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Parameter Estimation in the Burgers Equation Using Retrospective-Cost Model Refinement

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Abstract— **We apply retrospective cost model refinement to parameter estimation in a nonlinear partial differential equation. Specifically, for the scalar Burgers equation, we estimate the viscosity from measurements of flow velocity at a single grid point. We also consider the analogous problem for a modified Burgers equation as a proxy for a large eddy simulation to estimate a parameter that relates subgrid-scale stresses to the resolved strain rate.**

I. INTRODUCTION

In many science and engineering applications, a physically realistic model of a system is available, but key parameter values are often unknown. The lack of knowledge of these values may be due to measurement limitations, where the parameters are embedded in the model in such a way that standard regression techniques cannot be used. For example, estimating a friction coefficient is not straightforward without the benefit of measurements of either the friction force or the relative velocity of the contacting surfaces. In such cases, we say that the parameter is *inaccessible*.

Aside from inaccessibility as an impediment to parameter estimation, it may be difficult to estimate a parameter that does not have a true value, but rather represents the aggregate effect of phenomena that are too complex for detailed modeling. For example, artificial viscosity represents the net effect of spatial and temporal discretization. In such cases, we say that the parameter is *representational*.

Estimation of inaccessible parameters requires techniques for subsystem identification, where the unknown subsystem is inaccessible through the available measurements. The unknown subsystem may be either dynamic or static. This problem is addressed in [1], [2] using a specialized regression technique called *retrospective cost model refinement* (RCMR).

RCMR is a subsystem identification technique that recursively updates an estimate of the unknown subsystem by retrospectively optimizing the coefficients of the subsystem model. This technique is based on retrospective cost adaptive control (RCAC), which is developed in [3], [4]. RCMR uses the machinery of RCAC for adaptive feedback controller to instead adapt a model of an inaccessible subsystem. RCMR differs from standard regression due to the use of signal estimates in the regressor in place of unavailable signal data. This technique allows RCMR to re-optimize the subsystem model based on performance error data only. The RCMR technique is developed in [2] in relation to input reconstruction, and is applied to problems in space physics in [5], [6].

The goal of the present paper is to apply RCMR to the problem of parameter estimation in the Burgers equation. The Burgers equation is one of the fundamental partial differential equations in applied mathematics, and has been widely used to qualitatively describe physical phenomena in various disciplines of engineering. The Burgers equation has been used for feedback control in [7]–[10], and numerical methods are given in [11], [12].

In gas dynamics, the form of the Burgers equation is similar to the Navier–Stokes equations and can qualitatively represent phenomena such as shock waves and viscous diffusion. In nonlinear acoustics, the Westervelt equation can be transformed into the Burgers equation with a coordinate transformation and the assumption of strictly forward propagating waves [13]. The Burgers equation has also been used to model traffic flow [14] and nonhysteretic infiltration in non-swelling soil [15].

In this work, RCMR is used to estimate kinematic and subgrid viscosity coefficients in the Burgers equation. Since these coefficients relate the velocity gradient to shear stress, it is not directly accessible if the only available measurement is the flow velocity at a particular position.

We also consider the analogous problem for a modified Burgers equation as a proxy for a large eddy simulation to estimate a parameter that relates subgrid-scale stresses to the resolved strain rate. The study in this paper can thus be viewed as a precursor to the longer range goal of using model refinement to construct representational models of transport processes in large-scale computational fluid dynamics models to capture unmodeled processes and subgrid-scale features.

II. MODEL REFINEMENT

Consider the main system

$$x(k+1) = F(x(k), u(k), w(k)), \tag{1}$$

$$y_0(k) = G(x(k)), \tag{2}$$

$$y(k) = H(x(k)), \tag{3}$$

where $x(k) \in \mathbb{R}^{l_x}$ is the main system state, $u(k) \in \mathbb{R}^{l_u}$ is the main system input, $w(k) \in \mathbb{R}^{l_w}$ is the known excitation signal, $y_0(k) \in \mathbb{R}^{l_z}$ is the main system measurement, and $y(k) \in \mathbb{R}^{l_y}$ is the main system output. Note that (1)–(3) may be nonlinear.

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The main system (1)–(3) is interconnected with the unknown subsystem modeled by

$$x_{\rm s}(k+1) = A_{\rm s}x_{\rm s}(k) + B_{\rm s}y(k), \tag{4}$$

$$u(k) = C_{\rm s}x_{\rm s}(k) + D_{\rm s}y(k), \tag{5}$$

where $x_{\rm s}(k) \in \mathbb{R}^{l_{x_{\rm s}}}$. Note that (4)-(5) can be represented as $u(k) = G_{\rm s}(\mathbf{q})y(k)$, where $\mathbf{q}$ is the forward shift operator. Together, (1)–(5) represents the true system.

