frac{N_2}{D} u,$$

and the transmissibility from $y_1$ to $y_2$ is

$$y_2 = \frac{N_2}{D} y_1. \quad (1)$$

However, the PTF from $y_1$ to $y_2$ in Figure 1(b) does not assume that $y_1$ and $u$ are colocated, and hence

$$y_1 = \frac{N_1}{D} u.$$

The PTF from $y_1$ to $y_2$ is then [8]

$$y_2 = \frac{N_2}{N_1} y_1. \quad (2)$$

The transmissibility (1) contains pole information, while the PTF (2) does not. Pole information appears in the output-to-output relationship only if the excitation and sensor measurement are colocated, and hence a transmissibility is a special type of PTF.

![Fig. 1. Difference between the transmissibility from $y_1$ to $y_2$ (a) and the pseudo transfer function (PTF) from $y_1$ to $y_2$ (b).](image)

Given multiple excitation signals, it is shown in [8] that additional sensors can be used to obtain a MIMO PTF that is independent of the excitation signals. For example, the system shown in Figure 2 is excited by both $u_1$ and $u_2$ so that the sensor measurements $y_1$, $y_2$, and $y_3$ contain contributions from both $u_1$ and $u_2$. For $j = 1, 2, 3$, as shown in Figure 2(a), $y_{j,2}$ appears as sensor noise corrupting measurement $y_j$ if the SISO PTF from $y_j$ to $y_2$ is identified. Furthermore, identification of the SISO PTF from $y_1$ to $y_2$ involves estimation in the presence of a disturbance due to $u_2$. However, as shown in Figure 2(b), identification of the MIMO PTF from $[y_1 \ y_2]^T$ to $y_3$ is noise-free in the sense that, in the absence of additional noise sources, exact identification of the MIMO PTF is possible using finite data.

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\textsuperscript{1}Sr. Flight Controls & Navigation Engineer, Rockwell Collins, Gainesville, VA.
\textsuperscript{2}Research Scientist, NASA Dryden Flight Research Center, Dryden, CA
\textsuperscript{3}Professor, Dept. of Mech. Engineering, University of Michigan, Ann Arbor, MI
\textsuperscript{4}Professor, Dept. of Aerosp. Engineering, University of Michigan, Ann Arbor, MI

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The contribution of the present paper is to analyze MIMO PTFs in terms of the conditions on the sensors under which a MIMO PTF can be defined. In particular, we consider the normal rank of the PTF as well its order and relative degree. We also go beyond the results of [8] by considering the case in which the sensors are corrupted by noise that is not due to an excitation signal. Since both sensors may be corrupted by noise, we consider an errors-in-variables identification problem. To address this problem, we apply quadratically constrained least squares [13]. Since the results of [13] are confined to SISO systems with ARMAX model structures, we develop and apply an extension to MIMO systems with $\mu$-Markov model structures. Unlike an ARMAX model structure, a $\mu$-Markov model structure requires only a lower bound on the estimated model order.

II. PROBLEM FORMULATION

Consider system $\mathcal{J}$ with unknown inputs $\bar{u}_1, \ldots, \bar{u}_m$ and measured outputs $\bar{y}_1, \ldots, \bar{y}_p$ in Figure 3. Define $\bar{u} \triangleq [\bar{u}_1 \cdots \bar{u}_m]^T$ and $\bar{y} \triangleq [\bar{y}_1 \cdots \bar{y}_p]^T$. For $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, and $D \in \mathbb{R}^{p \times m}$, a state space representation of $\mathcal{J}$ is denoted by $(A, B, C, D)$, where

$$\dot{x}(t) = A\bar{x}(t) + B\bar{u}(t), \quad x(0) = x_0, \quad (3)$$

$$\bar{y}(t) = C\bar{x}(t) + D\bar{u}(t). \quad (4)$$

For $k \in \{0, 1, \ldots\}$ and sampling interval $h > 0$, we solve (3) for $\bar{x}(t)$ from $t = kh$ to $t = kh + h$, which yields

$$\bar{x}(kh+h) = e^{Akh}\bar{x}(kh) + \int_{kh}^{kh+h} e^{A(kh+h-\tau)}B\bar{u}(\tau)d\tau. \quad (5)$$

