Time-Domain Identification Using ARMARKOV/Toeplitz Models

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Nomenclature

$\circ$	Hadamard product
$\|\cdot\|_2$, $\|\cdot\|_{\rm F}$	spectral norm, Frobenius norm
$0_{l\times m}$, 0_l	$l\times m$ zero matrix, $0_{l\times l}$
I_l, $1_{l\times m}$	$l\times l$ identity matrix, $l\times m$ ones matrix

1. Introduction

Frequency-domain identification methods, which are applied off line, require fast Fourier transforms of measured time histories to construct an estimate of the frequency response function (FRF) of the system. Recently, frequency-domain methods have been studied in the framework of $\mathcal{H}_\infty$ control theory to obtain models with error bounds that are suitable for robust controller design [2, 3, 4, 5].

A widely used identification technique is the eigensystem realization algorithm (ERA) [9, pp. 133-137] which uses Markov parameters to obtain a state-space realization of the system. Estimates of the Markov parameters, obtained from an inverse of the FRF, are used to construct a block-Hankel matrix whose singular value decomposition provides an estimate of the system order and which is used to construct state space realizations.

While frequency-domain methods and ERA are off-line identification methods, recursive on-line methods have been developed using time-domain data. Recursive time-domain identification techniques using a recursive least-squares algorithm with an ARMA representation [10, pp. 305-310], [12, pp. 320-324] have been used to identify the transfer function coefficients of SISO systems. More recently, recursive subspace methods [11, 13] have been used to construct minimal realizations of MIMO systems from time-domain data. Recursive time-domain identification techniques have also been derived in the context of neural networks and learning processes [7].

In this paper we derive an alternative recursive time-domain identification technique that is based upon recursive identification of the Markov parameters of a system. Our approach is based upon the ideas proposed in [7] and further developed in [1, 8]. The present paper introduces the *recursive ARMARKOV/Toeplitz identification algorithm*, which estimates the Markov parameters recursively from time-domain input-output data. This identification technique is based upon linear time-invariant finite-dimensional systems having an ARMARKOV representation that relates the current output of a system to past outputs as well as current and past inputs. The ARMARKOV representation is an overparameterized and nonminimal representation. Appropriate "stacking" of time-delayed ARMARKOV representations yields a block-Toeplitz weight matrix which contains Markov parameters and which maps a vector of past outputs and inputs to a vector of current and past outputs. A recursive update law based upon a gradient that preserves the block-zero structure of the block-Toeplitz weight matrix and which, in the presence of a persistent input sequence, guarantees that the estimated weight matrix converges to the actual weight matrix. This algorithm provides the first step in the two-step identification algorithm we refer to as the *recursive ARMARKOV/Toeplitz/ERA identification algorithm* where ERA is used as the second step to construct minimal realizations from the estimated Markov parameters extracted from the converged weight matrix.

An advantage of using an ARMARKOV representation to estimate the Markov parameters instead of using an ARMA representation to estimate the transfer function coefficients is the ability to avoid the sensitivity of the poles and zeros with respect to the estimated coefficients. Another advantage is that the singular value decomposition of a block Hankel matrix constructed from the estimated Markov parameters provides an effective model order indicator as in ERA. The recursive ARMARKOV/Toeplitz/ERA identification algorithm can be used either on line or in an off-line batch mode.

2. ARMARKOV Representations

Consider the discrete-time finite-dimensional linear time-invariant system

$$x(k+1) = Ax(k)+Bu(k), \tag{2.1}$$

$$y(k) = Cx(k)+Du(k), \tag{2.2}$$

where $A\in\mathcal{R}^{n\times n}$, $B\in\mathcal{R}^{n\times m}$, $C\in\mathcal{R}^{l\times n}$, and $D\in\mathcal{R}^{l\times m}$. The *Markov parameters* H_j are defined by

$$H_j \triangleq D, \qquad j=-1, \tag{2.3}$$

$$\triangleq CA^jB, \qquad j\geq 0,$$

and satisfy

$$G(z)\triangleq C(zI-A)^{-1}B+D=\sum_{j=-1}^{\infty}H_jz^{-(j+1)}. \tag{2.4}$$

We refer to $G(z)=\sum_{j=-1}^{\infty}H_jz^{-(j+1)}$ as the *Markov parameter representation of* $G(z)$.

