Proceedings of the 2007 American Control Conference
Marriott Marquis Hotel at Times Square
New York City, USA, July 11-13, 2007

What Are the Physical Dimensions of the A Matrix?

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1. Introduction

Physical dimensions and units, such as mass (kg), length (m), time (s), and charge (C), provide the link between mathematics and the physical world. It is well known that careful attention to physical dimensions can provide valuable insight into relationships among physical quantities. In this regard, the Buckingham Pi Theorem, which is essentially an application of the fundamental theorem of linear algebra on the sum of the rank and defect of a matrix, has been extensively applied [1–8].

In the control literature, with its historically strong mathematical influence, it is not unusual to see expressions such as

$$V(x, \dot{x}) = x^2 + \dot{x}^2,$$

where x and $\dot{x}$ denote position and velocity states, respectively. Although this expression appears to be dimensionally incorrect, the reader usually assumes that unlabeled coefficients are present to convert units from squared position to squared velocity or vice versa.

A related issue concerns the appearance of nondimensional units. For example, for a stiffness k and a mass m, the expression $\sqrt{k/m}$ has the dimensions of reciprocal time. However, when used within the context of harmonic solutions of an oscillator, the same expression has the interpretation of rad/s, where the nondimensional unit "rad" is inserted to facilitate the use of trigonometric functions. Although this insertion is ad hoc, the recognition that radians are nondimensional provides reasonable justification.

A publication of special note is the book [6], which takes an in-depth look at the role of dimensions including matrices populated with dimensioned quantities. Although this text provides no situations in which the "usual" rules of dimensional analysis lead to incorrect answers, the careful reexamination in [6] of the treatment of dimensions motivates the present article.

The main objective of this article is to examine the dimensional structure of the dynamics matrix A that arises in the linear state space system $\dot{x} = Ax$. To do this, we extend results of [6] and provide a self-contained treatment of the dimensional structure of A and its exponential. Our investigation of the physical dimensions of A motivates us to look at the algebraic structure of dimensioned quantities. This development forces us to define multiple, distinct, group identity elements, which are the dimensionless units. One such dimensionless unit is the radian. However, to make the analysis complete, we introduce an additional dimensionless quantity for each physical dimension and each product of dimensions.

This approach immediately clarifies the mysterious appearance of radians in the example above. Specifically, $[\sqrt{k/m}] = ([k]/[m])^{1/2} = ((\mathrm{N/m})/\mathrm{kg})^{1/2} = [\ ]_{\mathrm{m}}[\ ]_{\mathrm{kg}}/\mathrm{s}$, where $[a]$ denotes the physical dimensions of a, $[\ ]_{\mathrm{kg}} \triangleq \mathrm{kg}^0$ is the identity element in the group of mass dimensions, and $[\ ]_{\mathrm{m}} \triangleq \mathrm{m}^0$ is the identity element in the group of length dimensions. In fact, $[\ ]_{\mathrm{m}}$ is the traditional radian, whose appearance is natural and need not be inserted with the justification that "radians are dimensionless." Rather, $[\ ]_{\mathrm{m}}$ appears because the mathematical structure of physical dimensions requires that it be present.

As an additional example, consider the expression $\omega = v/r$, where ω is angular velocity, v is translational velocity, and r is radius. Then $[\omega] = [v]/[r] = (\mathrm{m/s})/\mathrm{m} = [\ ]_{\mathrm{m}}/\mathrm{s} = \mathrm{rad/s}$. Again, there is no need to insert the nondimensional unit "rad" in order to obtain the angular velocity in the expected units. We also note that, for an angle θ in radians,

$$[\sin\theta] = [\theta - \frac{\theta^3}{3!} + \cdots] = [\ ]_{\mathrm{m}},$$

which is consistent with the fact that both θ and $\sin\theta$ are ratios of lengths.

In real computations of physical quantities, that is, aside from pure theory, it is essential to keep track of physical dimensions and their associated units. Elucidation of the physical dimension structure of the state space matrix A can thus be useful for verifying the model structure and ensuring that the units are consistent within the context of state space computations.