We assume that $\bar{u}(t)$ changes sufficiently slowly that $\bar{u}(t) \approx \bar{u}(kh)$ for all $t \in [kh, kh+h]$. Hence, we rewrite (5) as

$$x(k+1) = Ax(k) + Bu(k), \quad (6)$$

where $x(k) \triangleq \bar{x}(kh)$, $u(k) \triangleq \bar{u}(kh)$, $A \triangleq e^{Akh}$, and

$$B \triangleq \int_0^h e^{A\tau} d\tau B.$$
where
\[ \Gamma(q) \triangleq \frac{d(q)}{x(q)+e(q)} L(q) \det(U(q)) \in \mathbb{R}^{[p-m] \times m} \] (14)
is the MIMO PTF from \( y[1:m] \) to \( y[m+1:p] \). For the case \( m = 1 \), the common factor \( \delta(q) \) can be cancelled [8].

Note that
\[ U(q) = PQ(q), \] (15)
where
\[ P \triangleq \begin{bmatrix} C_U & D_U \end{bmatrix} \in \mathbb{R}^{m \times (n+m)} \]
and
\[ Q(q) \triangleq \begin{bmatrix} \text{adj}(qI - A) & B \\ \delta(q)I_m \end{bmatrix} \in \mathbb{R}^{(n+m) \times m}. \]
The following result uses (15) to provide a necessary condition for \( \det U(q) \neq 0 \).

**Proposition 3.1:** If \( \det U(q) \neq 0 \), then rank \( P = m \).

**Proof 3.1:**

normal rank \( U(q) = \) normal rank \( PQ(q) \)

\[ = \min \{ \text{rank} P, \text{normal rank} Q(q) \} = \text{rank} P = m. \] □

Note that
\[ U(q) = R(q)S, \] (16)
where
\[ R(q) \triangleq \begin{bmatrix} C_U & \text{adj}(qI - A) \end{bmatrix} \delta(q)I_m \in \mathbb{R}^{m \times (n+m)} \]
and
\[ S \triangleq \begin{bmatrix} B & D_U \end{bmatrix} \in \mathbb{R}^{(n+m) \times m}. \]
The following result uses (16) to provide a necessary condition for \( \det U(q) \neq 0 \).

**Proposition 3.2:** If \( \det U(q) \neq 0 \), then rank \( S = m \).

**Proof 3.2:**

normal rank \( U(q) = \) normal rank \( R(q)S \)

\[ = \min \{ \text{normal rank} R(q), \text{rank} S \} = \text{rank} S = m. \] □

The following result provides necessary and sufficient conditions for \( \det U(q) \neq 0 \).

**Proposition 3.3:** \( \det U(q) \neq 0 \) if and only if \( G_U(q) \) has full normal rank.

**Proof 3.3:**

normal rank \( G_U(q) = \) normal rank \( C_U (qI - A)^{-1} B + D_U \)

\[ = \text{normal rank} U(q) = m. \] □

Propositions 3.1 and 3.2 provide necessary conditions for \( \det U(q) \neq 0 \). However, the following example shows that these conditions are not sufficient for \( G_U(q) \) to have full normal rank and thus for \( \det U(q) \neq 0 \).

**Example 3.1:** Let
\[ A = -\frac{1}{4} \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \]

\[ C_U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_U = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]
Note that \( A \) is asymptotically stable, \( B \) and \( C_U \) are full-rank, \( A, B, C_U, D_U \) is minimal, and rank \( P = \text{rank} S = m \).