For parameter estimation, $G_{\rm s} = D_{\rm s}$ is a static gain, and thus (4), (5) become

$$u(k) = D_{\rm s}y(k). \tag{6}$$

Next, we assume a model of the main system of the form

$$\hat{x}(k+1) = F(\hat{x}(k), \hat{u}(k), w(k)), \tag{7}$$

$$\hat{y}_0(k) = G(\hat{x}(k)), \tag{8}$$

$$\hat{y}(k) = H(\hat{x}(k)), \tag{9}$$

where $\hat{x}(k) \in \mathbb{R}^{l_x}$ is the main system model state, $\hat{u}(k) \in \mathbb{R}^{l_u}$ is the main system model input, $\hat{y}_0(k) \in \mathbb{R}^{l_z}$ is the main system model measurement, and $\hat{y}(k) \in \mathbb{R}^{l_y}$ is the main system model output. Note that the vector functions F, G, and H are assumed to be known.

The main system model is interconnected with the subsystem model

$$\hat{u}(k) = \hat{G}_{\rm s}(\mathbf{q})\hat{y}(k). \tag{10}$$

Equations (7)–(10) represent the modeled system.

We assume that the unknown subsystem input y and unknown subsystem output u are not measured, and thus $G_{\rm s}$ is inaccessible. The input $\hat{y}$ of the subsystem model $\hat{G}_{\rm s}$ is computed, and the input $\hat{u}$ of the main system model is computed.

To update the subsystem model $\hat{G}_{\rm s}$, we minimize a cost function based on the performance variable

$$z(k) \stackrel{\triangle}{=} \hat{y}_0(k) - y_0(k) \in \mathbb{R}^{l_z}. \tag{11}$$

The model refinement problem is represented by the block diagram in Figure 1.

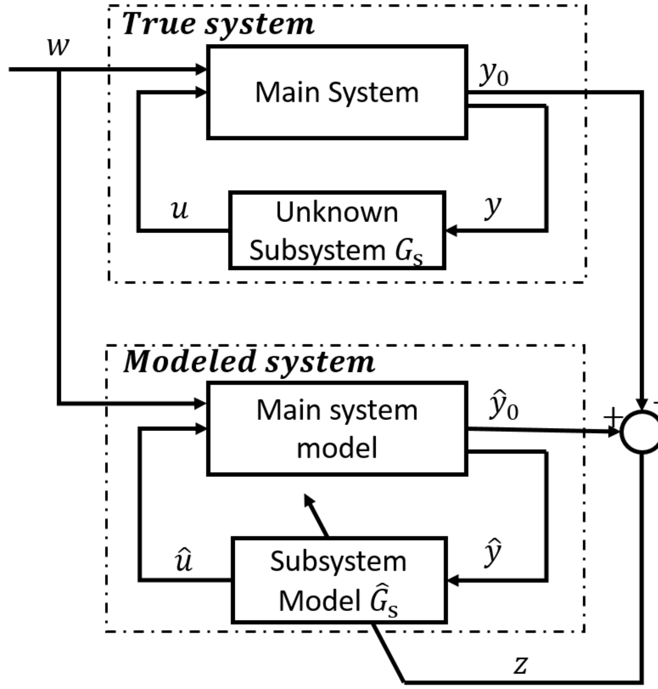

Fig. 1: Model refinement architecture. In the true system, the unknown subsystem input y and output u are not measured, and hence the unknown subsystem $G_{\rm s}$ is inaccessible. In the modeled system, the subsystem model input $\hat{y}$ is computed, and the subsystem model output $\hat{u}$ is computed using the estimate $\hat{G}_{\rm s}$ of the subsystem model. The performance $z(k) \stackrel{\triangle}{=} \hat{y}_0(k) - y_0(k)$ is used to estimate the subsystem model $\hat{G}_{\rm s}$.

III. RCMR Algorithm

In this section, we present the algorithm used to update the subsystem model $\hat{G}_{\rm s}$.

A. Subsystem Model

We represent the subsystem model $\hat{G}_{\rm s}$ by

$$\hat{u}(k) = \sum_{i=1}^{n_{\rm c}} M_i(k)\hat{u}(k-i) + \sum_{i=k_0}^{n_{\rm c}} N_i(k)\xi(k-i), \tag{12}$$

where $M_i(k) \in \mathbb{R}^{l_u \times l_u}, N_i(k) \in \mathbb{R}^{l_u \times l_\xi}$ are the coefficient matrices, $k_0 \geq 0$, and $\xi(k) \in \mathbb{R}^{l_\xi}$ consists of components of y, z, and w. We rewrite (12) as

$$\hat{u}(k) = \Phi(k)\theta(k), \tag{13}$$

where the regressor matrix $\Phi(k)$ is defined by

$$\Phi(k) \stackrel{\triangle}{=} \begin{bmatrix} \hat{u}(k-1) \\ \vdots \\ \hat{u}(k-n_{\rm c}) \\ \xi(k-k_0) \\ \vdots \\ \xi(k-n_{\rm c}) \end{bmatrix}^{\rm T} \otimes I_{l_u} \in \mathbb{R}^{l_u \times l_\theta},$$