2. Algebraic Structure of Units

For simplicity, we consider physical dimensions involving mass (kg), length (m), and time (s) only. For convenience, we use kg, m, and s to represent the respective physical dimension as well as the associated unit. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. Define $G_{\mathrm{kg}} \triangleq \{\mathrm{kg}^\alpha : \alpha \in \mathbb{R}\}$, $G_{\mathrm{m}} \triangleq \{\mathrm{m}^\beta : \beta \in \mathbb{R}\}$, and $G_{\mathrm{s}} \triangleq \{\mathrm{s}^\gamma : \gamma \in \mathbb{R}\}$. Note that G_{kg}, G_{m}, and G_{s} are Abelian (commutative) groups with the identity elements $[\ ]_{\mathrm{kg}}$, $[\ ]_{\mathrm{m}}$, and $[\ ]_{\mathrm{s}}$, respectively, which are dimensionless units referred to as the *massian*, *lengthian*, and *timian*. The lengthian $[\ ]_{\mathrm{m}}$ in G_{m}, when interpreted within the context of a circle, is traditionally called a radian. Note that $[\ ]_{\mathrm{kg}}^\alpha = (\mathrm{kg}^0)^\alpha = \mathrm{kg}^0 = [\ ]_{\mathrm{kg}}$ for all $\alpha \in \mathbb{R}$, and likewise for $[\ ]_{\mathrm{m}}$ and $[\ ]_{\mathrm{s}}$. Next, define the set G of all mixed units

$$G \triangleq \{\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma : \alpha,\, \beta,\, \gamma \in \mathbb{R}\}. \tag{2.1}$$

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Since, for all $\alpha, \beta, \gamma \in \mathbb{R}$, $\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma = \mathrm{kg}^\alpha \mathrm{s}^\gamma \mathrm{m}^\beta = \mathrm{m}^\beta \mathrm{kg}^\alpha \mathrm{s}^\gamma = \mathrm{m}^\beta \mathrm{s}^\gamma \mathrm{kg}^\alpha = \mathrm{s}^\gamma \mathrm{m}^\beta \mathrm{kg}^\alpha = \mathrm{s}^\gamma \mathrm{kg}^\alpha \mathrm{m}^\beta$, we have the following result.

Fact 2.1. G is an Abelian group with the identity element $[\ ]_{\mathrm{kg}}[\ ]_{\mathrm{m}}[\ ]_{\mathrm{s}}$.

The four products of the identity elements are represented by $[\ ]_{\mathrm{kg,m}} \triangleq [\ ]_{\mathrm{kg}}[\ ]_{\mathrm{m}}$, $[\ ]_{\mathrm{kg,s}} \triangleq [\ ]_{\mathrm{kg}}[\ ]_{\mathrm{s}}$, $[\ ]_{\mathrm{m,s}} \triangleq [\ ]_{\mathrm{m}}[\ ]_{\mathrm{s}}$, and $[\ ]_{\mathrm{kg,m,s}} \triangleq [\ ]_{\mathrm{kg}}[\ ]_{\mathrm{m}}[\ ]_{\mathrm{s}}$, of which only the last is an element of G. Note that the dimensionless Reynolds number in fluid dynamics defined by

$$\mathrm{Re} \triangleq \frac{v_s L}{\nu},$$

where v_s is the mean fluid velocity, L is the characteristic length of the flow, and ν is the kinematic fluid viscosity, has the units

$$[\mathrm{Re}] = [\ ]_{\mathrm{kg,m,s}}.$$

Similarly the dimensionless Froude number in fluid mechanics defined by

$$\mathrm{Fr} \triangleq \frac{v_s}{Lg},$$

where g is acceleration due to gravity, has the units

$$[\mathrm{Fr}] = [\ ]_{\mathrm{m,s}}.$$

Table I classifies several dimensionless quantities based on their units.

The set $\mathcal{D}$ of *dimensioned scalars* consists of elements of the form $a\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma$, where $a \in \mathbb{C}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. We define the units operator $[\ ]$ as

$$[a\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma] \triangleq \mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma.$$