However,
\[ G_U(q) = \frac{1}{q^3 + q^2 + \frac{2}{3}q + \frac{1}{3}} \cdot \begin{bmatrix} \frac{-(q + \frac{1}{3})(2q + 1)}{3} & \frac{-(q + \frac{1}{3})(2q + 1)}{3} \\ \frac{-(q + \frac{1}{3})(2q + 1)}{3} & \frac{-(q + \frac{1}{3})(2q + 1)}{3} \end{bmatrix}. \] (17)

**IV. MIMO PTF ORDER AND RELATIVE DEGREE**

For \( i = 1, \ldots, p \), we write
\[ y_i(k) = C_i x(k) + D_i u(k), \]
and thus
\[ \delta(q)y_i = N_i(q)u, \]
where \( N_i(q) \) is the \( i \)th row of \( N(q) \), \( C_i \) is the \( i \)th row of \( C \), and \( D_i \) is the \( i \)th row of \( D \). For all \( i \in \{1, \ldots, p\} \) and all \( j \in \{1, \ldots, m\} \),
\[ \eta_{i,j}(q) \triangleq (N(q))_{(i,j)} = C_i \text{adj}(qI - A)B_j + D_i \delta(q). \]

**Proposition 4.1:** Let \( G_U(q) \) have full normal rank. Then
\[ \deg (\det (U(q))) \leq nm, \]
and, for all \( i, j \in \{1, \ldots, m\} \),
\[ \deg (\text{adj} (U(q)))_{(i,j)} \leq n(m-1). \]

**Proof 4.1:** For all \( r \in \{1, \ldots, p\} \) the degree of \( \eta_{r,j}(q) \) is \( n \) if \( D_{r,j} \neq 0 \) and \( n-1 \) otherwise [8]. Hence, computing \( \det U(q) \) using the cofactor expansion yields
\[ \deg (\det (U(q))) \leq m \left( \max_{i,j} \eta_{i,j}(q) \right) = mn. \]

Furthermore,
\[ \deg (\text{adj} (U(q)))_{(i,j)} \leq (m-1) \left( \max_{i,j} \eta_{i,j}(q) \right) = (m-1)n. \] □

**Proposition 4.2:** Let \( G(q) \) have full normal rank. Then, for all \( i \in \{1, \ldots, p\} \) and all \( j \in \{1, \ldots, m\} \),
\[ \deg \{L(q) \text{adj}(U(q))\}_{(i,j)} \leq nm. \]

**Proof 4.2:** For all \( r \in \{1, \ldots, p\} \) the degree of \( \eta_{r,j}(q) \) is \( n \) if \( D_{r,j} \neq 0 \) and \( n-1 \) otherwise [8]. Hence, for all \( k \in \{1, \ldots, m\} \),
\[ \deg \{L(q) \text{adj}(U(q))\}_{(i,j)} \leq \max_{i,j} \left( \deg \{L(q)\}_{(i,j)} \right) + \max_{j,k} \left( \deg \{\text{adj}(U(q))\}_{(j,k)} \right) \]
\[ = n + n(m-1) = nm. \] □

The following result characterizes the order and relative degree of each of the entries of the MIMO PTF (14).

**Theorem 4.1:** Let \( G(q) \) have full normal rank and assume that the common factor \( \delta(q) \) in (14) can be cancelled. Then, for all \( i \in \{1, \ldots, p-m\} \) and for all \( j \in \{1, \ldots, m\} \),
the order of $\Gamma(q)_{(i,j)}$ is less than or equal to $nm$. Furthermore, the relative degree $d_{(i,j)}$ of $\Gamma(q)_{(i,j)}$ is given by

$$d_{(i,j)} \triangleq \deg \left[ \det \left( U(q) \right) \right] - \deg \left[ L(q) \det \left( U(q) \right) \right].$$

Proof 4.3: The result follows from Prop. 4.1 and 4.2.

