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Reduced-Order Kalman Filtering for Time-Varying Systems

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I. INTRODUCTION

Since the classical Kalman filter provides optimal least-squares estimates of all of the states of a linear time-varying system, there is longstanding interest in obtaining simpler filters that estimate only a subset of states. This objective is of particular interest when the system order is extremely large, which occurs for systems arising from discretized partial differential equations [1].

One approach to this problem is to consider reduced-order Kalman filters. These reduced-complexity filters provide state estimates that are suboptimal [2–5]. Alternative variants of the classical Kalman filter have been developed for computationally demanding applications such as weather forecasting [6–9], where the filter gain and covariance are modified so as to reduce the computational requirements. A comparison of various techniques is given in [10]. An alternative approach to reducing complexity is to restrict the data-injection subspace to obtain a spatially localized Kalman filter. This approach is developed in [11].

In the present paper we revisit the approach of [2, 12], which considers the problem of fixed-order steady-state reduced-order estimation. For a linear time-invariant system, the optimal steady-state fixed-order filter is characterized by coupled Riccati and Lyapunov equations, whose solution requires iterative techniques.

This paper extends the results of [2, 12] by adopting the finite-horizon optimization technique used in [11] to obtain reduced-order filters that are applicable to time-varying systems. This technique also avoids the periodicity constraint associated with the multirate filter derived in [13]. Related techniques are used in [14].

In addition to the reduced-order filter considered in [2, 12], we also consider a fixed-structure subspace observer constrained to estimate a specified collection of states. This problem is considered in [3, 15]. The difference between the reduced-order filter and subspace observer is apparent in the the distinct oblique projectors τ and μ that characterize the filter and observer gains, respectively.

II. FINITE-HORIZON DISCRETE-TIME OPTIMAL REDUCED-ORDER ESTIMATOR

Consider the system

$$x_{k+1} = A_k x_k + D_{1,k} w_k, \tag{2.1}$$
$$y_k = C_k x_k + D_{2,k} w_k, \tag{2.2}$$

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where $x_k \in \mathbb{R}^{n_k}$, $y_k \in \mathbb{R}^{p_k}$, and $w_k \in \mathbb{R}^{d_k}$ is a white noise process with zero mean and unit covariance. We assume for convenience that $D_{1,k} D_{2,k}^{\mathrm{T}} = 0$.

We consider a reduced-order estimator with dynamics

$$x_{\mathrm{e},k+1} = A_{\mathrm{e},k} x_{\mathrm{e},k} + B_{\mathrm{e},k} y_k, \tag{2.3}$$

where $x_{\mathrm{e},k} \in \mathbb{R}^{n_{\mathrm{e},k}}$. Define the combined state variance $\tilde{Q}_k$ by

$$\tilde{Q}_k \triangleq \mathcal{E}[\tilde{x}_k \tilde{x}_k^{\mathrm{T}}], \tag{2.4}$$

where $\tilde{x}_k \in \mathbb{R}^{\tilde{n}_k}$, $\tilde{n}_k \triangleq n_k + n_{\mathrm{e},k}$ is defined by

$$\tilde{x}_k \triangleq \begin{bmatrix} (x_k)^{\mathrm{T}} & (x_{\mathrm{e},k})^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}. \tag{2.5}$$

Consider the cost function

$$J_k \triangleq \mathcal{E}\Big[\big(L_k x_{k+1} - x_{\mathrm{e},k+1}\big)^{\mathrm{T}}\big(L_k x_{k+1} - x_{\mathrm{e},k+1}\big)\Big], \tag{2.6}$$

where $L_k \in \mathbb{R}^{n_{\mathrm{e},k} \times n_k}$ determines the subspace of the state x that is weighted. It follows from (2.4) and (2.5) that J_k can be expressed as $J_k = \operatorname{tr}\left(\tilde{Q}_{k+1}\tilde{R}_k\right)$, where $\tilde{R}_k \in \mathbb{R}^{n+n_{\mathrm{e}}}$ is defined by

$$\tilde{R}_k \triangleq \begin{bmatrix} L_k^{\mathrm{T}} L_k & -L_k^{\mathrm{T}} \\ -L_k & I \end{bmatrix}. \tag{2.7}$$

Note that (2.1) and (2.3) imply that

$$\tilde{x}_{k+1} = \tilde{A}_k \tilde{x}_k + \tilde{D}_{1,k} w_k, \tag{2.8}$$

where

$$\tilde{A}_k \triangleq \begin{bmatrix} A_k & 0 \\ B_{\mathrm{e},k} C_k & A_{\mathrm{e},k} \end{bmatrix}, \quad \tilde{D}_{1,k} \triangleq \begin{bmatrix} D_{1,k} \\ B_{\mathrm{e},k} D_{2,k} \end{bmatrix}. \tag{2.9}$$

Therefore,

$$\tilde{Q}_{k+1} = \tilde{A}_k \tilde{Q}_k \tilde{A}_k^{\mathrm{T}} + \tilde{V}_{1,k}, \tag{2.10}$$

where

$$\tilde{V}_{1,k} \triangleq \begin{bmatrix} V_{1,k} & 0 \\ 0 & B_{\mathrm{e},k} V_{2,k} B_{\mathrm{e},k}^{\mathrm{T}} \end{bmatrix}, \tag{2.11}$$

and

$$V_{1,k} \triangleq D_{1,k} D_{1,k}^{\mathrm{T}}, \quad V_{2,k} \triangleq D_{2,k} D_{2,k}^{\mathrm{T}}. \tag{2.12}$$

Partition $\tilde{Q}_k$ as

$$\tilde{Q}_k = \begin{bmatrix} \tilde{Q}_{1,k} & \tilde{Q}_{12,k} \\ \tilde{Q}_{12,k}^{\mathrm{T}} & \tilde{Q}_{2,k} \end{bmatrix}. \tag{2.13}$$

Hence, it follows from (2.10) that

$$\tilde{Q}_{1,k+1} = A_k \tilde{Q}_{1,k} A_k^{\mathrm{T}} + V_{1,k}, \tag{2.14}$$
$$\tilde{Q}_{12,k+1} = A_k \tilde{Q}_{1,k} C_k^{\mathrm{T}} B_{\mathrm{e},k}^{\mathrm{T}} + A_k \tilde{Q}_{12,k} A_{\mathrm{e},k}^{\mathrm{T}}, \tag{2.15}$$
$$\tilde{Q}_{2,k+1} = B_{\mathrm{e},k}\left(C_k \tilde{Q}_{1,k} C_k^{\mathrm{T}} + V_{2,k}\right) B_{\mathrm{e},k}^{\mathrm{T}} \tag{2.16}$$
$$+ A_{\mathrm{e},k} \tilde{Q}_{12,k}^{\mathrm{T}} C_k^{\mathrm{T}} B_{\mathrm{e},k}^{\mathrm{T}} + B_{\mathrm{e},k} C_k \tilde{Q}_{12,k} A_{\mathrm{e},k}^{\mathrm{T}} + A_{\mathrm{e},k} \tilde{Q}_{2,k} A_{\mathrm{e},k}.$$

Proposition 2.1: Assume that $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ minimize