The ARMA transfer function representation of $G(z)$ is given by

*This research was supported in part by the Air Force Office of Scientific Research under grants F49620-95-1-0019 and F49620-94-1-0409.

$$G(z) = \frac{(B_0 z^n + B_1 z^{n-1} + \cdots + B_n)}{z^n + a_1 z^{n-1} + \cdots + a_n}, \tag{2.5}$$

where $\det(zI - A) = z^n + a_1 z^{n-1} + \cdots + a_n$ and $B_i \in \mathcal{R}^{l\times m}$, $i = 0, \ldots, n$. Although (2.5) provides a rational representation of $G(z)$, it contains only the first Markov parameter $B_0 = H_{-1}$. Our goal is to blend (2.4) and (2.5) to obtain rational representations of $G(z)$ that explicitly involve additional Markov parameters. The ARMA time-domain representation of $G(z)$ corresponding to (2.5) is given by

$$\begin{aligned} y(k) &= -a_1 y(k-1) - \cdots - a_n y(k-n) + B_0 u(k) \\ &\quad + \cdots + B_n u(k-n). \end{aligned} \tag{2.6}$$

Next we express $y(k)$ in terms of past inputs, past outputs, and additional Markov parameters. Replacing k by $k-1$ in (2.6) and substituting the resulting relation back into (2.6) yields

$$\begin{aligned} y(k) &= (a_1^2 - a_2) y(k-2) + (a_1 a_2 - a_3) y(k-3) \\ &\quad + \cdots + (a_1 a_{n-1} - a_n) y(k-n) \\ &\quad + a_1 a_n y(k-n-1) + B_0 u(k) \\ &\quad + (B_1 - a_1 B_0) u(k-1) \\ &\quad + (B_2 - a_1 B_1) u(k-2) \\ &\quad + \cdots + (B_n - a_1 B_{n-1}) u(k-n) \\ &\quad - a_1 B_n u(k-n-1). \end{aligned} \tag{2.7}$$

Equating (2.4) and (2.5) and multiplying both sides by $z^n + a_1 z^{n-1} + \cdots + a_n$ yields

$$\begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} H_{-1} & 0 & \cdots & 0 \\ H_0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ H_{n-1} & \cdots & H_0 & H_{-1} \end{bmatrix} \begin{bmatrix} I_m \\ a_1 I_m \\ \vdots \\ a_n I_m \end{bmatrix} \tag{2.8}$$

and

$$H_{n+j} = -\sum_{i=1}^{n} a_i H_{n+j-i}, \quad j \geq 0. \tag{2.9}$$

Noting from (2.8) that $H_{-1} = B_0$ and $H_0 = B_1 - a_1 B_0$, then (2.7) yields

$$\begin{aligned} y(k) &= \alpha_{2,1} y(k-2) + \cdots + \alpha_{2,n} y(k-n-1) \\ &\quad + H_{-1} u(k) + H_0 u(k-1) + \mathcal{B}_1 u(k-2) \\ &\quad + \cdots + \mathcal{B}_n u(k-n-1), \end{aligned} \tag{2.10}$$

where $\alpha_{2,1}, \ldots, \alpha_{2,n} \in \mathcal{R}$ and $\mathcal{B}_{2,1}, \ldots, \mathcal{B}_{2,n} \in \mathcal{R}^{l\times m}$ are given by

$$\begin{aligned} \alpha_{2,i} &\triangleq a_1 a_i - a_{i+1}, \quad i = 1, \ldots, n-1, \\ \alpha_{2,n} &\triangleq a_1 a_n, \\ \mathcal{B}_{2,i} &\triangleq B_{i+1} - a_1 B_i, \quad i = 1, \ldots, n-1, \\ \mathcal{B}_{2,n} &\triangleq a_1 B_n. \end{aligned}$$