$$\theta(k) \stackrel{\triangle}{=} {\rm vec}\begin{bmatrix} M_1(k)\cdots M_{n_{\rm c}}(k) & N_{k_0}(k)\cdots N_{n_{\rm c}}(k) \end{bmatrix} \in \mathbb{R}^{l_\theta},$$

where $l_\theta \stackrel{\triangle}{=} l_u^2 n_{\rm c} + l_u l_\xi(n_{\rm c}+1-k_0)$, "$\otimes$" is the Kronecker product, and "vec" is the column-stacking operator. Note that for static parameter estimation, $\hat{G}_{\rm s}$ is a zeroth order controller, and $\hat{u}(k) = (y^{\rm T}(k) \otimes I_{l_u}){\rm vec}(N_0(k))$.

B. Retrospective Performance Variable

We define the retrospective input as

$$\tilde{u}(k) = \Phi(k)\hat{\theta} \tag{14}$$

and the corresponding retrospective performance variable as

$$\hat{z}(k) \stackrel{\triangle}{=} z(k) + \Phi_{\rm f}(k)\hat{\theta} - \hat{u}_{\rm f}(k), \tag{15}$$

where $\hat{\theta} \in \mathbb{R}^{l_\theta}$ is determined by optimization below, and $\Phi_{\rm f}(k) \in \mathbb{R}^{l_z \times l_\theta}$ and $\hat{u}_{\rm f}(k) \in \mathbb{R}^{l_z}$ are filtered versions of $\Phi(k)$ and $\hat{u}(k)$, defined by

$$\Phi_{\rm f}(k) \stackrel{\triangle}{=} G_{\rm f}(\mathbf{q})\Phi(k). \tag{16}$$

$$\hat{u}_{\rm f}(k) \stackrel{\triangle}{=} G_{\rm f}(\mathbf{q})\hat{u}(k). \tag{17}$$

The filter G_{f} has the form

$$G_{\mathrm{f}}(\mathbf{q}) \stackrel{\triangle}{=} D_{\mathrm{f}}^{-1}(\mathbf{q})N_{\mathrm{f}}(\mathbf{q}), \tag{18}$$

where D_{f} and N_{f} are polynomial matrices, and D_{f} is monic.

C. Retrospective Cost Function

Using the retrospective performance variable $\hat{z}(k)$, we define the retrospective cost function

$$J(k,\hat{\theta}) \stackrel{\triangle}{=} \sum_{i=1}^{k} \left[\hat{z}^{\mathrm{T}}(i)R_z\hat{z}(i) + (\Phi_{\mathrm{f}}(i)\hat{\theta})^{\mathrm{T}}R_{\mathrm{f}}\Phi_{\mathrm{f}}(i)\hat{\theta}\right] + (\hat{\theta}-\theta(0))^{\mathrm{T}}R_\theta(\hat{\theta}-\theta(0)), \tag{19}$$

where R_z and R_θ are positive definite, and R_{f} is positive semidefinite.

Proposition: Let $P(0)=R_\theta^{-1}$. Then, for all $k\geq 1$, the retrospective cost function (19) has the unique global minimizer $\theta(k)$ given by

$$\theta(k) = \theta(k-1) - P(k-1)\Phi_{\mathrm{f}}^{\mathrm{T}}(k)\Gamma^{-1}(k) \cdot[\Phi_{\mathrm{f}}(k)\theta(k-1)+(R_z+R_{\mathrm{f}})^{-1}R_z(z_{\mathrm{f}}(k)-u_{\mathrm{f}}(k))], \tag{20}$$

$$P(k) = P(k-1) - P(k-1)\Phi_{\mathrm{f}}^{\mathrm{T}}(k)\Gamma^{-1}(k)\Phi_{\mathrm{f}}(k)P(k-1), \tag{21}$$

where

$$\Gamma(k) \stackrel{\triangle}{=} (R_z+R_{\mathrm{f}})^{-1} + \Phi_{\mathrm{f}}(k)P(k-1)\Phi_{\mathrm{f}}^{\mathrm{T}}(k).$$

IV. PROBLEM FORMULATION

In this section, we formulate the problem of estimating the viscosity in the Burgers equation in terms of the model refinement architecture described in Section II.

In particular, we consider two problems. In the first problem, we estimate the viscosity in the Burgers equation using RCMR. In the second problem, we estimate a parameter relating the subgrid-scale stresses to the strain rate in the modified Burgers equation using RCMR. In this paper, we study the solutions of the Burgers equation on a periodic domain.

We discretize the Burgers equation, and organize the discretized equation such that the parameter to be estimated is present only in the subsystem. Appropriate boundary conditions are applied to make the domain periodic. Since we are estimating a parameter, the modeled subsystem is a static gain, which is updated by RCMR.