J_k. Then, $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy

$$A_{\mathrm{e},k}\tilde{Q}_{2,k} = (L_kA_k - B_{\mathrm{e},k}C_k)\,\tilde{Q}_{12,k}, \tag{2.17}$$

$$B_{\mathrm{e},k} = \left(L_kA_k\tilde{Q}_{1,k} - A_{\mathrm{e},k}\tilde{Q}_{12,k}^{\mathrm{T}}\right)C_k^{\mathrm{T}}\left(C_k\tilde{Q}_{1,k}C_k^{\mathrm{T}} + V_{2,k}\right)^{-1}. \tag{2.18}$$

Proof. Setting $\frac{\partial J_k}{\partial A_{\mathrm{e},k}} = 0$ and $\frac{\partial J_k}{\partial B_{\mathrm{e},k}} = 0$ yields the result. □

Next, we assume that $\tilde{Q}_{2,k}$ is invertible, define $Q_k, \hat{Q}_k \in \mathbb{R}^{n_k\times n_k}$ by

$$Q_k \triangleq \tilde{Q}_{1,k} - \tilde{Q}_{12,k}\tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}},\quad \hat{Q}_k \triangleq \tilde{Q}_{12,k}\tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}}, \tag{2.19}$$

$\tilde{V}_{2,k} \in \mathbb{R}^{p_k\times p_k}$ by $\tilde{V}_{2,k} \triangleq C_kQ_kC_k^{\mathrm{T}} + V_{2,k}$, and $G_k \in \mathbb{R}^{n_{\mathrm{e},k}\times n_k}$ by $G_k \triangleq \tilde{Q}_{2,k}^{-1}\tilde{Q}_{12,k}^{\mathrm{T}}$.

Proposition 2.2: Assume that $\tilde{Q}_{2,k}$ is positive definite and $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ minimize J_k. Then, $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy

$$A_{\mathrm{e},k} = L_kA_k\left(I - Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_k\right)G_k^{\mathrm{T}}, \tag{2.20}$$

$$B_{\mathrm{e},k} = L_kA_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}. \tag{2.21}$$

Proof. It follows from (2.17) that

$$A_{\mathrm{e},k} = (L_kA_k - B_{\mathrm{e},k}C_k)\,\tilde{Q}_{12,k}\tilde{Q}_{2,k}^{-1}. \tag{2.22}$$

Substituting (2.22) into (2.18) yields (2.21). Finally, substituting (2.21) into (2.22) yields (2.20). □

Proposition 2.3: Assume that $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy Proposition 2.2. Then,

$$L_k\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}, \tag{2.23}$$

$$\tilde{Q}_{12,k+1} = \hat{Q}_{k+1}L_k^{\mathrm{T}}, \tilde{Q}_{2,k+1} = L_k\hat{Q}_{k+1}L_k^{\mathrm{T}}. \tag{2.24}$$

Proof. Substituting (2.20) and (2.21) into (2.15) and (2.16) yields

$$\tilde{Q}_{12,k+1} = A_k\left[\hat{Q}_k + Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_k\right]A_k^{\mathrm{T}}L_k^{\mathrm{T}}, \tag{2.25}$$

$$\tilde{Q}_{2,k+1} = L_kA_k\left[\hat{Q}_k + Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_k\right]A_k^{\mathrm{T}}L_k^{\mathrm{T}}. \tag{2.26}$$

Pre-multiplying (2.25) by L_k yields $L_k\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}$. Using (2.19) and $L_k\tilde{Q}_{12,k+1} = \tilde{Q}_{2,k+1}$ yields $\tilde{Q}_{12,k+1} = \hat{Q}_{k+1}L_k^{\mathrm{T}}$ and $\tilde{Q}_{2,k+1} = L_k\hat{Q}_{k+1}L_k^{\mathrm{T}}$. □

Next, define $M_k \in \mathbb{R}^{n_k\times n_k}$ by

$$M_k \triangleq A_k\left(\hat{Q}_k + Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_k\right)A_k^{\mathrm{T}}, \tag{2.27}$$

and define $\tau_k, \tau_{k\perp} \in \mathbb{R}^{n_k\times n_k}$ by

$$\tau_k \triangleq G_k^{\mathrm{T}}L_{k-1},\quad \tau_{k\perp} \triangleq I - \tau_k. \tag{2.28}$$

Proposition 2.4: Assume that $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy Proposition 2.2. Then, $\tau_{k+1}^2 = \tau_{k+1}$, that is, τ_{k+1} is an oblique projector.

Proof. It follows from (2.27) that (2.25) and (2.26) can be expressed as

$$\tilde{Q}_{12,k+1} = M_kL_k^{\mathrm{T}}, \tilde{Q}_{2,k+1} = L_kM_kL_k^{\mathrm{T}}. \tag{2.29}$$

Hence, (2.28) implies that

$$\tau_{k+1} = M_kL_k^{\mathrm{T}}\left(L_kM_kL_k^{\mathrm{T}}\right)^{-1}L_k. \tag{2.30}$$

Therefore, $\tau_{k+1}^2 = \tau_{k+1}$.

Proposition 2.5: Assume that $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy Proposition 2.2. Then,

$$\tau_{k+1}\hat{Q}_{k+1} = \hat{Q}_{k+1}. \tag{2.31}$$

Proof. It follows from (2.19) that

$$\hat{Q}_{k+1} = \tilde{Q}_{12,k+1}\tilde{Q}_{2,k+1}^{-1}\tilde{Q}_{12,k+1}^{\mathrm{T}}. \tag{2.32}$$

Substituting (2.29) into (2.32) yields

$$\hat{Q}_{k+1} = M_kL_k^{\mathrm{T}}\left(L_kM_kL_k^{\mathrm{T}}\right)^{-1}L_kM_k. \tag{2.33}$$

Hence, pre-multiplying (2.33) by τ_{k+1} and substituting (2.30) into the resulting expression yields (2.31).

Proposition 2.6: Assume that $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ satisfy Proposition 2.2. Then,

$$\begin{aligned} Q_{k+1} = {} & A_kQ_kA_k^{\mathrm{T}} + V_{1,k} - A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}} \\ & + \tau_{k+1\perp}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1\perp}^{\mathrm{T}}, \end{aligned} \tag{2.34}$$

$$\hat{Q}_{k+1} = \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1}^{\mathrm{T}}, \tag{2.35}$$

$$\tau_{k+1} = M_kL_k^{\mathrm{T}}\left(L_kM_kL_k\right)^{-1}L_k. \tag{2.36}$$

Proof. It follows from (2.23) and (2.26) that

$$\begin{aligned} & L_k\hat{Q}_{k+1}L_k^{\mathrm{T}} \\ & = L_k\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]L_k^{\mathrm{T}}. \end{aligned} \tag{2.37}$$

Pre-multiplying and post-multiplying (2.37) by G_{k+1}^{T} and G_{k+1}, respectively, yields

$$\begin{aligned} & \tau_{k+1}\hat{Q}_{k+1}\tau_{k+1}^{\mathrm{T}} \\ & = \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1}^{\mathrm{T}}. \end{aligned} \tag{2.38}$$

Hence, (2.35) follows from Proposition 2.5.