Since (2.10) explicitly involves the first two Markov parameters H_{-1} and H_0 of $G(z)$, it is called an *ARMARKOV time-domain representation* of $G(z)$. Repeating this procedure $\mu - 1$ times yields the ARMARKOV time-domain representation of $G(z)$

$$\begin{aligned} y(k) &= \sum_{j=1}^{n} -\alpha_{\mu,j} y(k-\mu-j+1) \\ &\quad + \sum_{j=1}^{\mu} H_{j-2} u(k-j+1) \\ &\quad + \sum_{j=1}^{n} \mathcal{B}_{\mu,j} u(k-\mu-j+1). \end{aligned} \tag{2.11}$$

where $\alpha_{\mu,1}, \ldots, \alpha_{\mu,n} \in \mathcal{R}$ and $\mathcal{B}_{\mu,1}, \ldots, \mathcal{B}_{\mu,n} \in \mathcal{R}^{l\times m}$. This substitution procedure can be used to obtain recursive expressions for $\alpha_{\mu,1}, \ldots, \alpha_{\mu,n}$ and $\mathcal{B}_{\mu,1}, \ldots, \mathcal{B}_{\mu,n}$. However, the explicit form of these relationships will not be needed. Note that (2.11) involves the first μ Markov parameters $H_{-1}, \ldots, H_{\mu-2}$. Furthermore, note that the ARMA time-domain representation (2.6) is a specialized ARMARKOV time-domain representation (2.11) with $\mu = 1$.

Defining the *ARMARKOV regressor vector* $\Phi_\mu(k) \in \mathcal{R}^{(p+n-1)(l+m)+\mu m}$ by

$$\Phi_\mu(k) \triangleq \begin{bmatrix} y(k-\mu) \\ \vdots \\ y(k-\mu-p-n+2) \\ u(k) \\ \vdots \\ u(k-\mu-p-n+2) \end{bmatrix}, \tag{2.12}$$

it follows that

$$Y(k) = W_\mu \Phi_\mu(k), \tag{2.13}$$

where the *ARMARKOV/Toeplitz weight matrix* W_μ is the block-Toeplitz matrix defined by

$$W_\mu \triangleq \left[\begin{array}{cccccccccccc} -\mathcal{A}_\mu & 0_l & \cdots & 0_l & H_{-1} & \cdots & H_{\mu-2} & \mathcal{B}_\mu & 0_{l\times m} & \cdots & 0_{l\times m} \\ 0_l & \ddots & \ddots & \vdots & 0_{l\times m} & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_l & \vdots & \ddots & \ddots & & \ddots & \ddots & 0_{l\times m} \\ 0_l & \cdots & 0_l & -\mathcal{A}_\mu & 0_{l\times m} & \cdots & 0_{l\times m} & H_{-1} & \cdots & H_{\mu-2} & \mathcal{B}_\mu \end{array}\right] \tag{2.14}$$

where $\mathcal{A}_\mu \triangleq [\, \alpha_{\mu,1} I_l \ \ \cdots \ \ \alpha_{\mu,n} I_l \,] \in \mathcal{R}^{l\times nl}$ and $\mathcal{B}_\mu \triangleq [\, \mathcal{B}_{\mu,1} \ \ \cdots \ \ \mathcal{B}_{\mu,n} \,] \in \mathcal{R}^{l\times nm}$. Note that p determines the window of input-output data that appears in (2.12).

It follows from the ARMARKOV time-domain representation (2.11) that an *ARMARKOV transfer function representation* of $G(z)$ with μ Markov parameters is given by

$$G(z) = \frac{(H_{-1} z^{\mu+n-1} + \cdots + \mathcal{B}_{\mu,n})}{z^{\mu+n-1} + \alpha_{\mu,1} z^{n-1} + \cdots + \alpha_{\mu,n}}. \tag{2.15}$$

This representation of $G(z)$ can be viewed as a blending of the Markov parameter representation (2.4) and the ARMA transfer function representation (2.5), which correspond to $\mu = \infty$ and $\mu = 1$, respectively. The ARMA transfer function representation and the ARMARKOV transfer function representation are different representations of $G(z)$. However, the ARMARKOV transfer function representation, which is nonminimal when $\mu > 1$, provides greater flexibility in identifying $G(z)$ by allowing direct estimation of the Markov parameters. Note, however, that the ARMARKOV transfer function representation is not equivalent to an arbitrary nonminimal ARMA representation since the coefficients of $z^{\mu+n-2},\ldots,z^n$ in the denominator are constrained to be zero.

Henceforth, for convenience we omit the subscript μ and write W and $\Phi(k)$ for W_μ and $\Phi_\mu(k)$, respectively.