V. THREE-SENSOR, TWO-INPUT CASE

Let $p = 3$ and $m = 1$. From (54) of [8], the PTF from $y_1$ to $y_3$ is given by

$$y_3 = \frac{\eta_{3,1}(q)}{\eta_{1,1}(q)} y_1.$$  \hfill (18)

Next, let $m = 2$. Assuming $\delta(q)$ in (14) can be cancelled, the PTF from $Y_{[1:2]}$ to $y_3$ is given by

$$y_3 = \Gamma_{(1,1)}(q)y_1 + \Gamma_{(1,2)}y_2,$$ \hfill (19)

where

$$\Gamma_{(1,1)}(q) = \frac{\eta_{3,1}(q)\eta_{2,1}(q) - \eta_{3,2}(q)\eta_{1,1}(q)}{\eta_{1,1}(q)\eta_{2,1}(q) - \eta_{2,1}(q)\eta_{1,1}(q)}$$

and

$$\Gamma_{(1,2)}(q) = \frac{\eta_{3,1}(q)\eta_{1,1}(q) - \eta_{3,1}(q)\eta_{1,1}(q)}{\eta_{1,1}(q)\eta_{2,1}(q) - \eta_{2,1}(q)\eta_{1,1}(q)}.$$ \hfill (20)

It follows from (19) that, if two excitation signals are present, the SISO PTF from $y_1$ to $y_3$ given by (18) is incorrect. We see from (19) that both $y_1$ and $y_2$ contribute to $y_3$.

VI. EXAMPLES

Consider the mass-spring-damper structure in Figure 4, which has the equations of motion

$$M\ddot{q}(t) + C_d\dot{q}(t) + Kq(t) = F(t),$$ \hfill (21)

where

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix}, \quad M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$

$$C_d = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix},$$

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix},$$

$$F(t) = B\mu(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$ \hfill (22)

We express (20) in state-space form (3), where

$$\ddot{x}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix}^T, \quad \ddot{y}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ v_2(t) \end{bmatrix}^T,$$

$$\dot{\lambda} = \begin{bmatrix} 0 & I_3 \\ -M^{-1}K & -M^{-1}C_d \end{bmatrix}, \quad \dot{\bar{B}} = \begin{bmatrix} 0 & I_3 \\ M^{-1}C_d & -M^{-1}K \end{bmatrix},$$

and where $m_1 = \frac{1}{4} m_2 = \frac{1}{4} m_3 = 1$ kg, $k_1 = \frac{1}{4} k_2 = \frac{1}{4} k_3 = \frac{1}{4} k_4 = 4$ N/m, $c_1 = \frac{1}{4} c_2 = \frac{1}{4} c_3 = \frac{1}{4} c_3 = 0.1$ kg-m/s, and $h = 0.5$ s.

We use Markov parameter matrices to characterize the PTF estimate. To do this, we use the $\mu$-Markov model structure

$$A_{0\mu}y_2(k) = -A_{\mu}y_2(k - \mu) - \cdots - A_{\mu+n_{\text{mod}}-1}y_2(k - \mu - n_{\text{mod}} + 1) + H_0y_1(k) + \cdots + H_{\mu-1}y_1(k - \mu + 1) + B_{\mu}y_1(k - \mu) + \cdots + B_{\mu+n_{\text{mod}}-1}y_1(k - \mu - n_{\text{mod}} + 1)$$ \hfill (23)

of order $n_{\text{mod}}$. The absence of terms involving $y_2(k - 1), \ldots, y_2(k - \mu + 1)$ is responsible for the explicit presence of the Markov parameter matrices $H_0, \ldots, H_{\mu-1}$.

The $\mu$-Markov model structure has two principal advantages over the traditional ARMAX structure. First, within the context of least squares identification with white input, it is shown in [14] that the $\mu$-Markov model provides consistent estimates of the Markov parameters in the presence of arbitrary output noise. Furthermore, unlike parameter coefficients in an ARMAX model structure, the estimates of the Markov parameters are insensitive to the assumed model order $n_{\text{mod}}$ as long as $n_{\text{mod}}$ is larger than the true model order $n$. Consequently, only an upper bound on the true model order is needed.