RCMR uses measurements of the solution $u(x,t)$ of the Burgers equation at a single point in the domain to update the unknown parameter. Since RCMR utilizes the measurements from the true and modeled system to estimate the subsystem model, the accuracy and speed of model refinement are expected to improve as the spectral content of the measurements increases. This can be achieved by setting the initial profile $u(x,0)$ to be a sum of harmonics. This condition is similar to, but not same as, the persistency of excitation required for identification [16].

The Burgers equation is an unforced nonlinear convective equation. Hence, a non-constant initial profile subsequently convects and diffuses asymptotically to a constant solution, at a rate defined by the viscosity ν. However, two solution profiles with same initial profile, but with different viscosity, asymptotically reach the same solution. This means that the difference between the two solutions converges to zero, irrespective of the different values of viscosity. Therefore, an initial profile with high spectral content is advantageous for accurate estimation.

V. ESTIMATING THE VISCOSITY IN THE BURGERS EQUATION

In this section, we consider the one-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right), \tag{22}$$

where $u(x,t)$ is a function of space and time, and the constant ν is the viscosity. The goal is to estimate ν using measurements of u at a single arbitrary location in the domain.

A. Discretization of the Burgers equation

We discretize the Burgers equation using a forward Euler approximation for the time derivative, a second-order accurate upwind method for the convective term, and a second-order accurate central difference scheme for the viscous term. Note that $u(x,t)$ is a continuous variable whose domain is $[0,1]$ for $t>0$, while $u_j(k)$ is a discrete variable defined on the grid points $j\in\{1,...,N\}$ for $k\in\mathbb{N}$. Hence, at each grid point,

$$u_j(k+1) = u_j(k) - \frac{\Delta t}{2\Delta x}(1.5u_j(k)^2 - 2u_{j-1}(k)^2 + 0.5u_{j-2}(k)^2) + \nu\frac{\Delta t}{\Delta x^2}(u_{j+1}(k) - 2u_j(k) + u_{j-1}(k)). \tag{23}$$

In this work, we assume $u>0$, and hence the above scheme is stable for a sufficiently small time step. To make the domain infinite, periodic boundary conditions are enforced by defining one ghost node at each end of the grid. The ghost node 0 is juxtaposed with node 1 such that $u_0(k) \stackrel{\triangle}{=} u_{\mathrm{N}}(k)$, and ghost node N+1 is juxtaposed with node N such that $u_{\mathrm{N}+1}(k) \stackrel{\triangle}{=} u_1(k)$.

Defining

$$U(k) \stackrel{\triangle}{=} \begin{bmatrix} u_1(k) & u_2(k) & \dots & u_{\mathrm{N}}(k) \end{bmatrix}^{\mathrm{T}}, \tag{24}$$

we write (23) in vector form as

$$U(k+1) = F(U(k)) + W(k), \tag{25}$$

$$Y_0(k) = G(U(k)), \tag{26}$$

$$Y(k) = H(U(k)), \tag{27}$$

where $Y(k)$ is the input to the unknown subsystem, and $W(k)$ is the output of the unknown subsystem. H is defined such that $W(k)=\nu Y(k)$. Thus, F and H are vector functions, appropriately defined using (23). Note that $G_{\mathrm{s}}(\mathbf{q})=\nu$.

B. Numerical simulation of the Burgers equation

We consider the finite domain $x \in [0,1]$, and partition the domain by constructing a uniform grid with $N = 100$ nodes, so that $\Delta x = \frac{1}{N-1}$. The time step $\Delta t = 5 \times 10^{-5}$ is chosen such that the CFL condition $\frac{|u_{\max|\Delta t}}{\Delta x} < C_{\max}$ is satisfied. The Courant number $C_{\max}$ depends on the chosen discretization method [17]. In this paper, we set $C_{\max} = 0.25$. We assume the true value $\nu = 0.15$. The boundary conditions are periodic. Finally, the initial profile $u(x,0)$ is defined as

$$u(x,0) = 3 + \sin(2\pi \mathrm{x}) + \sin(4\pi \mathrm{x} + 3) + \sin(14\pi \mathrm{x} + 5). \tag{28}$$

Next, we define the measurement $Y_0(k) \stackrel{\triangle}{=} u_{98}(k)$, where the grid point is chosen arbitrarily. (25)–(27) along with the true value of ν constitutes the true system. Figure 2 shows the solution $U(k)$ at three time instants, as well as the measurement $Y_0(k)$ for the numerical simulation of the true system.