Since $\tilde{Q}_{12,k+1} = \hat{Q}_{k+1}L_k$, (2.25) and (2.28) imply that

$$\begin{aligned} & \tau_{k+1}\hat{Q}_{k+1} \\ & = \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]. \end{aligned} \tag{2.39}$$

Therefore, (2.35) imply that

$$\begin{aligned} & \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right] \\ & = \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1}^{\mathrm{T}}. \end{aligned} \tag{2.40}$$

Hence, $\hat{Q}_{k+1}$ can be expressed as

$$\begin{aligned} \hat{Q}_{k+1} = {} & A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}} \\ & - \tau_{k+1\perp}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1\perp}^{\mathrm{T}}. \end{aligned} \tag{2.41}$$

It follows from (2.14) and (2.19) that

$$Q_{k+1} = AQ_kA^{\mathrm{T}} + V_{1,k} + A\hat{Q}_kA^{\mathrm{T}} - \hat{Q}_{k+1}. \tag{2.42}$$

Therefore, substituting (2.41) into (2.42) yields (2.34). □

Note that although $A_{\mathrm{e},k}$ and $B_{\mathrm{e},k}$ depend on $\tilde{Q}_{12,k}$ and $\tilde{Q}_{2,k}$, it follows from Proposition 2.3 that $\tilde{Q}_{2,k}$ and $\tilde{Q}_{12,k}$ can be obtained from Q_k and $\hat{Q}_k$. Hence, it suffices to propagate Q_k and $\hat{Q}_k$ using (2.34) and (2.35), respectively.

Finally, we summarize the one-step reduced-order Kalman filter.

State update:

$$G_k = (L_k\hat{Q}_kL_k)^{-1}L_k\hat{Q}_k, \tag{2.43}$$

$$\begin{aligned} x_{\mathrm{e},k+1} = {} & L_kA_k\left(I - Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_k\right)G_k^{\mathrm{T}}x_{\mathrm{e},k} \\ & + L_kA_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}y_k. \end{aligned} \tag{2.44}$$

Covariance update:

$$M_k = A_k\left(\hat{Q}_k + Q_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_k\right)A_k^{\mathrm{T}}, \quad (2.45)$$

$$\tau_{k+1} = M_kL_k^{\mathrm{T}}\left(L_kM_kL_k\right)^{-1}L_k, \quad (2.46)$$

$$\hat{Q}_{k+1} = \tau_{k+1}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1}^{\mathrm{T}}, \quad (2.47)$$

$$Q_{k+1} = A_kQ_kA_k^{\mathrm{T}} + V_{1,k} - A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}} + \tau_{k+1\perp}\left[A_k\hat{Q}_kA_k^{\mathrm{T}} + A_kQ_kC_k^{\mathrm{T}}\tilde{V}_{2,k}^{-1}C_kQ_kA_k^{\mathrm{T}}\right]\tau_{k+1\perp}^{\mathrm{T}}. \quad (2.48)$$

III. Two-Step Estimator

Next, we consider a two-step estimator. The *data assimilation step* is given by

$$x_{\mathrm{e},k}^{\mathrm{da}} = C_{\mathrm{e},k}^{\mathrm{f}}x_{\mathrm{e},k}^{\mathrm{f}} + D_{\mathrm{e},k}^{\mathrm{f}}y_k, \quad (3.1)$$

where $x_{\mathrm{e},k}^{\mathrm{da}} \in \mathbb{R}^{n_{\mathrm{e},k}}$ is the *reduced-order data assimilation estimate* of Lx_k and $x_{\mathrm{e},k}^{\mathrm{f}} \in \mathbb{R}^{n_{\mathrm{e},k}}$ is the *reduced-order forecast estimate* of x_k. The *forecast step* or physics update of the estimator is given by

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = A_{\mathrm{e},k}^{\mathrm{da}}x_{\mathrm{e},k}^{\mathrm{da}}. \quad (3.2)$$

First, we define the combined state and forecast estimate covariance $\tilde{Q}_k^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_k\times\tilde{n}_k}$ and the combined state and data assimilation estimate covariance $\tilde{Q}_k^{\mathrm{da}} \in \mathbb{R}^{\tilde{n}_k\times\tilde{n}_k}$ by

$$\tilde{Q}_k^{\mathrm{f}} \triangleq \mathcal{E}\left[\tilde{x}_k^{\mathrm{f}}(\tilde{x}_k^{\mathrm{f}})^{\mathrm{T}}\right], \quad \tilde{Q}_k^{\mathrm{da}} \triangleq \mathcal{E}\left[\tilde{x}_k^{\mathrm{da}}(\tilde{x}_k^{\mathrm{da}})^{\mathrm{T}}\right], \quad (3.3)$$

where $\tilde{x}_k^{\mathrm{f}}, \tilde{x}_k^{\mathrm{da}} \in \mathbb{R}^{n+n_{\mathrm{e}}}$ are defined by

$$\tilde{x}_k^{\mathrm{f}} \triangleq \begin{bmatrix} x_k \\ x_{\mathrm{e},k}^{\mathrm{f}} \end{bmatrix}, \quad \tilde{x}_k^{\mathrm{da}} \triangleq \begin{bmatrix} x_k \\ x_{\mathrm{e},k}^{\mathrm{da}} \end{bmatrix}. \quad (3.4)$$

Define the *data assimilation cost* by

$$J_k^{\mathrm{da}} \triangleq \mathcal{E}\left[\left(L_kx_k - x_{\mathrm{e},k}^{\mathrm{da}}\right)^{\mathrm{T}}\left(L_kx_k - x_{\mathrm{e},k}^{\mathrm{da}}\right)\right]. \quad (3.5)$$

Hence, (3.3) implies that $J_k^{\mathrm{da}} = \mathrm{tr}(\tilde{Q}_k^{\mathrm{da}}\tilde{R}_k)$, where $\tilde{R}_k$ is defined by (2.7).

It follows from (2.1), (3.1), and (3.4) that

$$\tilde{x}_k^{\mathrm{da}} = \tilde{A}_k^{\mathrm{f}}\tilde{x}_k^{\mathrm{f}} + \tilde{D}_{1,k}^{\mathrm{f}}w_k, \quad (3.6)$$

where $\tilde{A}_k^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_k\times\tilde{n}_k}$ and $\tilde{D}_{1,k}^{\mathrm{f}} \in \mathbb{R}^{\tilde{n}_k\times d}$ are defined by

$$\tilde{A}_k^{\mathrm{f}} \triangleq \begin{bmatrix} I & 0 \\ D_{\mathrm{e},k}^{\mathrm{f}}C_k & C_{\mathrm{e},k}^{\mathrm{f}} \end{bmatrix}, \quad \tilde{D}_{1,k}^{\mathrm{f}} \triangleq \begin{bmatrix} 0 \\ D_{\mathrm{e},k}^{\mathrm{f}}D_{2,k} \end{bmatrix}. \quad (3.7)$$

Therefore,

$$\tilde{Q}_k^{\mathrm{da}} = \tilde{A}_k^{\mathrm{f}}\tilde{Q}_k^{\mathrm{f}}(\tilde{A}_k^{\mathrm{f}})^{\mathrm{T}} + \tilde{D}_{1,k}^{\mathrm{f}}(\tilde{D}_{1,k}^{\mathrm{f}})^{\mathrm{T}}. \quad (3.8)$$

Hence, J_k^{da} can be expressed as

$$J_k^{\mathrm{da}} = \mathrm{tr}\left[\left(\tilde{A}_k^{\mathrm{f}}\tilde{Q}_k^{\mathrm{f}}(\tilde{A}_k^{\mathrm{f}})^{\mathrm{T}} + \tilde{D}_{1,k}^{\mathrm{f}}(\tilde{D}_{1,k}^{\mathrm{f}})^{\mathrm{T}}\right)\tilde{R}_k\right]. \quad (3.9)$$

Finally, partition $\tilde{Q}_k^{\mathrm{f}}$ as

$$\tilde{Q}_k^{\mathrm{f}} = \begin{bmatrix} \tilde{Q}_{1,k}^{\mathrm{f}} & \tilde{Q}_{12,k}^{\mathrm{f}} \\ (\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}} & \tilde{Q}_{2,k}^{\mathrm{f}} \end{bmatrix}. \quad (3.10)$$

so that substituting (3.7) into (3.9) yields

The following result characterizes $C_{\mathrm{e},k}^{\mathrm{f}}$ and $D_{\mathrm{e},k}^{\mathrm{f}}$ that minimize J_k^{da}.