For MIMO PTF identification, neither of the sensor signals is white, and thus we use quadratically constrained least squares (QCLS) with the $\mu$-Markov model structure (21). MIMO QCLS with a $\mu$-Markov model structure is developed in the Appendix. QCLS yields consistent parameter estimates in the presence of both input and output noise as long as the noise autocorrelation matrices are known to within a scalar multiple [13]. When this assumption is not satisfied, instrumental variables methods can be used [15].

To quantify the difference between the estimated and actual Markov parameter matrices, we define

$$\varepsilon_T \triangleq \frac{||T - \hat{T}||_2}{||T||_2},$$

where

$$T \triangleq \begin{bmatrix} \text{vec} (H_0) \\ \vdots \\ \text{vec} (H_3) \end{bmatrix},$$

and $\hat{T}$ is the estimate of $T$ obtained from QCLS.

A. Effect of model order with noise-free measurements

We investigate the effect of the $\mu$-Markov model order $n_{\text{mod}}$ on the accuracy of the estimated Markov parameter matrices of the PTF. We simulate (20) with $x_0 \neq 0$ to obtain the position $q_1$ of the first mass, the position $q_2$ of the second mass, and the velocity $v_2$ of the second mass, where $u_1$ and
$u_2$ are realizations of white Gaussian processes with mean 0 and variance 1.

First we consider the SISO PTF from $q_1$ to $v_2$ and use LS with the $\mu$-Markov model structure (21) of relative degree 0 to estimate the first 4 (scalar) Markov parameters of the PTF from $q_1$ to $v_2$. For 10 realizations of 1000 samples of each $u$, we construct $\varepsilon_T$ and compute the average estimation error $\bar{\varepsilon}_T$ over all $u$. Plotting $\bar{\varepsilon}_T$ as a function of the $\mu$-Markov model order $n_{mod}$ in Figure 5 shows that the Markov parameters are not correctly estimated for all $n_{mod}$ from 1 to 15. This is expected because $q_1$ and $v_2$ are corrupted by contributions from both $u_1$ and $u_2$.

Next we consider the two-input, one-output PTF from $[q_1 \ q_2]^T$ to $v_2$ and use LS with the $\mu$-Markov model structure (21) of relative degree 0 to estimate the first 4 ($2 \times 2$) Markov parameters of the PTF from $[q_1 \ q_2]^T$ to $v_2$. For 10 realizations of 1000 samples of each $u$, we construct $\varepsilon_T$ and compute the average estimation error $\bar{\varepsilon}_T$ over all $u$. Plotting $\bar{\varepsilon}_T$ as a function of the $\mu$-Markov model order $n_{mod}$ in Figure 6 shows that the Markov parameters are correctly estimated for $n_{mod} \geq 4$.

Fig. 5. Error in the Markov parameters of the SISO PTF from $q_1$ to $v_2$, estimated using SISO LS with the $\mu$-Markov model structure (21), as the $\mu$-Markov model order increases. The error in estimated Markov parameters is nonzero for all values of $n_{mod}$ because the sensor measurements arise from two excitation signals.

Fig. 6. Error in the Markov parameters of the MIMO PTF from $[q_1 \ q_2]^T$ to $v_2$, estimated using MIMO LS with the $\mu$-Markov model structure (21), as the $\mu$-Markov model order increases. For $n_{mod} \geq 4$ the Markov parameter estimates equal the true values.

B. Consistency of the estimated MIMO PTF

We now investigate the effect of sensor noise on the accuracy of the identified PTF by adding noise to the sensor measurements. We assume that measurement $y_1$ is corrupted by white, zero-mean Gaussian noise $w_1$ and we assume that each measurement $y_j$ is corrupted by white, zero-mean Gaussian noise $v_{j-m}$, where $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, p\}$. Hence, the measured pseudo-inputs $\xi_{[1:m]}$ are

$$\xi_{[1:m]} = y_{[1:m]} + v_{[1:m]}.$$
defined, as well as results on the order and relative degree of each entry of a MIMO PTF.

