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Lyapunov-Stable Discrete-Time Model Reference Adaptive Control

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***Abstract*— Discrete-time model reference adaptive control (MRAC) has been studied extensively. Although the framework of the analyses is very general, the results obtained are restricted to boundedness and convergence and the important question of Lyapunov stability is not addressed. Lyapunov functions are an important tool for understanding and quantifying transient response, robustness and disturbance rejection, and thus merit attention. In this paper we investigate the use of a logarithmic Lyapunov function to establish Lyapunov stability of MRAC in the deterministic setting. A complete Lyapunov proof is given for stability and convergence. The results extend the approach of [8] to include Lyapunov stability for MRAC when the normalized projection algorithm is used for parameter identification.**

1. Introduction

In model reference adaptive control (MRAC) theory the objective is to have the plant emulate the dynamics of a specified model in response to a family of command signals. MRAC has been extensively developed for continuous-time systems [1] and discrete-time systems [2] where the boundedness of controller parameters and convergence of the tracking error are demonstrated using the *Gronwall-Bellman lemma* and the *key technical lemma* respectively. The objective of the present paper is to unify and extend the foundation of discrete-time MRAC by constructing Lyapunov functions to demonstrate Lyapunov stability as well as error convergence. The results of this paper are used in a companion paper [3] to prove Lyapunov stability for a discrete-time adaptive disturbance rejection algorithm.

Discrete-time MRAC algorithms have been based on a variety of parameter identification algorithms. For example, the recursive least squares (RLS) algorithm and the projection algorithm are used in [4], where convergence is based on the key technical lemma. This method of proof yields convergence but does not imply Lyapunov stability of the error system. MRAC is considered in the presence of additive noise in [2], [5–7]. In these results, convergence of the tracking error and parameters is guaranteed almost surely, but stochastic Lyapunov stability is not demonstrated.

Lyapunov stability of discrete-time MRAC and convergence of the error to a finite set is demonstrated in [8], where the RLS algorithm is used for parameter identification. A Lyapunov candidate is applied to the time-varying error system, which requires appropriate bounds on the Lyapunov difference. Stochastic Lyapunov stability of MRAC is addressed [11].

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The novel Lyapunov construction of [8–12] is of independent interest since it involves the logarithm of a quadratic form. A similar construction was used in [13] for full-state-feedback adaptive stabilization and extended in [14] to a more general class of gradient based gain update laws.

In view of these developments, in the present paper we extend the result of [8] by constructing a Lyapunov proof of MRAC for the projection algorithm. These constructs remove the need for the key technical lemma used in [4].

The contents of the paper are as follows. In Section 2 we present the solution to the model matching control problem in the case of a known plant. An adaptive control law with projection algorithm based parameter identification is presented in Section 4 and a proof of stability is given in Section 5. Finally, Section 6 presents simulation results.

2. Model Reference Control for a Known Plant

Consider a SISO process described by the DARMA model

$$y(k) = -\sum_{i=1}^{n} a_i y(k-i) + \sum_{j=0}^{m} b_j u(k-j), \quad k \geq 0. \quad (2.1)$$

The model (2.1) can be written in terms of the forward shift operator $\mathbf{q}$ as

$$\mathbf{A}(\mathbf{q})y(k) = \mathbf{B}(\mathbf{q})u(k), \quad (2.2)$$

where $\mathbf{A}$ and $\mathbf{B}$ are polynomials of degree n and m, respectively, defined by

$$\mathbf{A}(\mathbf{q}) \triangleq \mathbf{q}^n + a_1\mathbf{q}^{n-1} + \cdots + a_n \quad (2.3)$$

and

$$\mathbf{B}(\mathbf{q}) \triangleq b_0\mathbf{q}^m + b_1\mathbf{q}^{m-1} + \cdots + b_m, \quad (2.4)$$

where $b_0 \neq 0$. We define the delay $d \triangleq n - m$ and make the following assumptions about the plant.

Assumption 2.1. The realization (2.2) is minimal, i.e., $\mathbf{A}$ and $\mathbf{B}$ are coprime.

Assumption 2.2. All roots of $\mathbf{B}(\mathbf{q})$ are inside the unit circle.

Assumption 2.3. n and m are known, and $m < n$.

Assumption 2.4. b_0 is known.

To modify the dynamics (2.2) we consider the 2-DOF model matching control law

$$u(k) = \frac{\mathbf{T}(\mathbf{q})}{\mathbf{R}(\mathbf{q})}u_{\rm c}(k) - \frac{\mathbf{S}(\mathbf{q})}{\mathbf{R}(\mathbf{q})}y(k), \quad (2.5)$$

where $u_{\rm c}$ is the command signal. We want the response from the command signal $u_{\rm c}$ to the output y to be described by the reference model

$$y_{\mathrm{m}}(k) \quad = \quad \frac{\mathbf{B_m}(\mathbf{q})}{\mathbf{A_m}(\mathbf{q})}u_{\mathrm{c}}(k). \tag{2.6}$$

We make the following assumptions about the reference model.

Assumption 2.5. $\mathbf{A_m}(\mathbf{q})$ is monic and stable.

Assumption 2.6. $\deg \mathbf{A_m}(\mathbf{q}) - \deg \mathbf{B_m}(\mathbf{q}) = d$, i.e., the reference model has the same delay as the plant.