3. Recursive ARMARKOV/Toeplitz Algorithm

Let $\widehat{W}(k)$ denote an estimate of the ARMARKOV/Toeplitz weight matrix W at time k, where $\widehat{W}(k)$ has the same block-zero structure as W. Let $\widehat{Y}(k)$ denote the *estimated output vector* defined by $\widehat{Y}(k) \triangleq \widehat{W}(k)\Phi(k) \in \mathcal{R}^{pl}$. Furthermore, define the *output error* $\varepsilon(k) \in \mathcal{R}^{pl}$ by $\varepsilon(k) \triangleq Y(k) - \widehat{Y}(k)$, and the *output error cost function* $J(k)$ by $J(k) \triangleq \frac{1}{2}\varepsilon^{\mathrm{T}}(k)\varepsilon(k)$.

Lemma 3.1 The gradient of $J(k)$ with respect to the estimated weight matrix $\widehat{W}(k)$ is given by

$$\frac{\partial J(k)}{\partial \widehat{W}(k)} = -U \circ \left[\varepsilon(k)\Phi^{\mathrm{T}}(k)\right], \tag{3.1}$$

where $U \in \mathcal{R}^{pl\times[(p+n-1)(l+m)+\mu m]}$ is defined by

$$U \triangleq \left[\begin{array}{ccccc} \mathcal{I}_{l\times n} & 0_l & \cdots & 0_l & 1_{l\times(n+\mu)m} \\ 0_l & \ddots & \ddots & \vdots & 0_{l\times m} \\ \vdots & \ddots & \ddots & 0_l & \vdots \\ 0_l & \cdots & 0_l & \mathcal{I}_{l\times n} & 0_{l\times m} \end{array}\right.$$
$$\left.\begin{array}{ccc} 0_{l\times m} & \cdots & 0_{l\times m} \\ \ddots & \ddots & \vdots \\ \ddots & \ddots & 0_{l\times m} \\ \cdots & 0_{l\times m} & 1_{l\times(n+\mu)m} \end{array}\right]. \tag{3.2}$$

where $\mathcal{I}_{l\times n} = [\, I_l \quad \cdots \quad I_l \,] \in \mathcal{R}^{l\times nl}$.

We now consider the *estimated weight matrix update law*

$$\widehat{W}(k+1) = \widehat{W}(k) - \eta(k)\frac{\partial J(k)}{\partial \widehat{W}(k)}, \tag{3.3}$$

where $\eta(k) \geq 0$ is the *adaptive step size*. Furthermore, define the *estimated weight matrix error* by $E(k) \triangleq W - \widehat{W}(k)$, and the *estimated weight matrix error cost function* $\mathcal{J}(k,\eta(k)) \triangleq \|E(k+1)\|_{\mathrm{F}}^2 - \|E(k)\|_{\mathrm{F}}^2$. Then it follows from the estimated weight matrix update law (3.3) that $E(k+1) = E(k) + \eta(k)\frac{\partial J(k)}{\partial \widehat{W}(k)}$, and $\varepsilon(k) = E(k)\Phi(k)$.

To implement the estimated weight matrix update law (3.3) starting at time $k=0$ requires that $\Phi(k)$ be available for all $k \geq 0$. Since $\Phi(0)$ has the form

$$\Phi(0) = \begin{bmatrix} y(-\mu) \\ \vdots \\ y(-\mu-p-n+2) \\ u(0) \\ \vdots \\ u(-\mu-p-n+2) \end{bmatrix}, \tag{3.4}$$

it is assumed that measurements of $u(k)$ and $y(k)$ are available for all $k \geq -\mu-p-n+2$.

Let the *optimal adaptive step size* $\eta_{\mathrm{opt}}(k)$ be defined by

$$\eta_{\mathrm{opt}}(k) \triangleq \frac{\|\varepsilon(k)\|_2^2}{\left\|\frac{\partial J(k)}{\partial \widehat{W}(k)}\right\|_{\mathrm{F}}^2}. \tag{3.5}$$

The following result shows that $\eta_{\mathrm{opt}}(k)$ minimizes $\mathcal{J}(k,\eta(k))$.