Proposition 3.1: Assume that $C_{\mathrm{e},k}^{\mathrm{f}}$ and $D_{\mathrm{e},k}^{\mathrm{f}}$ minimize J_k^{da}. Then, $C_{\mathrm{e},k}^{\mathrm{f}}$ and $D_{\mathrm{e},k}^{\mathrm{f}}$ satisfy

$$C_{\mathrm{e},k}^{\mathrm{f}}\tilde{Q}_{2,k}^{\mathrm{f}} = \left(L_k - D_{\mathrm{e},k}^{\mathrm{f}}C_k\right)\tilde{Q}_{12,k}^{\mathrm{f}}, \quad (3.11)$$

$$D_{\mathrm{e},k}^{\mathrm{f}} = \left(L\tilde{Q}_{1,k}^{\mathrm{f}} - C_{\mathrm{e},k}^{\mathrm{f}}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}\right)C_k^{\mathrm{T}}\left(C_k\tilde{Q}_{1,k}^{\mathrm{f}}C_k^{\mathrm{T}} + V_{2,k}\right)^{-1}. \quad (3.12)$$

Proof. Setting $\frac{\partial J_k^{\mathrm{da}}}{\partial C_{\mathrm{e},k}^{\mathrm{f}}} = 0$ and $\frac{\partial J_k^{\mathrm{da}}}{\partial D_{\mathrm{e},k}^{\mathrm{f}}} = 0$ yields the result. □

Next, we assume that $\tilde{Q}_{2,k}^{\mathrm{f}}$ is invertible and define $Q_k^{\mathrm{f}}, \hat{Q}_k^{\mathrm{f}} \in \mathbb{R}^{n_k\times n_k}$ by

$$Q_k^{\mathrm{f}} \triangleq \tilde{Q}_{1,k}^{\mathrm{f}} - \tilde{Q}_{12,k}^{\mathrm{f}}(\tilde{Q}_{2,k}^{\mathrm{f}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}, \quad \hat{Q}_k^{\mathrm{f}} \triangleq \tilde{Q}_{12,k}^{\mathrm{f}}(\tilde{Q}_{2,k}^{\mathrm{f}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}. \quad (3.13)$$

Next, define $V_{2,k}^{\mathrm{f}} \in \mathbb{R}^{p_k\times p_k}$ by

$$V_{2,k}^{\mathrm{f}} \triangleq C_kQ_k^{\mathrm{f}}C_k^{\mathrm{T}} + V_{2,k}. \quad (3.14)$$

Also, define $G_k^{\mathrm{f}} \in \mathbb{R}^{n_{\mathrm{e},k}\times n_k}$ by

$$G_k^{\mathrm{f}} \triangleq (\tilde{Q}_{2,k}^{\mathrm{f}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}. \quad (3.15)$$

Proposition 3.2: Assume that $C_{\mathrm{e},k}^{\mathrm{f}}$ and $D_{\mathrm{e},k}^{\mathrm{f}}$ minimize J_k^{da} and assume that $\tilde{Q}_{2,k}^{\mathrm{f}}$ is positive definite. Then,

$$C_{\mathrm{e},k}^{\mathrm{f}} = L_k\left(I - Q_k^{\mathrm{f}}C_k^{\mathrm{T}}(V_{2,k}^{\mathrm{f}})^{-1}C_k\right)(G_k^{\mathrm{f}})^{\mathrm{T}}, \quad (3.16)$$

$$D_{\mathrm{e},k}^{\mathrm{f}} = L_kQ_k^{\mathrm{f}}C_k^{\mathrm{T}}(V_{2,k}^{\mathrm{f}})^{-1}. \quad (3.17)$$

Proof. It follows from (3.11) that

$$C_{\mathrm{e},k}^{\mathrm{f}} = \left(L_k - D_{\mathrm{e},k}^{\mathrm{f}}C_k\right)(G_k^{\mathrm{f}})^{\mathrm{T}}. \quad (3.18)$$

Substituting (3.18) into (3.12) yields

$$D_{\mathrm{e},k}^{\mathrm{f}} = [L_k\tilde{Q}_{1,k}^{\mathrm{f}} - L_k\tilde{Q}_{12,k}^{\mathrm{f}}(\tilde{Q}_{2,k}^{\mathrm{f}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}C_k^{\mathrm{T}} + D_{\mathrm{e},k}^{\mathrm{f}}C_k\tilde{Q}_{12,k}^{\mathrm{f}}(\tilde{Q}_{2,k}^{\mathrm{f}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}C_k^{\mathrm{T}}]\left(C_k\tilde{Q}_{1,k}^{\mathrm{f}}C_k^{\mathrm{T}} + V_{2,k}\right)^{-1}. \quad (3.19)$$

Therefore, (3.17) follows from (3.13) and (3.14). Finally, substituting (3.17) into (3.18) yields (3.16). □

Next, partition $\tilde{Q}_k^{\mathrm{da}}$ as

$$\tilde{Q}_k^{\mathrm{da}} = \begin{bmatrix} \tilde{Q}_{1,k}^{\mathrm{da}} & \tilde{Q}_{12,k}^{\mathrm{da}} \\ (\tilde{Q}_{12,k}^{\mathrm{da}})^{\mathrm{T}} & \tilde{Q}_{2,k}^{\mathrm{da}} \end{bmatrix}. \quad (3.20)$$

Proposition 3.3: Assume that $x_{\mathrm{e},k}^{\mathrm{da}}$ is given by (3.1), and $C_{\mathrm{e},k}^{\mathrm{f}}$ and $D_{\mathrm{e},k}^{\mathrm{f}}$ satisfy (3.16), (3.17). Then,

$$\tilde{Q}_{1,k}^{\mathrm{da}} = \tilde{Q}_{1,k}^{\mathrm{f}}, \quad (3.21)$$

$$\tilde{Q}_{12,k}^{\mathrm{da}} = \left(\hat{Q}_k^{\mathrm{f}} + Q_k^{\mathrm{f}}C_k^{\mathrm{T}}(V_{2,k}^{\mathrm{f}})^{-1}C_kQ_k^{\mathrm{f}}\right)L_k^{\mathrm{T}}, \quad (3.22)$$

$$\tilde{Q}_{2,k}^{\mathrm{da}} = L_k\left(\hat{Q}_k^{\mathrm{f}} + Q_k^{\mathrm{f}}C_k^{\mathrm{T}}(V_{2,k}^{\mathrm{f}})^{-1}C_kQ_k^{\mathrm{f}}\right)L_k^{\mathrm{T}}. \quad (3.23)$$