REFERENCES

reliability of reference-based damage identification techniques by

analysis approach based on parametrically identified multivariable

[3] L. Mevel, A. Benveniste, M. Basseville, M. Goursat, B. Peters,
H. Van der Auweraer, and A. Vecchio, “Input/output versus output
only data processing for structural identification—application to in-flight

only identification and dynamic analysis of time-varying mechanical
structures under random excitation: a comparative assessment of

concept in multi-degree-of-freedom systems,” Mechanical Systems

S. Vanlanduit, and P. Guillaume, “Structural health monitoring in
changing operational conditions using transmissibility measurements,”

from transmissibility measurements,” Journal of Sound and Vibration,

only fault detection using pseudo transfer function identification,” in

stein, “Sensor-only noncausal blind identification of pseudo transfer
functions,” in Proc. SYSID, (Saint-Malo, France), pp. 1698–1703, July
2009.


operational modal analysis:a review,” ASME Journal of Dyn. Sys.,

[12] D. E. Adams, Health Monitoring of Structural Materials and Com-
ponents: Methods with Applications. Chichester, UK: John Wiley &
Sons, Ltd., 2007.

stein, “Parameter consistency and quadratically constrained errors-in-
variables least-squares identification,” Int. J. Contr., vol. 83, no. 4,

parameters and time-delay/interactor matrix,” Chemical Engineering

identification,” Circuits Systems Signal Processing, vol. 21, no. 1,
pp. 1–9, 2002.

APPENDIX

We consider QCLS identification of a MIMO system with input
\( \tilde{u}(k) \) and output \( \tilde{y}(k) \) that satisfy
\[
\dot{\hat{A}}(\theta)\tilde{y} = \hat{B}(\theta)\tilde{u},
\]
where
\[
\hat{A}(\theta) \triangleq \hat{A}_0 \theta^n + \cdots + \hat{A}_n
\]
and
\[
\hat{B}(\theta) \triangleq \hat{B}_0 \theta^q + \cdots + \hat{B}_n.
\]
Assuming \( \hat{A}(\theta) \) has full normal rank, we can express the
identification problem (22) as the QCLS problem
\[
\min_{\theta \in \mathcal{D}(N)} \frac{1}{l} \theta^T \Phi \hat{\Phi} \theta,
\]
where
\[
\mathcal{D}(N) \triangleq \left\{ \theta \in \mathbb{R}^{\text{pm}(n+\mu+n+1)} \mid \text{pm}(n+\mu+n+1) \right\},
\]
\[
\Phi \triangleq \left[ \Phi_y - \Phi_u \right],
\]
\[
\Phi_y \triangleq \begin{bmatrix}
\mu(n + 1 - 1) & \mu(n - 1) & \cdots & \mu(0) \\
\vdots & \vdots & \ddots & \vdots \\
\mu(l) & \mu(l - \mu) & \cdots & \mu(l - \mu - n + 1)
\end{bmatrix},
\]
\[
\Phi_u \triangleq \begin{bmatrix}
\mu^T(n + 1) \otimes I_p & \cdots & \mu^T(0) \otimes I_p \\
\vdots & \ddots & \vdots \\
\mu^T(l) \otimes I_p & \cdots & \mu^T(l - \mu - n + 1) \otimes I_p
\end{bmatrix},
\]
and
\[
\theta \triangleq \begin{bmatrix}
\alpha_0 \\
\alpha_\mu \\
\vdots \\
\alpha_{\mu+n-1} \\
\vec{(H'_\mu)} \\
\vec{(H'_{\mu-1})} \\
\vdots \\
\vec{(B'_{\mu+n-1})}
\end{bmatrix} \in \mathbb{R}^{\text{pm}(n+\mu)+n+1}.
\]
The solution \( \theta \) of the QCLS problem (23) corresponds to the
generalized eigenvector associated with the smallest positive
generalized eigenvalue of \( \left( \Phi^T \Phi \right) \). Note that the
solution of the QCLS problem (23) is unbiased and consistent
under the persistency conditions given in [13].