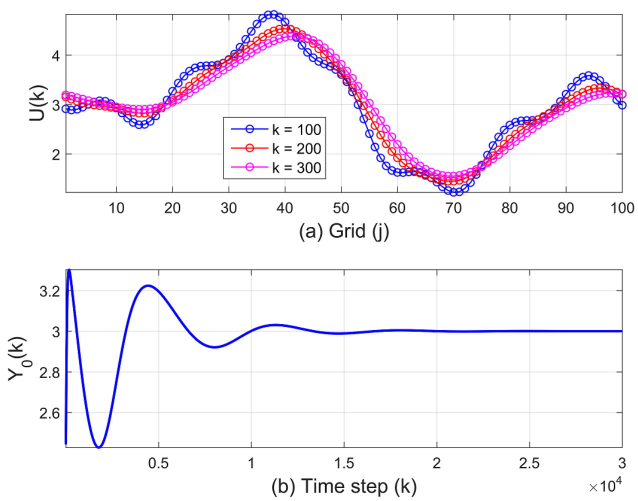

Fig. 2: Numerical simulation of the Burgers equation. The initial profile is defined in (28). (a) shows the solution $U(k)$ at $k = 100$, 200, and 300. Note that the solution advects towards the right and diffuses simultaneously. (b) shows the measurement $Y_0(k) = u_{98}(k)$. Note that the measurement converges to a finite value that depends on the initial profile $u(x,0)$ used in the simulation.

C. RCMR

We estimate the viscosity in the Burgers equation using RCMR. The main system model is defined as in (25)–(27). Since $W(k) = \nu Y(k)$, we define a zeroth-order subsystem model $\hat{W}(k) = \hat{Y}(k)\theta$, and update θ by using the RCMR algorithm in section III. G_{f} is chosen as a finite impulse filter so that $D_{\mathrm{f}}(\mathbf{q}) = 1$. Further, we use $N_{\mathrm{f}}(\mathbf{q}) = \mathbf{q}^{-2}$, $R_\theta = 10^{-5}$, and $\theta(0) = 0$. (25)–(27) along with the estimated viscosity θ constitutes the modeled system. Figure 3 shows the RCMR estimates of the unknown viscosity.

Next, we investigate the effect of the initial profile on accuracy of the estimate. We set the initial profile to be a sum of asynchronous harmonics as

$$u(x,0) = 4 + \frac{1}{n}\sum_{i=1}^{n} \sin(2\pi x i + i^2). \tag{29}$$

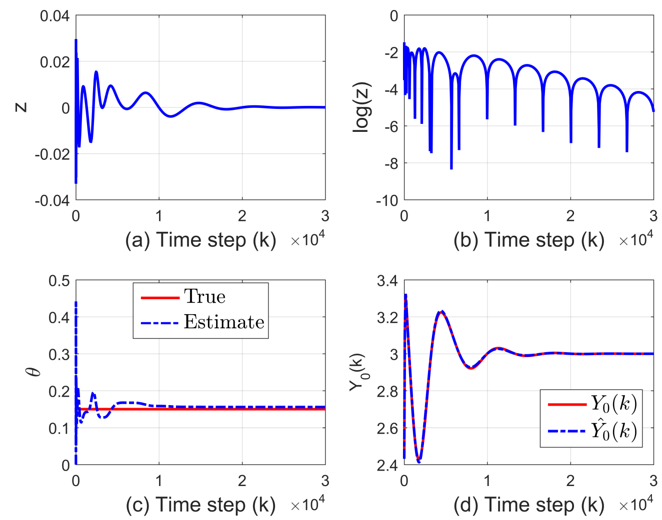

Fig. 3: Estimation of viscosity using RCMR in the Burgers equation. (a) shows the performance $z(k)$. RCMR minimizes a cost function based on $z(k)$ to estimate the subsystem model. (b) shows the performance $z(k)$ on logarithmic scale. (c) shows the estimate of viscosity θ. (d) shows the measurement $Y_0(k)$ from the true system as well as $\hat{Y}_0(k)$ computed by the main system model.

Note that the initial profile is centered at a value of 4. This choice is arbitrary as long as the CFL condition is satisfied and $u(x,t) > 0$ on the entire domain. The initial profile defined by (29) is a Schroeder-phased signal [18]. These signals minimize the peak-to-peak amplitude of multi-sine signals. This choice furnishes a large number of small amplitude peaks in the initial profile in the domain, instead of a small number of high amplitude peaks where the phase in (29) is zero.

Figure 4 shows the effect of various initial profiles on the estimate and suggests that increasing the spectral content of the initial profile increases the accuracy as well as convergence speed of the estimate.

In this section, we applied RCMR to estimate the kinematic viscosity in the Burgers equation using the measurement of the solution at one point. Further, we showed the effect of spectral content of the initial profile on the quality of the estimate.