The closed-loop system (2.2)-(2.5) the reference model (2.6) have the same forced response if

$$\frac{\mathbf{B}(\mathbf{q})\mathbf{T}(\mathbf{q})}{\mathbf{A}(\mathbf{q})\mathbf{R}(\mathbf{q}) + \mathbf{B}(\mathbf{q})\mathbf{S}(\mathbf{q})} = \frac{\mathbf{B_m}(\mathbf{q})}{\mathbf{A_m}(\mathbf{q})}, \tag{2.7}$$

which is equivalent to

$$\frac{\mathbf{T}(\mathbf{q})}{\mathbf{A}(\mathbf{q})\mathbf{R}(\mathbf{q}) + \mathbf{B}(\mathbf{q})\mathbf{S}(\mathbf{q})} = \frac{\mathbf{B_m}(\mathbf{q})}{\mathbf{A_m}(\mathbf{q})\mathbf{B}(\mathbf{q})}. \tag{2.8}$$

The roots of the closed-loop characteristic polynomial $\mathbf{A_m}(\mathbf{q})\mathbf{B}(\mathbf{q})$ consist of the roots of $\mathbf{A_m}(\mathbf{q})$ as well as the roots of $\mathbf{B}(\mathbf{q})$, all of which are stable by assumption. Let $n_{\mathrm{m}} \triangleq \deg \mathbf{A_m}(\mathbf{q})$ and define

$$\begin{aligned}\mathbf{P}(\mathbf{q}) &\triangleq \mathbf{A_m}(\mathbf{q})\mathbf{B}(\mathbf{q}) \\ &= b_0\mathbf{q}^{(n_{\mathrm{m}}+m)} + p_1\mathbf{q}^{(n_{\mathrm{m}}+m)-1} + \cdots + p_{n_{\mathrm{m}}+m}.\end{aligned}$$

To satisfy (2.8) it suffices to choose

$$\mathbf{T}(\mathbf{q}) = \mathbf{B_m}(\mathbf{q}) \tag{2.9}$$

and require that $\mathbf{R}(\mathbf{q})$ and $\mathbf{S}(\mathbf{q})$ satisfy

$$\mathbf{A}(\mathbf{q})\mathbf{R}(\mathbf{q}) + \mathbf{B}(\mathbf{q})\mathbf{S}(\mathbf{q}) = \mathbf{P}(\mathbf{q}). \tag{2.10}$$

Defining $n_{\mathbf{R}} \triangleq \deg \mathbf{R}(\mathbf{q})$, $n_{\mathbf{S}} \triangleq \deg \mathbf{S}(\mathbf{q})$, $n_e \triangleq n+n_{\mathbf{R}}+1$, and $n_u \triangleq n_{\mathbf{R}} + n_{\mathbf{S}} + 2$, (2.10) can be written as

$$M \begin{bmatrix} \mathcal{C}(\mathbf{R}) \\ \mathcal{C}(\mathbf{S}) \end{bmatrix} = \mathcal{C}(\mathbf{P}), \tag{2.11}$$

where $M \in \mathbb{R}^{n_e \times n_u}$ is the Sylvester matrix given by

$$\begin{bmatrix} 1 & 0 & & 0_{(n-1)\times 1} & 0_{d\times 1} & 0_{(d+1)\times 1} & & 0_{(n+d-1)\times 1} \\ a_1 & 1 & & 1 & b_0 & b_0 & & b_0 \\ a_2 & a_1 & & a_1 & b_1 & b_1 & & b_1 \\ \vdots & \vdots & \ddots & a_2 & \vdots & \vdots & \ddots & b_2 \\ a_n & a_n & & \vdots & b_m & b_m & & \vdots \\ 0_{(n-1)\times 1} & 0_{(n-2)\times 1} & & a_n & 0_{(n-1)\times 1} & 0_{(n-2)\times 1} & & b_m \end{bmatrix}$$

$\mathcal{C}(\mathbf{R}) = [r_0 r_1 \cdots r_{n_{\mathbf{R}}}]^{\mathrm{T}}$, $\mathcal{C}(\mathbf{R}) = [s_0 s_1 \cdots s_{n_{\mathbf{S}}}]^{\mathrm{T}}$, and $\mathcal{C}(\mathbf{R}) = [b_0 p_1 \cdots p_{n_{\mathrm{m}}+m}]^{\mathrm{T}}$ are vectors containing the coefficients of $\mathbf{R}(\mathbf{q})$, $\mathbf{S}(\mathbf{q})$, and $\mathbf{P}(\mathbf{q})$, respectively. In the remainder of the paper, we omit the explicit dependence of polynomial operators on $\mathbf{q}$.

Proposition 2.1. Assume that $n_{\mathrm{m}} = 2n-m-1$ and $n_{\mathbf{S}} = n-1$. Then, for each $n_{\mathbf{R}} \geq 0$ there exist unique polynomials $\mathbf{R}$ and $\mathbf{S}$ satisfying (2.11). Furthermore, if $n_{\mathbf{R}} \geq n_{\mathbf{S}}$ then the control law (2.5) is causal.