Note that $[0\,\mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma] \triangleq \mathrm{kg}^\alpha \mathrm{m}^\beta \mathrm{s}^\gamma$. Let $a_1\mathrm{kg}^{\alpha_1}\mathrm{m}^{\beta_1}\mathrm{s}^{\gamma_1}$ and $a_2\mathrm{kg}^{\alpha_2}\mathrm{m}^{\beta_2}\mathrm{s}^{\gamma_2}$ be dimensioned scalars. Then the product of two dimensioned scalars always exists and is defined to be $a_1\mathrm{kg}^{\alpha_1}\mathrm{m}^{\beta_1}\mathrm{s}^{\gamma_1} a_2\mathrm{kg}^{\alpha_2}\mathrm{m}^{\beta_2}\mathrm{s}^{\gamma_2} = a_1 a_2 \mathrm{kg}^{\alpha_1+\alpha_2}\mathrm{m}^{\beta_1+\beta_2}\mathrm{s}^{\gamma_1+\gamma_2}$. However, the sum $a_1\mathrm{kg}^{\alpha_1}\mathrm{m}^{\beta_1}\mathrm{s}^{\gamma_1} + a_2\mathrm{kg}^{\alpha_2}\mathrm{m}^{\beta_2}\mathrm{s}^{\gamma_2}$ is defined only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, and $\gamma_1 = \gamma_2$, in which case $a_1\mathrm{kg}^{\alpha_1}\mathrm{m}^{\beta_1}\mathrm{s}^{\gamma_1} + a_2\mathrm{kg}^{\alpha_2}\mathrm{m}^{\beta_2}\mathrm{s}^{\gamma_2} = (a_1 + a_2)\mathrm{kg}^{\alpha_1}\mathrm{m}^{\beta_1}\mathrm{s}^{\gamma_1}$. Furthermore, although quantities such as $a\mathrm{kg}^\alpha \in \mathbb{C} \times G_{\mathrm{kg}}$ and $b\mathrm{s}^\gamma \in \mathbb{C} \times G_{\mathrm{s}}$ are not elements of $\mathcal{D}$, we assume that all operations occur after these quantities are embedded in the appropriate group containing all the common units. For example, $(a\mathrm{kg}^\alpha)(b\mathrm{s}^\gamma) \triangleq (a\mathrm{kg}^\alpha[\ ]_{\mathrm{s}})(b\mathrm{s}^\gamma[\ ]_{\mathrm{kg}}) = ab\mathrm{kg}^\alpha \mathrm{s}^\gamma$.

Dimensioned vectors and *dimensioned matrices* are denoted by $\mathcal{D}^n$ and $\mathcal{D}^{n\times m}$, respectively, all of whose entries are dimensioned scalars. Let $P \in \mathcal{D}^{n\times m}$ and define

$$[P] \triangleq \begin{bmatrix} [P_{1,1}] & \cdots & [P_{1,m}] \\ \vdots & \ddots & \vdots \\ [P_{n,1}] & \cdots & [P_{n,m}] \end{bmatrix} \in G^{n\times m}, \quad (2.2)$$

where $P_{i,j}$ is the (i,j) entry of P and $G^{n\times m}$ denotes the set of $n\times m$ matrices with entries in G. Note that $[P^{\mathrm{T}}] = [P]^{\mathrm{T}}$. If $P \in \mathcal{D}^{n\times m}$ and $Q \in \mathcal{D}^{m\times p}$, then PQ *exists* if all addition operations required to form the product are defined.

Fact 2.2. Let $P \in \mathcal{D}^{n\times m}$ and $Q \in \mathcal{D}^{m\times p}$. Then PQ exists if and only if, for all $i = 1,\ldots,n$ and $j = 1,\ldots,p$,

$$[P_{i,1}][Q_{1,j}] = [P_{i,2}][Q_{2,j}] = \cdots = [P_{i,n}][Q_{n,j}]. \quad (2.3)$$

Furthermore, if PQ exists, then

$$[PQ] = [P][Q]. \quad (2.4)$$

Fact 2.3. Let $P \in \mathcal{D}^{n\times n}$. If P^2 exists, then

$$[P_{1,1}] = [P_{2,2}] = \cdots = [P_{n,n}]. \quad (2.5)$$

Proof. Since P^2 exists, it follows that, for all $i, j = 1,\ldots,n$,

$$[(P^2)_{i,i}] = [P_{i,1}][P_{1,i}] = [P_{i,2}][P_{2,i}] = \cdots = [P_{i,n}][P_{n,i}].$$