Proof. It follows from (3.8) that $\tilde{Q}_{1,k}^{\mathrm{da}} = \tilde{Q}_{1,k}^{\mathrm{f}}$ and

$$\tilde{Q}_{12,k}^{\mathrm{da}} = \tilde{Q}_{12,k}^{\mathrm{f}}(C_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}} + \tilde{Q}_{1,k}^{\mathrm{f}}C_k^{\mathrm{T}}(D_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}}. \quad (3.24)$$

Substituting (3.16) and (3.17) into (3.24) yields (3.22). Similarly, it follows from (3.8) and (3.20) that

$$\tilde{Q}_{2,k}^{\mathrm{da}} = C_{\mathrm{e},k}^{\mathrm{f}}\tilde{Q}_{1,k}^{\mathrm{f}}(C_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}} + C_{\mathrm{e},k}^{\mathrm{f}}(\tilde{Q}_{12,k}^{\mathrm{f}})^{\mathrm{T}}C_k^{\mathrm{T}}(D_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}} + D_{\mathrm{e},k}^{\mathrm{f}}C_k\tilde{Q}_{12,k}^{\mathrm{f}}(C_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}} + D_{\mathrm{e},k}^{\mathrm{f}}\left(C_k\tilde{Q}_{1,k}^{\mathrm{f}}C_k^{\mathrm{T}} + V_{2,k}\right)(D_{\mathrm{e},k}^{\mathrm{f}})^{\mathrm{T}}. \quad (3.25)$$

Finally, substituting (3.16) and (3.17) into (3.25) yields (3.23). □

Next, define Q_k^{da} and $\hat{Q}_k^{\mathrm{da}}$ by

$$Q_k^{\mathrm{da}} \triangleq \tilde{Q}_{1,k}^{\mathrm{da}} - \tilde{Q}_{12,k}^{\mathrm{da}}(\tilde{Q}_{2,k}^{\mathrm{da}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{da}})^{\mathrm{T}}, \quad \hat{Q}_k^{\mathrm{da}} \triangleq \tilde{Q}_{12,k}^{\mathrm{da}}(\tilde{Q}_{2,k}^{\mathrm{da}})^{-1}(\tilde{Q}_{12,k}^{\mathrm{da}})^{\mathrm{T}}. \quad (3.26)$$

Corollary 3.1: Assume that $C^{\rm f}_{{\rm e},k}$ and $D^{\rm f}_{{\rm e},k}$ satisfy Proposition 3.2. Then,

$$L_k\tilde{Q}^{\rm da}_{12,k} = \tilde{Q}^{\rm da}_{2,k},\ \tilde{Q}^{\rm da}_{12,k} = \hat{Q}^{\rm da}_k L_k^{\rm T},\ \tilde{Q}^{\rm da}_{2,k} = L_k\hat{Q}^{\rm da}_k L_k^{\rm T}. \quad (3.27)$$

Next, define $G^{\rm da}_k$ by

$$G^{\rm da}_k \triangleq (\tilde{Q}^{\rm da}_{2,k})^{-1}(\tilde{Q}^{\rm da}_{12,k})^{\rm T}. \quad (3.28)$$

Also, define $M^{\rm da}_k$ by

$$M^{\rm da}_k \triangleq \hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k \quad (3.29)$$

and define $\tau^{\rm da}_k$ and $\tau^{\rm da}_{k\perp}$ by

$$\tau^{\rm da}_k \triangleq (G^{\rm da}_k)^{\rm T} L_k,\ \ \tau^{\rm da}_{k\perp} \triangleq I - \tau^{\rm da}_k. \quad (3.30)$$

Proposition 3.4: Assume that $C^{\rm f}_{{\rm e},k}$ and $D^{\rm f}_{{\rm e},k}$ satisfy Proposition 3.2. Then, $\tau^{\rm da}_k$ is an oblique projector.

Proof. The proof is similar to that of Proposition 2.4. □

Proposition 3.5: Assume that $C^{\rm f}_{{\rm e},k}$ and $D^{\rm f}_{{\rm e},k}$ satisfy Proposition 3.2. Then,

$$\tau^{\rm da}_k\hat{Q}^{\rm da}_k = \hat{Q}^{\rm da}_k. \quad (3.31)$$

Proof. The proof is similar to that of Proposition 2.5. □

Proposition 3.6: Assume that $x^{\rm da}_{{\rm e},k}$ is given by (3.1), and $C^{\rm f}_{{\rm e},k}$ and $D^{\rm f}_{{\rm e},k}$ satisfy Proposition 3.2. Then,

$$\hat{Q}^{\rm da}_k = \tau^{\rm da}_k\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right)(\tau^{\rm da}_k)^{\rm T}, \quad (3.32)$$

$$Q^{\rm da}_k = Q^{\rm f}_k - Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k + \tau^{\rm da}_{k\perp}\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right)(\tau^{\rm da}_{k\perp})^{\rm T}. \quad (3.33)$$

Proof. It follows from (3.23) and (3.27) that

$$L_k\hat{Q}^{\rm da}_k L_k^{\rm T} = L_k\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right)L_k^{\rm T}. \quad (3.34)$$

Pre-multiplying and post-multiplying (3.34) by $(G^{\rm da}_k)^{\rm T}$ and $G^{\rm da}_k$, respectively, yields (3.32).

Next, it follows from (3.22), (3.27), and (3.30) that

$$\tau^{\rm da}_k\hat{Q}^{\rm da}_k = \tau^{\rm da}_k\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right). \quad (3.35)$$

Therefore, Proposition 3.4 and (3.32) imply that

$$\tau^{\rm da}_k\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right) = \tau^{\rm da}_k\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right)(\tau^{\rm da}_k)^{\rm T}. \quad (3.36)$$

Hence, $\hat{Q}^{\rm da}_k$ can be expressed as

$$\hat{Q}^{\rm da}_k = \hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k - \tau^{\rm da}_{k\perp}\left(\hat{Q}^{\rm f}_k + Q^{\rm f}_k C_k^{\rm T}(V^{\rm f}_{2,k})^{-1}C_k Q^{\rm f}_k\right)(\tau^{\rm da}_{k\perp})^{\rm T}. \quad (3.37)$$

Finally, note that (3.21) implies that $Q^{\rm da}_k = \tilde{Q}^{\rm f}_{1,k} - \hat{Q}^{\rm da}_k$. Hence, using (3.37) yields (3.33). □

Next, we define the *forecast cost* $J^{\rm f}_k$ by

$$J^{\rm f}_k \triangleq \mathcal{E}\left[\left(L_k x_{k+1} - x^{\rm f}_{{\rm e},k+1}\right)\left(L_k x_{k+1} - x^{\rm f}_{{\rm e},k+1}\right)^{\rm T}\right]. \quad (3.38)$$