Proof. Since $n_{\mathbf{S}} = n - 1$ it follows that $M \in \mathbb{R}^{(n+n_{\mathbf{R}}+1)\times(n+n_{\mathbf{R}}+1)}$ is square. Also, since $\mathbf{A}$ and $\mathbf{B}$ are relatively coprime, M is nonsingular and the solution to (2.11) is unique. From (2.9) we have

$$\deg \mathbf{T} = \deg \mathbf{B_m} = \deg \mathbf{A_m} - d = n-1. \tag{2.12}$$

The condition $n_{\mathbf{R}} \geq n_{\mathbf{S}}$ implies that $\deg \mathbf{R} \geq \deg \mathbf{S} = \deg \mathbf{T}$, and thus the model matching controller (2.5) is causal. □

Henceforth in accordance with Proposition 2.1 we assume that $\deg \mathbf{S} = n-1$ and $\deg \mathbf{A_m} = 2n-m-1$ so that $\deg \mathbf{P} = 2n-1$. Also, to obtain a minimum degree causal controller we assume that $\deg \mathbf{R} = n-1$ so that $M \in \mathbb{R}^{2n\times 2n}$. Hence we write

$$\mathbf{R} \triangleq r_0\mathbf{q}^{n-1} + r_1\mathbf{q}^{n-2} + \cdots + r_{n-1} \tag{2.13}$$

and

$$\mathbf{S} \triangleq s_0\mathbf{q}^{n-1} + s_1\mathbf{q}^{n-2} + \cdots + s_{n-1}, \tag{2.14}$$

where r_0 and s_0 are nonzero. In fact, it follows from (2.10) and (2.13) that $r_0 = b_0$.

Next to obtain a linear estimation model in terms of the controller we define the filtered output signal

$$y_{\mathrm{f}}(k) \triangleq \mathbf{q}^{-n-d+1}\mathbf{A_m}y(k) = \frac{\mathbf{q}^{-n-d+1}\mathbf{A_m}\mathbf{B}}{\mathbf{A}}u(k). \tag{2.15}$$

With the model matching condition (2.10), y_{f} satisfies

$$\begin{aligned} y_{\mathrm{f}}(k+d) &= \frac{\mathbf{q}^{-n+1}(\mathbf{AR}+\mathbf{BS})}{\mathbf{A}}u(k) \\ &= \mathbf{R}u(k-n+1) + \mathbf{S}y(k-n+1). \end{aligned} \tag{2.16}$$

Since $r_0 = b_0$ (2.16) can be written as the linear identification model

$$y_{\mathrm{f}}(k+d) = b_0 u(k) + \varphi^{\mathrm{T}}(k)\theta, \tag{2.17}$$

where the parameter vector $\theta \in \mathbb{R}^{2n-1}$ and the regressor $\varphi(k) \in \mathbb{R}^{2n-1}$ are defined by

$$\theta \triangleq [r_1 \cdots r_{n-1} s_0 \cdots s_{n-1}]^{\mathrm{T}} \tag{2.18}$$

and

$$\varphi(k) \triangleq [u(k-1) \cdots u(k-n+1) y(k) \cdots y(k-n+1)]^{\mathrm{T}}. \tag{2.19}$$

Using (2.16) and (2.17) the model matching control law (2.5) can be written as

$$u(k) = -\frac{1}{b_0}\left[\varphi^{\mathrm{T}}(k)\theta - \mathbf{q}^{-n+1}\mathbf{B_m}u_{\mathrm{c}}(k)\right]. \tag{2.20}$$

The filtered plant model (2.17) and the control law (2.20) are now in a form suitable for direct adaptive control.

3. Model Matching Error Dynamics

When the plant (2.2) is unknown we cannot solve (2.11) for the controller parameters $\mathbf{R}$ and $\mathbf{S}$. Hence, let $\hat{\mathbf{R}}(k)$ and $\hat{\mathbf{S}}(k)$ be polynomials in $\mathbf{q}$ that are estimates of $\mathbf{R}$ and $\mathbf{S}$ at time k. Then in place of (2.5), the estimated model matching controller is

$$u(k) = \frac{\mathbf{B_m}}{\hat{\mathbf{R}}(k)}u_{\mathrm{c}}(k) - \frac{\hat{\mathbf{S}}(k)}{\hat{\mathbf{R}}(k)}y(k). \tag{3.1}$$

With (3.1) the closed loop system has the form

$$y(k) \quad = \quad \frac{\mathbf{B}\mathbf{B_m}}{\mathbf{A}\hat{\mathbf{R}}(k) + \mathbf{B}\hat{\mathbf{S}}(k)}u_{\mathrm{c}}(k). \tag{3.2}$$