VI. Parameter Estimation in the Modified Burgers Equation

In this section, we consider the modified one-dimensional Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = \frac{\partial}{\partial x}\left(\nu \frac{\partial u}{\partial x}\right) - \frac{1}{2}\frac{\partial \tau}{\partial x}. \tag{30}$$

This equation has the additional term τ, which, in the context of large eddy simulations (LES), captures the effect of subgrid-scales (SGS) of the flow. LES is a technique that lies between direct numerical simulation (in which all scales are resolved), and averaging techniques (which model all

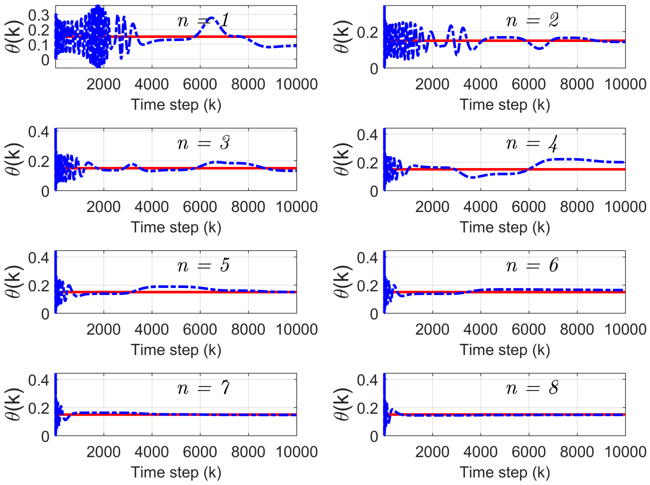

Fig. 4: Estimation of viscosity using RCMR in the Burgers equation with various initial profiles. The solid red line is the true value $\nu = 0.15$, and the dashed blue line is the estimate θ. n is the number of harmonics in the initial profile, which is defined by (29). This figure shows that increasing the spectral content of the initial profile by increasing the number of harmonics increases both the accuracy and speed of convergence of the estimate.

scales other than the mean). In LES, large-scale features are resolved through numerical computation of the underlying physics, while features that are smaller than the mesh size are modeled.

The SGS stress τ represents the contribution of unresolved scales to the total momentum transport. A classical model for SGS stress is the Smagorinsky-type eddy viscosity model [19], which relates the subgrid stress tensor to the resolved strain rate

$$\tau \triangleq -2(C_S\Delta_f)^2 \left|\frac{\partial u}{\partial x}\right| \frac{\partial u}{\partial x}, \tag{31}$$

where C_S is the Smagorinsky coefficient, and Δ_f is a characteristic width. Commonly, $\frac{\Delta_f}{\Delta x} = 1$ or 2, we assume $\Delta_f = \Delta x$. For details, see Chapter 1 of [20]. The goal is to estimate C_S using measurements of $u(x,t)$ at a single arbitrary location in the domain.

Note that (31) can be expressed as

$$\tau \triangleq -2\mu \frac{\partial u}{\partial x}, \tag{32}$$

where $\mu \triangleq (C_S\Delta_f)^2 \left|\frac{\partial u}{\partial x}\right|$. Rewriting (30) with τ given by (32) yields

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = \frac{\partial}{\partial x}\left((\nu+\mu)\frac{\partial u}{\partial x}\right). \tag{33}$$

The effect of including the SGS stresses is equivalent to adding artificial viscosity to the system. Note that the artificial viscosity depends on both spatially and temporally evolving variables.

Similar to the qualitative behavior of the Burgers equation, the artificial viscosity in the modified Burgers equation only regulates the rate of diffusion. Hence, two solution profiles with same initial profiles, but with different Smagorinsky coefficients, asymptotically reach the same solution. Therefore, it is necessary that the initial profile $u(x,0)$ have sufficient spectral content for accurate estimation of the viscosity, which can be realized by choosing the initial profile as a sum of harmonics.

A. Discretization of modified Burgers Equation

We discretize the Burgers equation using forward Euler approximation for the time derivative and the first spatial derivatives, a second-order scheme for the convective term, and a central difference scheme for the viscous term. Note that $u(x,t)$ is a continuous variable whose domain is $[0,1]$ for $t > 0$, while $u_j(k)$ is a discrete variable defined on the grid points $j \in \{1,\ldots,N\}$ for $k \in \mathbb{N}$. Hence, at each grid point,

$$\begin{aligned} u_j(k+1) = u_j(k) \\ &- \frac{\Delta t}{2\Delta x}(1.5u_j(k)^2 - 2u_{j-1}(k)^2 + 0.5u_{j-2}(k)^2) \\ &+ \nu\frac{\Delta t}{\Delta x^2}(u_{j+1}(k) - 2u_j(k) + u_{j-1}(k)) \\ &+ c\frac{\Delta t}{\Delta x}(\left|u_{j+1}(k)-u_j(k)\right|(u_{j+1}(k)-u_j(k)) \\ &- \left|u_j(k)-u_{j-1}(k)\right|(u_j(k)-u_{j-1}(k))), \end{aligned} \tag{34}$$

where $c \triangleq C_S^2$. To make the domain infinite, periodic boundary conditions are enforced by defining two ghost nodes at both ends of the grid as in Section V.