Hence, it follows from (3.3) that $J^{\rm f}_k = {\rm tr}\left(\tilde{Q}^{\rm f}_{k+1}\tilde{R}_k\right)$. It follows from (2.1) and (3.2) that

$$\tilde{x}^{\rm f}_{k+1} = \tilde{A}^{\rm da}_k\tilde{x}^{\rm da}_k + \tilde{D}^{\rm da}_{1,k}w_k, \quad (3.39)$$

where $\tilde{A}^{\rm da}_k \in \mathbb{R}^{\tilde{n}_k\times\tilde{n}_k}$ and $\tilde{D}^{\rm da}_{1,k} \in \mathbb{R}^{\tilde{n}_k\times d}$ are defined by

$$\tilde{A}^{\rm da}_k \triangleq \begin{bmatrix} A_k & 0 \\ 0 & A^{\rm da}_{{\rm e},k}\end{bmatrix},\ \ \tilde{D}^{\rm da}_{1,k} \triangleq \begin{bmatrix} D_{1,k} \\ 0 \end{bmatrix}. \quad (3.40)$$

Therefore,

$$\tilde{Q}^{\rm f}_{k+1} = \tilde{A}^{\rm da}_k\tilde{Q}^{\rm da}_k(\tilde{A}^{\rm da}_k)^{\rm T} + \tilde{D}^{\rm da}_{1,k}(\tilde{D}^{\rm da}_{1,k})^{\rm T}. \quad (3.41)$$

Proposition 3.7: Assume that $A^{\rm da}_{{\rm e},k}$ minimizes $J^{\rm f}_k$, and assume that $\tilde{Q}^{\rm da}_{2,k}$ is positive definite. Then

$$A^{\rm da}_{{\rm e},k} = L_k A_k (G^{\rm da}_k)^{\rm T}. \quad (3.42)$$

Proof. Setting $\frac{\partial J^{\rm f}_k}{\partial A^{\rm da}_{{\rm e},k}} = 0$ yields the result. □

Proposition 3.8: Assume that $A^{\rm da}_{{\rm e},k}$ satisfies (3.42). Then,

$$L_k\tilde{Q}^{\rm f}_{12,k+1} = \tilde{Q}^{\rm f}_{2,k+1},\ \tilde{Q}^{\rm f}_{12,k+1} = \hat{Q}^{\rm f}_{k+1}L_k^{\rm T}, \quad (3.43)$$

$$\tilde{Q}^{\rm f}_{2,k+1} = L_k\hat{Q}^{\rm f}_{k+1}L_k^{\rm T}. \quad (3.44)$$

Proof. The proof is similar to that of Proposition 2.3. □

Next, define $M^{\rm f}_k$ by $M^{\rm f}_k \triangleq A_k\hat{Q}^{\rm da}_k A_k^{\rm T}$ and define $\tau^{\rm f}_k$ and $\tau^{\rm f}_{k\perp}$ by $\tau^{\rm f}_k \triangleq (G^{\rm f}_k)^{\rm T}L_{k-1}$, $\tau^{\rm f}_{k\perp} \triangleq I - \tau^{\rm f}_k$.

Proposition 3.9: Assume that $A^{\rm da}_{{\rm e},k}$ satisfies (3.42). Then, $\tau^{\rm f}_{k+1}$ is an oblique projector, that is, $(\tau^{\rm f}_{k+1})^2 = \tau^{\rm f}_{k+1}$.

Proof. The proof is similar to that of Proposition 2.4. □

Proposition 3.10: Assume that $A^{\rm da}_{{\rm e},k}$ satisfies (3.42). Then,

$$\tau^{\rm f}_{k+1}\hat{Q}^{\rm f}_{k+1} = \hat{Q}^{\rm f}_{k+1}. \quad (3.45)$$

Proof. The proof is similar to that of Proposition 2.5. □

Proposition 3.11: Assume that $A^{\rm da}_{{\rm e},k}$ satisfies (3.42). Then,

$$\hat{Q}^{\rm f}_{k+1} = \tau^{\rm f}_{k+1}A_k\hat{Q}^{\rm da}_k A_k^{\rm T}(\tau^{\rm f}_{k+1})^{\rm T}, \quad (3.46)$$

$$Q^{\rm f}_{k+1} = A_k Q^{\rm da}_k A_k^{\rm T} + V_{1,k} + \tau^{\rm f}_{k+1\perp}\left(A_k\hat{Q}^{\rm da}_k A_k^{\rm T}\right)(\tau^{\rm f}_{k+1\perp})^{\rm T}. \quad (3.47)$$

Proof. The proof is similar to that of Proposition 2.6. □

IV. Finite-Horizon Discrete-Time Optimal Subspace Estimator

Next, we consider reduced-order estimator that focuses on a specific subspace of the state. Without any loss of generality, we partition the system (2.1), (2.2) as

$$\begin{bmatrix} x_{{\rm r},k+1} \\ x_{{\rm s},k+1}\end{bmatrix} = \begin{bmatrix} A_{{\rm r},k} & A_{{\rm us},k} \\ 0 & A_{{\rm s},k}\end{bmatrix}\begin{bmatrix} x_{{\rm r},k} \\ x_{{\rm s},k}\end{bmatrix} + \begin{bmatrix} D_{1{\rm r},k} \\ D_{1{\rm s},k}\end{bmatrix} w_k, \quad (4.1)$$

$$y_k = \begin{bmatrix} C_{{\rm r},k} & C_{{\rm s},k}\end{bmatrix}\begin{bmatrix} x_{{\rm r},k} \\ x_{{\rm s},k}\end{bmatrix} + D_{2,k}w_k, \quad (4.2)$$

we seek a reduced-order subspace estimator

$$x_{{\rm e},k+1} = A_{{\rm e},k}x_{{\rm e},k} + B_{{\rm e},k}y_k, \quad (4.3)$$

$$y_{{\rm e},k} = C_{{\rm e},k}x_{{\rm e},k}, \quad (4.4)$$

that minimizes

$$J(A_{{\rm e},k}, B_{{\rm e},k}, C_{{\rm e},k+1}) \triangleq \mathcal{E}\left([L_{k+1}x_{k+1} - y_{{\rm e},k+1}]^{\rm T} R_k [L_{k+1}x_{k+1} - y_{{\rm e},k+1}]\right). \quad (4.5)$$

In this formulation the plant state x_k is partitioned into subsystems for $x_{{\rm r},k}$ and $x_{{\rm s},k}$ of dimension $n_{{\rm r},k}$ and $n_{{\rm s},k}$, respectively. The state $x_{{\rm r},k}$ may contain the components of x_k of interest. Furthermore, the matrix state weighting matrix L_k is partitioned as $L_k \triangleq [L_{{\rm r},k}\ \ L_{{\rm s},k}]$, where $L_{{\rm r},k}$ and $L_{{\rm s},k}$ are $q_k\times n_{{\rm r},k}$ and $q_k\times n_{{\rm s},k}$ matrices, respectively. The order $n_{{\rm e},k}$ of the estimator state $x_{{\rm e},k}$ is fixed to be equal to the order of the $n_{\mathrm{r},k}$-dimensional subspace for $x_{\mathrm{r},k}$. Thus, the goal of the optimal reduced-order subspace estimator problem is to design an estimator of order $n_{\mathrm{r},k}$ that yields least-squares estimates of specified linear combinations of the states of the system.