Next let $\hat{\theta}(k)$ denote an estimate of θ at time k and define the parameter error

$$\tilde{\theta}(k) \triangleq \hat{\theta}(k) - \theta \tag{3.3}$$

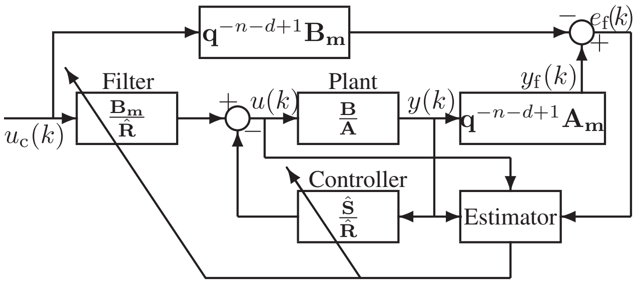

Fig. 1. Model Reference Adaptive Control Block Diagram

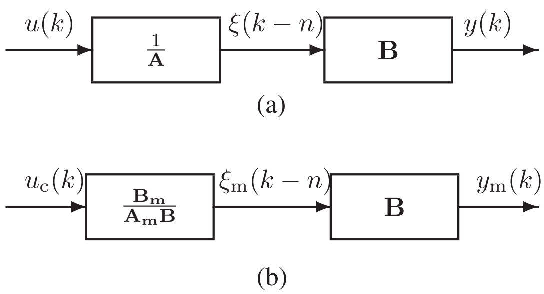

Fig. 2. Fraction Forms of the Plant and the Reference Model

and the filtered output error signal (see Figure 1)

$$e_{\rm f}(k) \triangleq y_{\rm f}(k) - \mathbf{q}^{-n-d+1}\mathbf{B_m}u_{\rm c}(k). \tag{3.4}$$

To express $e_{\rm f}(k)$ in terms of $\tilde{\theta}$, note that

$$\begin{aligned} y_{\rm f}(k+d) &= \mathbf{q}^{-n+1}\frac{\mathbf{A_m B B_m}}{\mathbf{A}\hat{\mathbf{R}}(k)+\mathbf{B}\hat{\mathbf{S}}(k)}u_{\rm c}(k) \\ &= \frac{b_0 u(k)+\varphi^{\rm T}(k)\theta}{\mathbf{q}^{n-1}\left[b_0 u(k)+\varphi^{\rm T}(k)\hat{\theta}(k)\right]}\mathbf{B_m}u_{\rm c}(k). \end{aligned} \tag{3.5}$$

Combining (2.17) and (3.5) yields

$$b_0 u(k)+\varphi^{\rm T}(k)\hat{\theta}(k) = \mathbf{q}^{-n+1}\mathbf{B_m}u_{\rm c}(k). \tag{3.6}$$

From (2.17), (3.4) and (3.6) it follows that

$$\begin{aligned} e_{\rm f}(k+d) &= y_{\rm f}(k+d) - \mathbf{q}^{-n+1}\mathbf{B_m}u_{\rm c}(k) \\ &= -\varphi^{\rm T}(k)\tilde{\theta}(k). \end{aligned} \tag{3.7}$$

To formulate the model matching error dynamics we note that the plant (2.2) can be written in the $n^{\rm th}$ order fraction form as [15] (see Figure 2(a))

$$\mathbf{A}\xi(k-n) = u(k), \tag{3.8}$$

$$y(k) = \mathbf{B}\xi(k-n). \tag{3.9}$$

From (2.15) and (3.9) it follows that

$$\begin{aligned} y_{\rm f}(k+d) &= \mathbf{q}^{-n+1}\mathbf{A_m}y(k) = \mathbf{q}^{-n+1}\mathbf{A_m B}\xi(k-n) \\ &= \mathbf{q}^{-2n+1}\mathbf{P}\xi(k). \end{aligned} \tag{3.10}$$

In a similar manner the reference model can be written in the $(2n-1)^{\rm th}$ order non-minimal fraction form (see Figure 2(b))

$$\mathbf{A_m B}\xi_{\rm m}(k-n) = \mathbf{B_m}u_{\rm c}(k), \tag{3.11}$$

$$y_{\rm m}(k) = \mathbf{B}\xi_{\rm m}(k-n). \tag{3.12}$$

From (3.11) it follows that

$$\begin{aligned} \mathbf{q}^{-n+1}\mathbf{B_m}u_{\rm c}(k) &= \mathbf{q}^{-n+1}\mathbf{A_m B}\xi_{\rm m}(k-n) \\ &= \mathbf{q}^{-2n+1}\mathbf{P}\xi_{\rm m}(k). \end{aligned} \tag{3.13}$$

Using (3.10) and (3.13), the d-step ahead filtered output error can now be written as

$$\begin{aligned} e_{\rm f}(k+d) &= y_{\rm f}(k+d) - \mathbf{q}^{-n+1}\mathbf{B_m}u_{\rm c}(k) \\ &= \mathbf{q}^{-2n+1}\mathbf{P}\xi_{\rm e}(k), \end{aligned} \tag{3.14}$$

where

$$\xi_{\rm e}(k) \triangleq \xi(k) - \xi_{\rm m}(k). \tag{3.15}$$

Next define plant, reference model, and model-matching error states by

$$x(k) \triangleq \left[\xi(k-1)\cdots\xi(k-2n+1)\right]^{\rm T}, \tag{3.16}$$

$$x_{\rm m}(k) \triangleq \left[\xi_{\rm m}(k-1)\cdots\xi_{\rm m}(k-2n+1)\right]^{\rm T}, \tag{3.17}$$

and

$$x_{\rm e}(k) \triangleq x(k) - x_{\rm m}(k). \tag{3.18}$$

Then

$$x_{\rm e}(k) = \left[\xi_{\rm e}(k-1)\cdots\xi_{\rm e}(k-2n+1)\right]^{\rm T}. \tag{3.19}$$