Theorem 3.1 Let $\widehat{W}(0)$ have the same block-zero structure as W and consider the estimated weight matrix update law (3.3). Assume that $\frac{\partial J(k)}{\partial \widehat{W}(k)} \neq 0$, $k \geq 0$, and assume that the adaptive step size $\eta(k)$ satisfies

$$0 < \eta(k) < 2\eta_{\mathrm{opt}}(k)\ , \quad k \geq 0. \tag{3.6}$$

Then $\{\|E(k)\|_{\mathrm{F}}\}_{k=0}^{\infty}$ is decreasing, and thus $\mathcal{J}(k,\eta(k)) < 0$, $k \geq 0$. Furthermore, for all $k \geq 0$, $\eta(k) = \eta_{\mathrm{opt}}(k)$ minimizes $\mathcal{J}(k,\eta(k))$, and $\mathcal{J}(k,\eta_{\mathrm{opt}}(k)) = -\|\varepsilon(k)\|_2^2\eta_{\mathrm{opt}}(k)$. If, in addition,

$$\sup_{k\geq 0}\left|\frac{\eta(k)}{\eta_{\mathrm{opt}}(k)} - 1\right| < 1 \tag{3.7}$$

and

$$\sup_{k\geq 0}\|\Phi(k)\|_2 < \infty, \tag{3.8}$$

then

$$\sum_{k=0}^{\infty}\|\varepsilon(k)\|_2^2 < \infty. \tag{3.9}$$

Consequently,

$$\lim_{k\to\infty}\varepsilon(k) = 0, \tag{3.10}$$

and

$$\lim_{k\to\infty}\frac{\partial J(k)}{\partial \widehat{W}(k)} = 0. \tag{3.11}$$

Remark 3.1 Note that (3.7) implies (3.6), whereas the converse is not true. Note also that (3.8) is equivalent to $\sup_{k\geq-\mu-p-n+2}\|u(k)\|_2 < \infty$ and $\sup_{k\geq-\mu-p-n+2}\|y(k)\|_2 < \infty$. If we choose $\eta(k) = \eta_{\mathrm{opt}}(k)$, $\alpha \in (0,2)$, then Theorem 3.1 holds.

Next, assuming $\Phi(k) \neq 0$, define the *computationally efficient step size* $\eta_{\mathrm{eff}}(k)$ by

$$\eta_{\mathrm{eff}}(k) \triangleq \frac{1}{\|\Phi(k)\|_2^2}\ . \tag{3.12}$$

Note that if $\frac{\partial J(k)}{\partial \widehat{W}(k)} \neq 0$, then $\eta_{\mathrm{eff}}(k) \leq \eta_{\mathrm{opt}}(k)$. In the following corollary of Theorem 3.1, $\eta_{\mathrm{opt}}(k)$ is replaced by $\eta_{\mathrm{eff}}(k)$. In this case (3.15) is needed along with (3.14) to verify that (3.7) is satisfied.

Corollary 3.1 Let $\widehat{W}(0)$ have the same block-zero structure as W and consider the estimated weight matrix update law (3.3). Assume that $\frac{\partial J(k)}{\partial \widehat{W}(k)} \neq 0$, $k \geq 0$, and assume that the adaptive step size $\eta(k)$ satisfies

$$0 < \eta(k) < 2\eta_{\mathrm{eff}}(k)\ , \quad k \geq 0. \tag{3.13}$$

Then (3.6) holds and $\{\|E(k)\|_{\mathrm{F}}\}_{k=0}^{\infty}$ is decreasing. If, in addition,

$$\sup_{k\geq 0}\left|\frac{\eta(k)}{\eta_{\mathrm{eff}}(k)} - 1\right| < 1, \tag{3.14}$$

$$\inf_{k\geq 0}\left\|\begin{bmatrix} u(k) \\ \vdots \\ u(k-\mu-p-n+2)\end{bmatrix}\right\|_2 > 0\ ,\quad k\geq 0, \qquad (3.15)$$

and (3.8) is satisfied, then (3.7), (3.9), (3.10), and (3.11) hold.

Remark 3.2 If we choose $\eta(k) = \alpha\eta_{\rm eff}(k)$, $\alpha \in (0,2)$, then Corollary 3.1 holds.

4. Convergence of the Weight Matrix

Theorem 3.1 provides sufficient conditions that guarantee that the output error vector $\varepsilon(k)$ converges to zero. However, condition (3.8), which implies that the input and output sequences are bounded (see Remark 3.1), is not strong enough to guarantee that $\widehat{W}(k)$ converges to W. To guarantee that $\widehat{W}(k)$ converges to W we require a persistent excitation condition.