We define $U(k)$ as defined in Section V, and express the true system similar to (25)–(27), where F, G, and H are defined appropriately using (34). Note that $G_s(\mathbf{q}) = \mathrm{c}$.

B. Numerical simulation of the modified Burgers equation

We consider the same domain as defined in Section (V). The time step $\Delta t = 5\times 10^{-4}$ is according to the CFL condition. We assume the true value $\nu = 0.05$, and $c = 0.05$. The boundary conditions are periodic. Finally, the initial profile $u(x,0)$ is arbitrarily defined as

$$u(x,0) = 4 + \frac{1}{6}\sum_{i=1}^{6}\sin(2\pi xi + i^2). \tag{35}$$

We define the measurement $Y_0(k) \triangleq u_{98}(k)$. (25)–(27) along with the true values of ν and c constitutes the true system.

C. RCMR

We estimate the parameter c in the modified Burgers equation using RCMR. The main system model is (25)–(27), defined appropriately using (34). Since $W(k) = cY(k)$, we define a zeroth order subsystem model $\hat{W}(k) = \hat{Y}(k)\theta$, and update θ by using the RCMR algorithm in section III. G_f is chosen as a finite impulse filter, thus, $D_f(\mathbf{q}) = 1$. $N_f(\mathbf{q}) = \mathbf{q}^{-2}$, $R_\theta = 10^{-5}$. Figure 5 shows the estimate.

Next, we investigate the effect of initial profiles on accuracy of the estimate. The initial profiles are defined by (29). Figure 6 shows the effect of various initial profiles on the estimate.

In this section, we applied RCMR to estimate the parameter relating the SGS stresses and the strain tensor in the modified Burgers equation using the measurement of

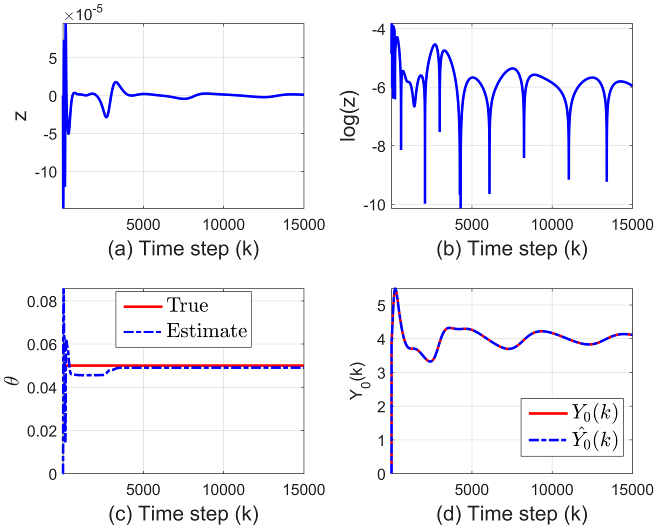

Fig. 5: Estimation of viscosity using RCMR in the modified Burgers equation. (a) shows the performance $z(k)$. RCMR minimizes a cost function based on $z(k)$ to estimate the subsystem model. (b) shows the performance $z(k)$ on logarithmic scale. (c) shows the estimate of viscosity, θ. (d) shows the measurement $Y_0(k)$ from the true system and $\hat{Y}_0(k)$ computed by the main system model.

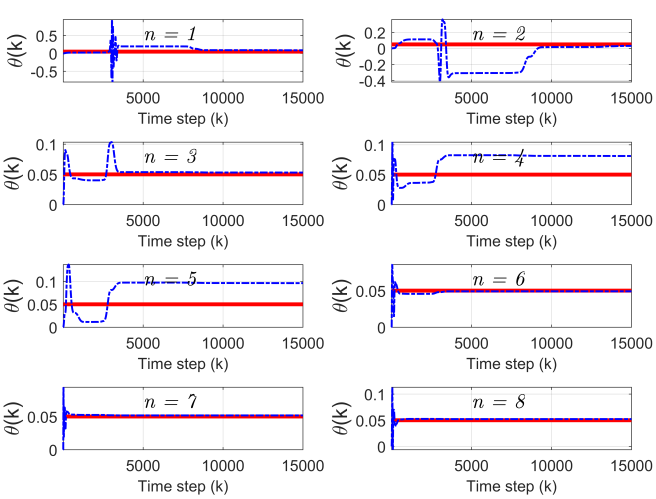

Fig. 6: Estimation of the parameter c using RCMR in the modified Burgers equation with different initial profiles. The solid red line is the true value $c = 0.05$, and the dashed blue line is the estimate θ. n is the number of harmonics in the initial profile.This figure shows that increasing the spectral content of the initial profile increases the accuracy as well as convergence speed of the estimate.

the solution at one point. Further, we showed the effect of spectral content of the initial profile on the quality of the estimate.