Next, we define the error state $z_k \triangleq x_{\mathrm{r},k} - x_{\mathrm{e},k}$ satisfying

$$\begin{aligned} z_{k+1} = (A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k})x_{\mathrm{r},k} - A_{\mathrm{e},k}x_{\mathrm{e},k} \\ + (A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k})x_{\mathrm{s},k} + (D_{1\mathrm{u},k} - B_{\mathrm{e},k}D_{2,k})w_k. \end{aligned} \tag{4.6}$$

By constraining $A_{\mathrm{e},k} = A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k}$, (4.6) becomes

$$\begin{aligned} z_{k+1} =& (A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k})z_k + (A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k})x_{\mathrm{s},k} \\ &+ (D_{1\mathrm{u},k} - B_{\mathrm{e},k}D_{2,k})w_k. \end{aligned} \tag{4.7}$$

Furthermore, the explicit dependence of the estimation error in (4.5) on the $x_{\mathrm{r},k}$ subsystem can be eliminated by constraining $C_{\mathrm{e},k} = L_{\mathrm{r},k}$. Now, from (4.1)-(4.4) it follows that

$$\tilde{x}_{k+1} = \tilde{A}_k\tilde{x}_k + \tilde{D}_k w_k \tag{4.8}$$

where

$$\tilde{x}_k \triangleq \begin{bmatrix} z_k \\ x_{\mathrm{s},k} \end{bmatrix}, \quad \tilde{A}_k \triangleq \begin{bmatrix} A_{\mathrm{r},k} - B_{\mathrm{e},k}C_{\mathrm{r},k} & A_{\mathrm{us},k} - B_{\mathrm{e},k}C_{\mathrm{s},k} \\ 0 & A_{\mathrm{s},k} \end{bmatrix},$$

$$\tilde{D}_k \triangleq \begin{bmatrix} D_{1\mathrm{r},k} - B_{\mathrm{e},k}D_{2,k} \\ D_{1\mathrm{s},k} \end{bmatrix}. \tag{4.9}$$

Then, the problem can be restated as finding $B_{\mathrm{e},k}$ that minimizes $J(B_{\mathrm{e},k}) = \mathrm{tr}\left(Q_{k+1}\tilde{R}_{k+1}\right)$, where $\tilde{R}_{k+1} \triangleq L_{k+1}^{\mathrm{T}}R_kL_{k+1}$ and Q_k is the $n \times n$ state-error covariance defined as $Q_k \triangleq \mathcal{E}\left[\tilde{x}_k\tilde{x}_k^{\mathrm{T}}\right]$.

Following the procedure in Section 2, we obtain the finite-horizon discrete-time optimal reduced-order subspace estimator given by

$$\begin{aligned} x_{\mathrm{e},k+1} = \Phi_kA_k(I - Q_kC_k^{\mathrm{T}}\hat{V}_k^{-1}C_k)F_k^{\mathrm{T}}x_{\mathrm{e},k} \\ + \Phi_kA_kQ_kC_k^{\mathrm{T}}\hat{V}_k^{-1}y_k, \end{aligned} \tag{4.10}$$

$$\begin{aligned} Q_{k+1} = A_kQ_kA^{\mathrm{T}} - A_kQ_kC^{\mathrm{T}}\hat{V}_k^{-1}C_kQ_kA_k^{\mathrm{T}} + V_{1,k} \\ + \mu_{k+1\perp}A_kQ_kC^{\mathrm{T}}\hat{V}_k^{-1}C_kQ_kA_k^{\mathrm{T}}\mu_{k+1\perp}^{\mathrm{T}}, \end{aligned} \tag{4.11}$$

where $\mu_{k\perp} \triangleq I - \mu_k$, $\hat{V}_k \triangleq C_kQ_kC_k^{\mathrm{T}} + V_{2,k}$, $F_k \triangleq \left[I_{n_{\mathrm{r},k}} \; 0_{n_{\mathrm{r},k}\times n_{\mathrm{s},k}}\right]$,

$$\Phi_k \triangleq [I_{n_\mathrm{r}} \quad (L_{\mathrm{r},k}^{\mathrm{T}}R_kL_{\mathrm{r},k})^{-1}(L_{\mathrm{r},k}^{\mathrm{T}}R_kL_{\mathrm{s},k})], \tag{4.12}$$

$$\mu_k \triangleq F_k^{\mathrm{T}}\Phi_k = \begin{bmatrix} I_{n_\mathrm{r}} & (L_{\mathrm{r},k}^{\mathrm{T}}R_kL_{\mathrm{r},k})^{-1}(L_{\mathrm{r},k}^{\mathrm{T}}R_kL_{\mathrm{s},k}) \\ 0 & 0 \end{bmatrix}. \tag{4.13}$$

Furthermore, defining the data-assimilation cost J_k^{da} and the forecast cost J_k^{f} separately such that

$$J_k^{\mathrm{da}} \triangleq \mathcal{E}\left(\left[L_kx_k - y_{\mathrm{e},k}^{\mathrm{da}}\right]^{\mathrm{T}}R_k\left[L_kx_k - y_{\mathrm{e},k}^{\mathrm{da}}\right]\right), \tag{4.14}$$

$$J_k^{\mathrm{f}} \triangleq \mathcal{E}\left(\left[L_kx_k - y_{\mathrm{e},k}^{\mathrm{f}}\right]^{\mathrm{T}}R_{k-1}\left[L_kx_k - y_{\mathrm{e},k}^{\mathrm{f}}\right]\right), \tag{4.15}$$

we obtain the following two-step finite-horizon discrete-time optimal subspace estimator:

Data assimilation step:

$$\begin{aligned} x_{\mathrm{e},k}^{\mathrm{da}} = \Phi_k(I - Q_k^{\mathrm{f}}C_k^{\mathrm{T}}\hat{V}_{2,k}^{-1}C_k)F_k^{\mathrm{T}}x_k^{\mathrm{da}} \\ + \Phi_kQ_k^{\mathrm{f}}C_k^{\mathrm{T}}\hat{V}_{2,k}^{-1}y_k, \end{aligned} \tag{4.16}$$

$$\begin{aligned} Q_k^{\mathrm{da}} = Q_k^{\mathrm{f}} - Q_k^{\mathrm{f}}C^{\mathrm{T}}\hat{V}_{2,k}^{-1}C_kQ_k^{\mathrm{f}} \\ + \mu_{k\perp}Q_k^{\mathrm{f}}C^{\mathrm{T}}\hat{V}_{2,k}^{-1}C_kQ_k^{\mathrm{f}}\mu_{k\perp}^{\mathrm{T}}, \end{aligned} \tag{4.17}$$

$$\hat{V}_{2,k} = C_kQ_k^{\mathrm{f}}C_k^{\mathrm{T}} + V_{2,k}, \; \mu_k = \Phi_kF_k^{\mathrm{T}}. \tag{4.18}$$

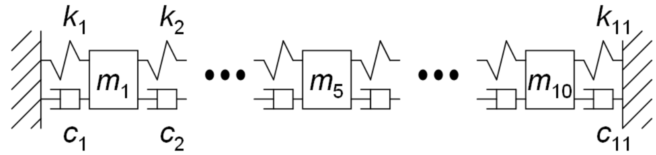


Fig. 1. Mass-spring-dashpot system.