Since

$$\begin{aligned} \xi_{\rm e}(k) &= \mathbf{q}\xi_{\rm e}(k-1) \\ &= -\mathbf{q}\left[\frac{1}{b_0}q^{-2n+1}\mathbf{P}-1\right]\xi_{\rm e}(k-1) + \frac{1}{b_0}e_{\rm f}(k+d) \end{aligned}$$

a state equation for $x_{\rm e}$ in controllable canonical form is given by

$$x_{\rm e}(k+1) = Ax_{\rm e}(k) + \frac{1}{b_0}Be_{\rm f}(k+d), \quad k \geq 0, \tag{3.20}$$

where

$$A \triangleq \left[\begin{array}{c|c} \multicolumn{2}{c}{-p_1/b_0 \quad \cdots \quad -p_{2n-1}/b_0} \\ \hline & 0 \\ I_{(2n-2)\times(2n-2)} & \vdots \\ & 0 \end{array}\right], B \triangleq \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Note that A is asymptotically stable. Alternatively, using (3.7) the model matching dynamics (3.20) can be written as

$$x_{\rm e}(k+1) = Ax_{\rm e}(k) - \frac{1}{b_0}B\varphi^{\rm T}(k)\tilde{\theta}(k), \qquad k \geq 0. \tag{3.21}$$

Next we show that the state x defined in (3.16) is related to φ through a nonsingular transformation.

Lemma 3.1. The plant state x defined by (3.16) and the regressor (3.16) are related by

$$\varphi(k) = M_0 x(k), \tag{3.22}$$

where the nonsingular matrix $M_0 \in \mathbb{R}^{(2n-1)\times(2n-1)}$ is given by

$$\begin{bmatrix} 1 \;\; a_1 \;\; a_2 \;\; \cdots \;\; a_n & & 0_{1\times(n-2)} \\ \ddots & \ddots & \ddots \\ 0_{1\times(n-2)} & & 1 \;\; a_1 \;\; a_2 \;\; \cdots \;\; a_n \\ 0_{1\times(d-1)} \; b_0 \;\; b_1 \;\; \cdots \;\; b_m & & 0_{1\times(n-1)} \\ \ddots & \ddots & \ddots \\ 0_{1\times(n+d-2)} & & b_0 \;\; b_1 \;\; \cdots \;\; b_m \end{bmatrix}.$$

Proof. It follows from (2.19), (3.8) and (3.9) that

$$\varphi(k)=\begin{bmatrix} [1 a_1 \cdots a_n]\,[\xi(k-1)\cdots\xi(k-n-1)]^{\mathrm{T}} \\ \vdots \\ [1 a_1 \cdots a_n]\,[\xi(k-n+1)\cdots\xi(k-2n+1)]^{\mathrm{T}} \\ [b_0 b_1 \cdots b_m]\,[\xi(k-d)\cdots\xi(k-n)]^{\mathrm{T}} \\ \vdots \\ [b_0 b_1 \cdots b_m]\,[\xi(k-n-d+1)\cdots\xi(k-2n+1)]^{\mathrm{T}} \end{bmatrix}. \tag{3.23}$$

From (3.16) and (3.23) it follows that $\varphi(k) = M_0 x(k)$. It can be seen that M_0 is the $(2n-1)\times(2n-1)$ submatrix of M^{T}, formed by omitting the first row and first column of M^{T}. Note that $\det M_0 = \det M$. Since $\mathbf{A}$ and $\mathbf{B}$ are relatively co-prime by assumption, it follows that M is nonsingular, and thus M_0 is nonsingular. $\square$

4. Projection Adaptive Control Algorithm

Consider the cost function

$$J_\lambda\left(\hat\theta(k),\hat\theta(k-1),k\right) \triangleq \frac{1}{2}\|\hat\theta(k)-\hat\theta(k-1)\|_2^2 + \lambda\left[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat\theta(k)\right], \tag{4.1}$$

where λ is the Lagrange multiplier. A recursive expression for $\hat\theta(k)$ that minimizes (4.1) is given by [2, p. 51]

$$\hat\theta(k) = \hat\theta(k-1) + \frac{\varphi(k-d)\left[y_{\mathrm{f}}(k) - b_0 u(k-d) - \varphi^{\mathrm{T}}(k-d)\hat\theta(k-1)\right]}{\varphi^{\mathrm{T}}(k-d)\varphi(k-d)}. \tag{4.2}$$

Therefore, the adaptive control law is

$$u(k) \;\triangleq\; -\frac{1}{b_0}\left[\varphi^{\mathrm{T}}(k)\hat\theta(k) - q^{-n+1}\mathbf{B_m}u_{\mathrm{c}}(k)\right] \tag{4.3}$$

with the parameter update

$$\hat\theta(k) = \begin{cases} \hat\theta(k-1)+\frac{\varphi(k-d)\left[y_{\mathrm{f}}(k)-b_0u(k-d)-\varphi^{\mathrm{T}}(k-d)\hat\theta(k-1)\right]}{\varphi^{\mathrm{T}}(k-d)\varphi(k-d)}, \varphi(k-d)\neq 0, \\ \hat\theta(k-1), \varphi(k-d) = 0. \end{cases} \tag{4.4}$$

5. Lyapunov Stability

Define

$$\tilde\Theta(k) \triangleq \begin{bmatrix} \tilde\theta(k) \\ \vdots \\ \tilde\theta(k+d-1) \end{bmatrix}. \tag{5.1}$$