For convenience, define

$$\Theta(k) \triangleq \begin{bmatrix} y(k-\mu) \\ \vdots \\ y(k-\mu-n+1) \\ u(k) \\ \vdots \\ u(k-\mu-n+1)\end{bmatrix} \in \mathcal{R}^q, \qquad (4.1)$$

where $q \triangleq n(l+m)+\mu m$. Note that if $p=1$ then $\Theta(k) = \Phi(k)$. Furthermore, comparing (3.4) and (4.1) it can be seen that, for all $p\geq 1$, if data for $\Phi(0)$ is available, then data for $\Theta(0)$ is available. For $F\in\mathcal{R}^{q\times q}$ let $\sigma_1 \geq \cdots \geq \sigma_q \geq 0$ denote the singular values of F.

Definition 4.1 The input sequence $\{u(k)\}_{k=0}^{\infty}$ is *persistent with respect to* $G(z)$ if there exists $\delta > 0$ such that, for all $k>0$, there exist $k \leq r_1(k) < \cdots < r_q(k)$ such that $\sigma_q\left[\Theta(r_1(k)) \ \cdots \ \Theta(r_q(k))\right] > \delta$.

The following result provides conditions that guarantee convergence of the estimated weight matrix $\widehat{W}(k)$ to the weight matrix W. In particular, this result shows that if (3.8) is satisfied then the estimated weight matrix update law (3.3) with the adaptive step size $\eta(k)$ satisfying (3.7) and a persistent input sequence $\{u(k)\}_{k=0}^{\infty}$ guarantees that $\widehat{W}(k)$ converges to W. Alternatively, if (3.15) is satisfied and the adaptive step size $\eta(k)$ satisfies (3.14) then convergence of $\widehat{W}(k)$ to W is guaranteed.

Theorem 4.1 Let $\widehat{W}(0)$ have the same block-zero structure as W and consider the estimated weight matrix update law (3.3). Assume that (3.8) is satisfied, $\{u(k)\}_{k=-\mu-p-n+2}^{\infty}$ is persistent with respect to $G(z)$, and $\frac{\partial J(k)}{\partial \widehat{W}(k)} \neq 0$, $k\geq 0$. If $\eta(k)$ satisfies (3.7) then

$$\lim_{k\to\infty}\widehat{W}(k) = W. \qquad (4.2)$$

Alternatively, if (3.15) is satisfied and $\eta(k)$ satisfies (3.14), then (4.2) holds.

In the following corollary of Theorem 4.1 we choose $\eta(k)$ so that $\frac{\eta(k)}{\eta_{\rm opt}(k)}$ is a constant or alternatively $\frac{\eta(k)}{\eta_{\rm eff}(k)}$ is a constant.

Corollary 4.1 Let $\widehat{W}(0)$ have the same block-zero structure as W and consider the estimated weight matrix update law (3.3). Assume that (3.8) is satisfied, $\{u(k)\}_{k=-\mu-p-n+2}^{\infty}$ is persistent with respect to $G(z)$, and $\frac{\partial J(k)}{\partial \widehat{W}(k)} \neq 0$, $k\geq 0$. If $\eta(k) = \alpha\eta_{\rm opt}(k)$, $k\geq 0$, then (4.2) holds. Furthermore, if (3.15) is satisfied and $\eta(k) = \alpha\eta_{\rm eff}(k)$, $k\geq 0$, then (4.2) holds.

5. Numerical Example

In the following numerical example the first six Markov parameters of a second-order asymptotically stable SISO system are estimated using the recursive ARMARKOV/Toeplitz identification algorithm. The estimated Markov parameters are obtained after each update of the estimated ARMARKOV/Toeplitz weight matrix $\widehat{W}(k)$ by averaging over the corresponding entries of $\widehat{W}(k)$. Consider the continuous-time single-degree-of-freedom oscillator

$$\widehat{G}(s) = \frac{\omega_{\rm n}^2}{s^2+2\zeta\omega_{\rm n}s+\omega_{\rm n}^2}\ , \qquad (5.1)$$

with a natural frequency $f_{\rm n} = 10$ Hz ($\omega_{\rm n} = 6.28$ rad/sec)