Forecast step:

$$x_{\mathrm{e},k+1}^{\mathrm{f}} = \Phi_kA_kF_k^{\mathrm{T}}x_{\mathrm{e},k}^{\mathrm{da}}, \tag{4.19}$$

$$Q_{k+1}^{\mathrm{f}} = A_kQ_k^{\mathrm{da}}A_k^{\mathrm{T}} + V_{1,k}. \tag{4.20}$$

V. Asymptotically Stable Mass-Spring-Dashpot Example

We consider a zero-order hold discretized model of the mass-spring-dashpot structure consisting of 10 masses shown in Figure 1 so that $n = 20$. For $i = 1, \ldots, 10$, $m_i = 1.0$ kg, while, for $j = 1, \ldots, 11$, $k_j = 1.0$ N/m and $c_j = 0.05$ Ns/m. We set the initial error covariance $P_0 = 100I$ and assume that $V_{1,k} = I$, $V_{2,k} = I$ for all $k \geqslant 0$.

Let q_i denote the position of the ith mass so that

$$x \triangleq \left[\begin{array}{ccccc} q_1 & \dot{q}_1 & \cdots & q_{10} & \dot{q}_{10} \end{array}\right]. \tag{5.1}$$

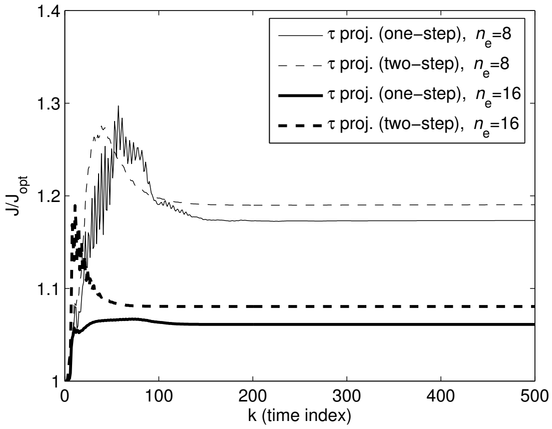
We assume that measurements of position and velocities of $m_1, \ldots, m_4$ are available so that $C_k = [I_8 \; 0_{8\times 12}]$ for all $k \geqslant 0$. Next, we obtain state estimates from the reduced-order estimator with $n_\mathrm{e} = 8$. For the subspace estimator, we consider a change of basis so that the system has a block upper-triangular structure. Recall that the costs for the estimators are defined as (2.6) and (4.5) with $R_k = I$. The ratio of the cost J_k to the best achievable cost when a full-order Kalman filter is used is shown in Figure 2. As expected, the performance of the reduced-order filter is never better than the full-order Kalman filter (indicated by ratios greater than 1). Next, we assume that measurements of positions and velocities of $m_1, \ldots, m_8$ are available so that $C_k = [I_{16} \; 0_{16\times 4}]$ for all $k \geqslant 0$. The performance of the reduced-order estimator with $n_\mathrm{e} = 16$ is shown in Figure 2. The objective in both the cases is to obtain estimates of Lx_k, where for $i = 1, \ldots, n_\mathrm{e}$, $j = 1, \ldots, n$, the (i,j)th entry of $L \in \mathbb{R}^{n_\mathrm{e}\times(n-n_\mathrm{e})}$ is given by

$$L_{(i,i)} = \begin{cases} 1, & \text{if } i = j, \\ 0.05, & \text{else.} \end{cases} \tag{5.2}$$

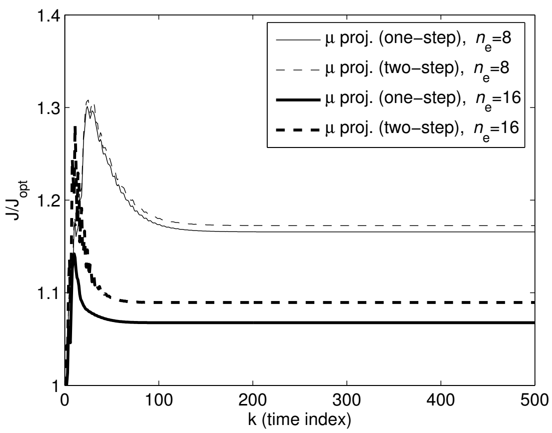
The plots also demonstrate that the one-step and two-step estimators are not equivalent.

VI. Mass-Spring-Dashpot Example with Rigid-Body Mode

Next, we consider the case in which both ends of the mass-spring-dashpot structure are free, that is, $k_1 = k_{11} = 0.0$ and $c_1 = c_{11} = 0.0$, and thus the structure has an unstable rigid-body mode. We consider only the subspace estimator with $x_\mathrm{r} = \left[\begin{array}{cc} q_{11} & \dot{q}_{11} \end{array}\right]$ so that $x = \left[\begin{array}{cccccc} q_{11} & \dot{q}_{11} & q_1 & \dot{q}_1 & \cdots & q_{20} & \dot{q}_{20} \end{array}\right]$. We assume



(a)



(b)

Fig. 2. Ratios of J to the corresponding full-order costs when the (a) reduced-order and (b) subspace estimators are applied to the asymptotically stable mass-spring-dashpot system. (a) and (b) demonstrate the performance of both estimators for $n_{\mathrm{e}} = 8, 16$. The plots also demonstrate that the one-step and two-step estimators are not equivalent.

that measurements of the position and velocity of m_{11} are available so that $C_k = \left[\begin{array}{cc} I_2 & 0_{2\times 18} \end{array}\right]$ for all $k \geqslant 0$ and L is given by (5.2) with $n_{\mathrm{e}} = 4, 8$. The performance of the reduce-order subspace estimator with $n_{\mathrm{e}} = 4, 8$ is shown in Figure 3. The subspace estimator is effectively able to handle the unstable modes in the system.

VII. CONCLUSION

Using the finite-horizon optimization, an optimal reduced-order estimator and optimal fixed-structure subspace estimator have been obtained in the form of recursive update equations for time-varying systems. These estimators are characterized by the τ and μ oblique projectors. Moreover, we derived one-step and two-step update equations for each estimator. When the order of the each estimator is equal to the order of the system, the oblique projections become the identity and the estimators are equivalent to the classical optimal recursive full-order filter. We demonstrated the performance of the reduced-order and the subspace estimators for lumped-structures.

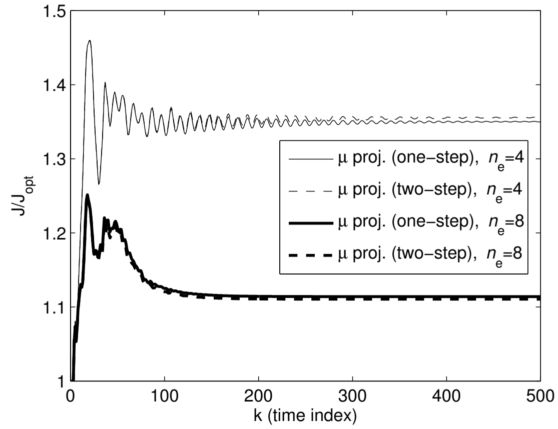


Fig. 3. Ratios of J to the corresponding optimal costs when the subspace estimator is applied to the unstable mass-spring-dashpot system. The subspace estimator is able to handle the unstable modes in its filter structure.

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