Then the error state vector consisting of the model matching error states and the parameter identification error states is defined by

$$X(k) \triangleq \begin{bmatrix} x_{\mathrm{e}}(k) \\ \tilde\Theta(k) \end{bmatrix}^{\mathrm{T}}. \tag{5.2}$$

Using (3.3),(3.7), (3.21) and (4.4) the closed-loop error dynamics with projection identification can be represented in the state space form, for $k \geq 0$, by

$$x_{\mathrm{e}}(k+1) = Ax_{\mathrm{e}}(k) - \frac{1}{b_0}B\varphi^{\mathrm{T}}(k)\tilde\theta(k), \tag{5.3}$$

$$\tilde\theta(j+1) = \left[I - \frac{\varphi(j-d+1)\varphi^{\mathrm{T}}(j-d+1)}{\varphi^{\mathrm{T}}(j-d+1)\varphi(j-d+1)}\right]\tilde\theta(j), \tag{5.4}$$

where $j = k, \ldots k+d$.

Remark 5.1. The error system (5.3)-(5.4) is time varying since the regressor $\varphi(k)$ is a function of the exogenous signal $u_{\mathrm{c}}(k)$. Also, the origin is an equilibrium of the error system.

Remark 5.2. Although the future parameter errors $\tilde\theta(k+1)$ to $\tilde\theta(k+d-1)$ are not computed by the algorithm at time k, they are included in the state vector $X(k)$ to facilitate the stability analysis.

To demonstrate that the origin of the system (5.3)-(5.4) is Lyapunov stable, the following results are required.

Fact 5.1. Define

$$V_{\tilde\theta}(\tilde\theta) \triangleq \tilde\theta^{\mathrm{T}}\tilde\theta \tag{5.5}$$

and

$$\triangle V_{\tilde\theta}(k) \;\triangleq\; \tilde\theta^{\mathrm{T}}(k+1)\tilde\theta(k+1) - \tilde\theta^{\mathrm{T}}(k)\tilde\theta(k).$$

Then for $k \geq 0$ (See [2, p. 51])

$$\triangle V_{\tilde\theta}(k) \leq \frac{-\left[\varphi^{\mathrm{T}}(k-d+1)\tilde\theta(k)\right]^2}{\varphi^{\mathrm{T}}(k-d+1)\varphi(k-d+1)} \leq 0. \tag{5.6}$$

Lemma 5.1. For all $k > 0$,

$$\frac{\left[\varphi^{\mathrm{T}}(k)\tilde\theta(k)\right]^2}{\varphi^{\mathrm{T}}(k)\varphi(k)} \leq \sum_{i=k}^{k+d-1}\frac{\left[\varphi^{\mathrm{T}}(i-d+1)\tilde\theta(i)\right]^2}{\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)}.$$

Proof. From successive self substitutions of (5.4) it follows that

$$\tilde\theta(k+d-1) = \tilde\theta(k) - \sum_{i=k}^{k+d-2}\frac{\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde\theta(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]}. \tag{5.7}$$

Dividing both sides of (5.7) by $\varphi^{\mathrm{T}}(k)/\sqrt{\varphi^{\mathrm{T}}(k)\varphi(k)}$ yields

$$\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} = \frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}} + \sum_{i=k}^{k+d-2} \frac{\varphi^{\mathrm{T}}(k)\varphi(i-d+1)\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]}.$$

The triangle inequality implies

$$\left|\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}}\right| \le \left|\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k+d-1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}}\right| + \sum_{i=k}^{k+d-2}\left|\frac{\varphi^{\mathrm{T}}(k)\varphi(i-d+1)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}}\right| \cdot \left|\frac{\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}}\right|.$$

Now using the Cauchy-Schwarz inequality we have

$$\left|\frac{\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)}{[\varphi^{\mathrm{T}}(k)\varphi(k)]^{1/2}}\right| = \sum_{i=k}^{k+d-1}\left|\frac{\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)}{[\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)]^{1/2}}\right|. \qquad \square$$

Lemma 5.2. Define

$$V_{\tilde{\Theta}}(\tilde{\Theta}) \triangleq \tilde{\Theta}^{\mathrm{T}}\tilde{\Theta} \tag{5.8}$$

and

$$\triangle V_{\tilde{\Theta}}(k) \triangleq \tilde{\Theta}^{\mathrm{T}}(k+1)\tilde{\Theta}(k+1) - \tilde{\Theta}^{\mathrm{T}}(k)\tilde{\Theta}(k). \tag{5.9}$$

Then

$$\triangle V_{\tilde{\Theta}}(k) \quad \le \quad -\frac{\left[\varphi(k)\tilde{\theta}(k)\right]^2}{\varphi^{\mathrm{T}}(k)\varphi(k)}, \quad k \ge 0. \tag{5.10}$$

Proof. From (5.1) and (5.9) it follows that

$$\triangle V_{\tilde{\Theta}}(k) = \sum_{i=k+1}^{k+d}\tilde{\theta}^{\mathrm{T}}(i)\tilde{\theta}(i) - \sum_{i=k}^{k+d-1}\tilde{\theta}^{\mathrm{T}}(i)\tilde{\theta}(i). \tag{5.11}$$