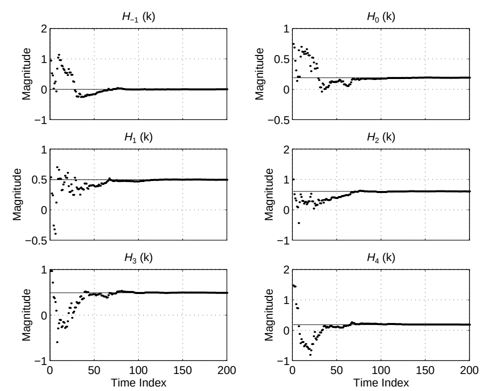

Figure 1: Markov parameter estimates obtained from the recursive ARMARKOV/Toeplitz identification algorithm with $\mu = 6$, $n = 2$, and $p = 4$.

and a damping ratio $\zeta = 1\%$. A balanced realization of

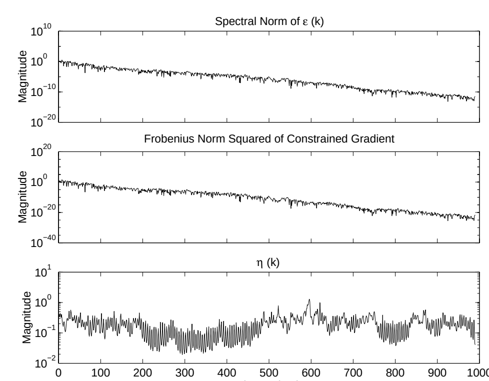

Figure 2: Error, gradient, and step size of the ARMARKOV/Toeplitz identification algorithm with $\mu = 6$, $n = 2$, and $p = 4$.

a zero-order-hold discretization of (5.1) at a sampling frequency of 100 Hz is given by

$$x(k+1) = \begin{bmatrix} 0.80595 & 0.58408 \\ -0.58408 & 0.80199 \end{bmatrix} x(k) + \begin{bmatrix} 0.63417 \\ 0.46041 \end{bmatrix} u(k), \quad (5.2)$$

$$y(k) = [\,0.63417 \quad -0.46041\,]\, x(k). \quad (5.3)$$

The input $u(k)$ is chosen to be zero-mean white noise uniformly distributed on $[-1,1]$. The input was applied to (5.2), (5.3) for 10 seconds ($-10 \le k \le 990$) with zero initial conditions so that data for $\Phi(0)$ is available. Although not shown, $u(k)$ is persistent with respect to $G(z)$ for the available data.

The six estimated Markov parameters obtained from the recursive ARMARKOV/Toeplitz identification algorithm with $\mu = 6$, $n = 2$, $p = 4$, and $\eta(k) = \eta_{\rm opt}(k)$ for 10 seconds of data are shown in Figure 1. The values of $\|\varepsilon(k)\|_2$ and $\left\|\frac{\partial J(k)}{\partial \widehat{W}(k)}\right\|_{\rm F}^2$, shown in Figure 2, can be seen to be decreasing nearly exponentially. The six estimated Markov parameters, which were obtained after each update by averaging over the corresponding entries of $\widehat{W}(k)$, converge to within 1% of their respective true values after 1 second or 100 time steps (see Figure 1).

6. Identification of an Acoustic Duct

In this section the recursive ARMARKOV/Toeplitz/ERA identification algorithm was used to obtain a realization of the dynamics of an acoustic duct. The acoustic duct is constructed from a 19.75

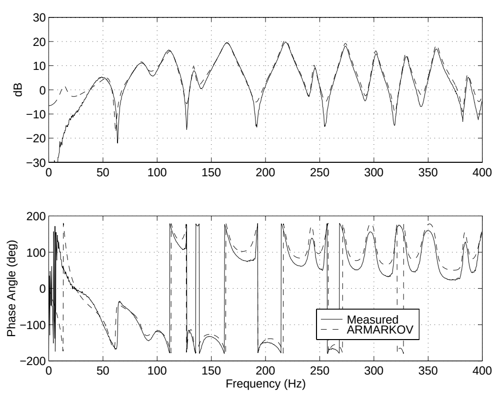

Figure 3: Time-domain identification of the dynamics of an acoustic duct using the recursive ARMARKOV/Toeplitz/ERA identification algorithm yielding a 31st-order realization.