VII. CONCLUSIONS

In this paper, we showed that retrospective model refinement can be used to estimate inaccessible parameters in a nonlinear system with limited measurements. This is a particularly useful application of RCMR, as it can be used to refine large scale CFD models, which are usually based on nonlinear models of physical phenomena.

We also showed that the spectral content of the initial profile is particularly important for the estimation in unforced systems where the asymptotic behavior of the system is relatively insensitive to the choice of parameters to be estimated. In fact, asymptotic solutions of the Burgers equation considered in this paper are insensitive to the estimated parameters.

Continuing work is focused on estimating the *functional* relationship between the subgrid stress tensor and the resolved strain rate using measurements from direct numerical simulations. Obtaining such an estimate is expected to result in more accurate, data-driven subgrid-scale models.

REFERENCES

[1] A. M. D'Amato, A. J. Ridley, and D. S. Bernstein, "Retrospective-cost-based adaptive model refinement for the ionosphere and thermosphere," *Statistical Analysis and Data Mining*, vol. 4, pp. 446–458, 2011.

[2] A. M. D'Amato, S. L. Kukreja, and D. S. Bernstein, "Data-based model refinement using retrospective cost optimization," in *Proc. AIAA Guid. Nav. Contr. Conf.*, Toronto, August 2010, aIAA-2010-7889-784.

[3] M. A. Santillo and D. S. Bernstein, "Adaptive Control Based on Retrospective Cost Optimization," *J. Guid. Contr. Dyn.*, vol. 33, pp. 289–304, 2010.

[4] J. B. Hoagg and D. S. Bernstein, "Retrospective Cost Model Reference Adaptive Control for Nonminimum-Phase Systems," *J. Guid. Contr. Dyn.*, vol. 35, pp. 1767–1786, 2012.

[5] A. A. Ali, A. Goel, A. J. Ridley, and D. S. Bernstein, "Retrospective-cost-based adaptive input and state estimation for the ionosphere–thermosphere," *J. of Aerospace Information Systems*, pp. 1–17, 2015.

[6] A. G. Burrell, A. Goel, A. Ridley, and D. Bernstein, "Correction of the photoelectron heating efficiency within the global ionosphere-thermosphere model using retrospective cost model refinement," *J. of Atmospheric and Solar-Terrestrial Physics*, vol. 124, pp. 30–38, 2015.

[7] N. Smaoui, "Boundary and distributed control of the viscous Burgers equation," *Journal of Computational and Applied Mathematics*, vol. 182, no. 1, pp. 91–104, 2005.

[8] J. Marburger and R. Pinnau, "Optimal control for Burgers equation using particle methods," *arXiv preprint arXiv:1309.7619*, 2013.

[9] J. A. Burns and S. Kang, "A control problem for Burgers equation with bounded input/output," *Nonlinear Dynamics*, vol. 2, no. 4, pp. 235–262, 1991.

[10] M. Krstic, "On global stabilization of Burgers equation by boundary control," *Systems & Control Letters*, vol. 37, no. 3, pp. 123–141, 1999.

[11] S. Kutluay, A. Bahadir, and A. Özdeş, "Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods," *Journal of Computational and Applied Mathematics*, vol. 103, no. 2, pp. 251–261, 1999.

[12] B. Inan and A. R. Bahadir, "Numerical solution of the one-dimensional Burgers equation: Implicit and fully implicit exponential finite difference methods," *Pramana*, vol. 81, no. 4, pp. 547–556, 2013.

[13] M. F. Hamilton, D. T. Blackstock *et al.*, *Nonlinear acoustics*. Academic press, 1998, vol. 427.

[14] A. Aw and M. Rascle, "Resurrection of "second order" models of traffic flow," *SIAM journal on applied mathematics*, vol. 60, no. 3, pp. 916–938, 2000.

[15] C.-K. Cho, S. Kang, and Y. Kwon, "Parameter estimation problem for nonhysteretic infiltration in soil," *Journal of Korean Society for Industrial and Applied Mathematics*, vol. 4, no. 1, pp. 1–22, 2000.

[16] N. Shimkin and A. Feuer, "Persistency of excitation in continuous-time systems," *Systems & control letters*, vol. 9, no. 3, pp. 225–233, 1987.

[17] R. Courant, K. Friedrichs, and H. Lewy, "On the partial difference equations of mathematical physics," *IBM journal of Research and Development*, vol. 11, no. 2, pp. 215–234, 1967.

[18] M. R. Schroeder, "Synthesis of low-peak-factor signals and binary sequences with low autocorrelation," *IEEE Trans. Infor. Theory*, vol. 16, no. 1, pp. 85–89, January 1970.

[19] J. Smagorinsky, "General circulation experiments with the primitive equations: I. the basic experiment*," *Monthly weather review*, vol. 91, no. 3, pp. 99–164, 1963.

[20] B. J. Geurts, *Elements of Direct and Large-Eddy simulation*. R.T. Edwards, 2004.