Use of Lemma 5.1 and Lemma 5.1 yields

$$\triangle V_{\tilde{\Theta}}(k) = -\sum_{i=k}^{k+d-1}\frac{\left[\varphi^{\mathrm{T}}(i-d+1)\tilde{\theta}(i)\right]^2}{\varphi^{\mathrm{T}}(i-d+1)\varphi(i-d+1)} \le -\frac{\left[\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]^2}{\varphi^{\mathrm{T}}(k)\varphi(k)}. \qquad \square$$

Lemma 5.3. Let $P, R \in \mathbb{R}^{n\times n} > 0$ be positive-definite matrices that satisfy

$$P = A^{\mathrm{T}}PA + R + I, \tag{5.12}$$

and let

$$\sigma \triangleq \sqrt{\lambda_{\max}(A^{\mathrm{T}}PA)}. \tag{5.13}$$

Furthermore let $\mu > 0$ and define

$$V_{x_{\mathrm{e}}}(x_{\mathrm{e}}) \triangleq \ln(1+\mu x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}}) \tag{5.14}$$

and

$$\triangle V_{x_{\mathrm{e}}}(k) \triangleq V_{x_{\mathrm{e}}}(x_{\mathrm{e}}(k+1)) - V_{x_{\mathrm{e}}}(x_{\mathrm{e}}(k)).$$

Then for $k \ge 0$

$$\triangle V_{x_{\mathrm{e}}}(k) \le \mu\frac{-x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_e(k) + b_0^{-2}(\sigma^2+1)B^{\mathrm{T}}PB\left[\varphi(k)\tilde{\theta}(k)\right]^2}{1+\mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}.$$

Proof. Define

$$F \triangleq \frac{1}{\sigma}P^{1/2}A, \quad G \triangleq \sigma P^{1/2}B, \quad \mathcal{J}(x_{\mathrm{e}}) \triangleq x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}}.$$

Then,

$$\triangle\mathcal{J}_{x_{\mathrm{e}}}(k) \triangleq x_{\mathrm{e}}^{\mathrm{T}}(k+1)Px_{\mathrm{e}}(k+1) - x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k).$$

Omitting the explicit dependence on k we have

$$\begin{aligned}\triangle\mathcal{J}_{x_{\mathrm{e}}}(k) &= x_{\mathrm{e}}^{\mathrm{T}}A^{\mathrm{T}}PAx_{\mathrm{e}} - x_{\mathrm{e}}^{\mathrm{T}}A^{\mathrm{T}}PBb_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta} \\ &\quad - b_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta}B^{\mathrm{T}}PAx_{\mathrm{e}} \\ &\quad + b_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta}B^{\mathrm{T}}PBb_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta} - x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}}\end{aligned}$$

Adding and subtracting $b_0^{-2}(\varphi^{\mathrm{T}}\tilde{\theta})^2G^{\mathrm{T}}G$ yields

$$\begin{aligned}\triangle\mathcal{J}_{x_{\mathrm{e}}}(k) &= x_{\mathrm{e}}^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)x_{\mathrm{e}} \\ &\quad + \left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)b_0^{-2}(\varphi^{\mathrm{T}}\tilde{\theta})^2 \\ &\quad - \left[\begin{array}{cc} x_{\mathrm{e}}^{\mathrm{T}} & b_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta}\end{array}\right]\begin{bmatrix}F^{\mathrm{T}}F & F^{\mathrm{T}}G \\ G^{\mathrm{T}}F & G^{\mathrm{T}}G\end{bmatrix}\begin{bmatrix}x_{\mathrm{e}}^{\mathrm{T}} \\ b_0^{-1}\varphi^{\mathrm{T}}\tilde{\theta}\end{bmatrix} \\ &\le x_{\mathrm{e}}^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)x_{\mathrm{e}} \\ &\quad + \left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)b_0^{-2}(\varphi^{\mathrm{T}}\tilde{\theta})^2.\end{aligned} \tag{5.15}$$

Noting that $G^{\mathrm{T}}G = \sigma^2B^{\mathrm{T}}PB$ and

$$F^{\mathrm{T}}F \le \frac{\lambda_{\max}\left(A^{\mathrm{T}}PA\right)I_n}{\lambda_{\max}(A^{\mathrm{T}}PA)} = I_n,$$

it follows from (5.12) and (5.15) that

$$\triangle\mathcal{J}_{x_{\mathrm{e}}}(k) \le - x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_e(k) + (\sigma^2+1)B^{\mathrm{T}}PBb_0^{-2}\left[\varphi(k)\tilde{\theta}(k)\right]^2.$$

Now, since $\ln\lambda \le \lambda - 1$ for all $\lambda > 0$,

$$\triangle V_{x_{\mathrm{e}}}(k) \le \mu\frac{-x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_e(k) + b_0^{-2}(\sigma^2+1)B^{\mathrm{T}}PB\left[\varphi(k)\tilde{\theta}(k)\right]^2}{1+\mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}.$$

$\square$

We now present the main stability result.

Theorem 5.1. Assume that the reference signal $u_{\mathrm{c}}(k)$ is bounded . Then the origin of the error system (5.3)-(5.4) is Lyapunov stable, and $y(k) - y_{\mathrm{m}}(k) \to 0$ as $k \to \infty$.