foot long 4 inch diameter PVC pipe with open-closed boundary conditions and a colocated microphone and speaker mounted on the side. The speaker input and microphone output were recorded at a sampling frequency of 1024 Hz with a time-record length of 4096 data points spanning 4.0 seconds. The input $u(k)$ was chosen to be white noise. The experimentally measured frequency response was obtained using a spectrum analyzer with the frequency range chosen to be 0 - 400 Hz with 1601 spectral lines of resolution. Hence the measured FRF is an estimate of the frequency response of the duct at the discrete frequencies $0, 0.25, 0.5, \ldots, 399.5, 399.75, 400$ Hz.

Based upon an analytical model of the acoustic duct [6] the system order was estimated to be approximately 40, hence μ was chosen to be 210. Three ARMARKOV/Toeplitz representations were evaluated, all having $\mu = 210$ and $n = 40$, with $p = 4$, $p = 20$, and $p = 50$, respectively. For each ARMARKOV representation, estimates of the first 210 Markov parameters were obtained. The 210 estimated Markov parameters were used within ERA to obtain a minimal realization. The frequency response of these minimal realizations was compared to the experimentally measured frequency response. Based upon this comparison, it can be seen that the accuracy of the minimal realizations increases as p increases. The ARMARKOV representation with $p = 50$ yields a 31st-order realization whose frequency response is shown in Figure 3.

References

[1] C. Ahn, "A Recursive Algorithm for Identification of IIR Systems," *Proc. IEEE Conf. Dec. Contr.*, pp. 2595-2600, San Antonio, TX, December, 1993.

[2] J. Friedman and P. Khargonekar, "Application of Identification in $\mathcal{H}_\infty$ to Lightly Damped Systems: Two Case Studies," *IEEE Trans. Autom. Contr.*, Vol. 3, no. 3, pp. 279 - 289, 1995.

[3] G. Gu and P. Khargonekar, "Linear and Nonlinear Algorithms for Identification in $\mathcal{H}_\infty$ with Error Bounds," *IEEE Trans. Autom. Contr.*, Vol. 37, no. 7, pp. 953 - 963, 1992.

[4] A. Helmicki, C. Jacobson, and C. Nett, "Control-Oriented System Identification: A Worst-Case/Deterministic Approach in $\mathcal{H}_\infty$," *IEEE Trans. Autom. Contr.*, Vol. 3, no. 10, pp. 1163 - 1176, 1991.

[5] A. Helmicki, C. Jacobson, and C. Nett, "Identification in $\mathcal{H}_\infty$: The Continuous-Time Case," *IEEE Trans. Autom. Contr.*, Vol. 37, pp. 604 - 610, 1992.

[6] J. Hong, J. C. Akers, R. Venugopal, M. Lee, A. G. Sparks, P. D. Washabaugh, and D. S. Bernstein, " Modeling, Identification, and Feedback Control of Noise in an Acoustic Duct," *IEEE Trans. Autom. Contr.*, Vol. 4, pp. 283 - 291, 1996.

[7] D. C. Hyland, "Neural Network Architecture for On-Line System Identification and Recursively Optimized Control," *Proc. IEEE Conf. Dec. Contr.*, pp. 2552-2557, Brighton, UK, December, 1991.

[8] D. C. Hyland, E. G. Collins Jr., W. M. Haddad, and D. L. Hunter, "Neural Network System Identification for Improved Noise Rejection," *Proc. Amer. Contr. Conf.*, pp. 345-349, Seattle, WA, June, 1995.

[9] J. Juang, *Applied System Identification*, Prentice-Hall, Englewood Cliffs, NJ, 1994.

[10] L. Ljung, *System Identification: Theory for the User*, Prentice-Hall, Englewood Cliffs, NJ, 1987.

[11] M. Moonen et al.,"On- and Off-line Identification of Linear State-Space Models," *Int. J. Contr.*, Vol. 49, no. 1, pp. 219 - 232, 1989.

[12] T. Söderström and P. Stoica, *System Identification*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[13] M. Verhaegen and P. Dewilde, "Subspace Model Identification Part 1 & 2," *Int. J. Contr.*, Vol. 56, no. 5, pp. 1187 - 1241, 1992.