Proof. Consider the Lyapunov function candidate

$$V(X) \triangleq aV_{x_{\mathrm{e}}}(x_{\mathrm{e}}) + V_{\tilde{\Theta}}(\tilde{\Theta}) \tag{5.16}$$

Let $P, R \in \mathbb{R}^{n\times n} > 0$ be positive definite and satisfy (5.12), and let $a > 0$. Then using lemmas 5.3 and 5.2 it follows that

$$\begin{aligned}&\triangle V(k) \stackrel{\triangle}{=} V(X(k+1)) - V(X(k))\\&\leq a\mu \frac{-x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{e}(k) + (\sigma^2+1)B^{\mathrm{T}}PB\left[\varphi(k)\tilde{\theta}(k)\right]^2/b_0^2}{1+\mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}\\&\quad - \frac{\left[\varphi^{\mathrm{T}}(k)\tilde{\theta}(k)\right]^2}{\varphi^{\mathrm{T}}(k)\varphi(k)}.\end{aligned}$$

Now from (3.22) it follows that

$$\begin{aligned}\varphi^{\mathrm{T}}(k)\varphi(k) &= x^{\mathrm{T}}M_0^{\mathrm{T}}M_0x\\&\leq x_{\mathrm{e}}^{\mathrm{T}}M_0^{\mathrm{T}}M_0x_{\mathrm{e}} + x_{\mathrm{m}}^{\mathrm{T}}M_0^{\mathrm{T}}M_0x_{\mathrm{m}}. \qquad (5.17)\end{aligned}$$

Let $\mu_1 > 0$ satisfy

$$\mu_1 P > M_0^{\mathrm{T}}M_0. \qquad (5.18)$$

By assumption, the command signal $u_{\mathrm{c}}(k)$ is bounded and the $\mathbf{A_m}$ is stable. It thus follows that there exists $\beta > 0$ such that

$$x_m^{\mathrm{T}}(k)x_{\mathrm{m}}(k) \leq \beta. \qquad (5.19)$$

Using (5.17)-(5.19) we have

$$\varphi^{\mathrm{T}}(k)\varphi(k) \leq \mu_1 x_{\mathrm{e}}^{\mathrm{T}}Px_{\mathrm{e}} + \beta\mu_1\lambda_{\max}(P). \qquad (5.20)$$

Defining $\mu \stackrel{\triangle}{=} \mu_1/(1+\beta\mu_1\lambda_{\max}(P))$ and $a \stackrel{\triangle}{=} b_0^2/(1+\mu_1\left(\sigma^2+1\right)B^{\mathrm{T}}PB)$ we have

$$\triangle V \leq -\mu a\frac{x_{\mathrm{e}}^{\mathrm{T}}(k)Rx_{\mathrm{e}}(k)}{1+\mu x_{\mathrm{e}}^{\mathrm{T}}(k)Px_{\mathrm{e}}(k)}. \qquad (5.21)$$

Since $V(X)$ is positive definite and radially unbounded it follows from (5.21) that the origin of the error system (5.3)-(5.4) is Lyapunov stable. Furthermore, using Theorem A1 of [12] it follows that $x_{\mathrm{e}}(k) \to 0$ as $k \to \infty$. Then using (3.9) and (3.12) we have that $y(k) - y_{\mathrm{m}}(k) \to 0$ as $k \to \infty$. □

6. **Example**

Example 6.1. Consider the unstable minimum phase SISO plant with relative degree $d = 2$ given by

$$\frac{y(\mathbf{q})}{u(\mathbf{q})} = \frac{\mathbf{q}+0.5}{\mathbf{q}^3+\mathbf{q}^2+\mathbf{q}+1.5} \qquad (6.1)$$

To track reference signals we choose an FIR filter as the reference model. We require $\deg \mathbf{A_m} = 2n - m - 1 = 4$ and $\deg \mathbf{B_m} = \deg \mathbf{A_m} - d = 2$. Let the reference model be

$$\frac{y_{\mathrm{m}}(\mathbf{q})}{u_{\mathrm{c}}(\mathbf{q})} = \frac{\mathbf{q}^2}{\mathbf{q}^4} \qquad (6.2)$$

and let $u_{\mathrm{c}}(k)$ be a square wave with period of 100 samples. The plant (6.1) with the control law (4.3) and the parameter update (4.4) is simulated in MATLAB . The simulation results are shown in Figure 3.

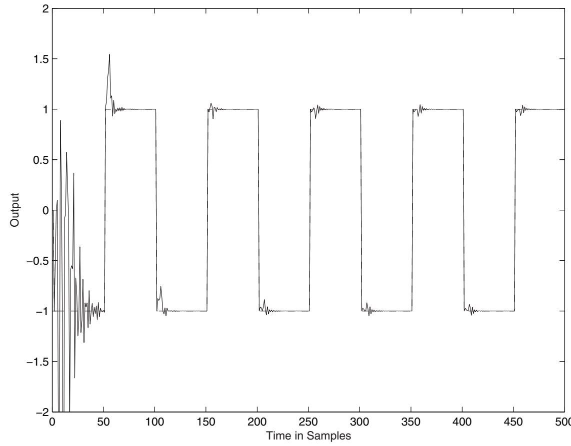

Fig. 3. Tracking Performance

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