Now, let $i, j \in \{1,\ldots,n\}$. Then $[P_{i,i}][P_{i,i}] = [P_{i,j}][P_{j,i}] = [P_{j,i}][P_{i,j}] = [P_{j,j}][P_{j,j}]$. Hence $[P_{i,i}] = [P_{j,j}]$. □

Dimensionless Unit	Name	Examples
$[\ ]_{\mathrm{kg}}$	massian	air-fuel ratio stoichiometric mass ratio
$[\ ]_{\mathrm{m}}$	lengthian	radian strain Poisson's ratio Fresnel number aspect ratio
$[\ ]_{\mathrm{s}}$	timian	Courant-Friedrichs-Lewy (CFL) number Damkohler numbers
$[\ ]_{\mathrm{kg,m}}$	densian	density ratio moment-of-inertia ratio
$[\ ]_{\mathrm{kg,s}}$	flowian	mass-flow ratio stiffness ratio
$[\ ]_{\mathrm{m,s}}$	velocian	Froude number Fourier number Mach number Stokes number
$[\ ]_{\mathrm{kg,m,s}}$	forcian	Reynolds number Weber number coefficient of friction lift coefficient drag coefficient

TABLE I
CLASSIFICATION OF DIMENSIONLESS UNITS AND EXAMPLES.

Fact 2.4. Let $P \in \mathcal{D}^{n\times n}$. If P^2 exists, then, for all positive integers k, P^k exists and $[P^k] = [P]^k$. Furthermore, for all $i = 1,\ldots,n$ and for all positive integers k,

$$[P^k] = [(P_{i,i})^{k-1}][P]. \quad (2.6)$$

Proof. Since, for all $i, j = 1,\ldots,n$,

$$[(P^2)_{i,j}] = [P_{i,1}][P_{1,j}] = [P_{i,2}][P_{2,j}] = \cdots = [P_{i,n}][P_{n,j}],$$

it follows that

$$[(P^2)_{i,j}] = [P_{i,i}][P_{i,j}].$$

Hence $[P^2] = [P_{i,i}][P]$. Induction yields (2.6). □

Fact 2.5. Let $P \in \mathcal{D}^{n\times n}$. Then P^2 exists if and only if there exist $z_1,\, z_2 \in G^n$ such that $z_2^{\mathrm{T}} z_1$ exists and

$$[P] = z_1 z_2^{\mathrm{T}}. \tag{2.7}$$

Proof. Sufficiency is immediate. To prove necessity, define

$$z_1 \triangleq \begin{bmatrix} [P_{1,1}] \\ [P_{2,1}] \\ \vdots \\ [P_{n,1}] \end{bmatrix}, \quad z_2 \triangleq \begin{bmatrix} [P_{1,1}]/[P_{1,1}] \\ [P_{1,2}]/[P_{1,1}] \\ \vdots \\ [P_{1,n}]/[P_{1,1}] \end{bmatrix}.$$

Since P^2 exists it follows that $[(P^2)_{1,1}]/[P_{1,1}] = z_2^{\mathrm{T}} z_1$ exists. Furthermore, let $k \in \{1,\ldots,n\}$ and define $z_3 \in G^n$ by

$$z_3 \triangleq \begin{bmatrix} [P_{1,k}] & [P_{2,k}] & \cdots & [P_{n,k}] \end{bmatrix}^{\mathrm{T}}.$$

Then, $z_2^{\mathrm{T}} z_3$ exists and thus the rows of $[P]$ are dimensioned scalar multiples of each other. Hence

$$[P] = \begin{bmatrix} [P_{1,1}] & [P_{1,2}] & \cdots & [P_{1,n}] \\ [P_{2,1}] & [P_{1,2}][P_{2,1}]/[P_{1,1}] & \cdots & [P_{1,n}][P_{2,1}]/[P_{1,1}] \\ \vdots & \vdots & \ddots & \vdots \\ [P_{n,1}] & [P_{1,2}][P_{n,1}]/[P_{1,1}] & \cdots & [P_{1,n}][P_{n,1}]/[P_{1,1}] \end{bmatrix} = z_1 z_2^{\mathrm{T}}.$$

□

Fact 2.6. Let $P \in \mathcal{D}^{n\times n}$. Then $e^P \in \mathcal{D}^{n\times n}$ exists if and only if P^2 exists and $[P] = [P^2]$. Furthermore, if e^P exists then

$$[e^P] = [P]. \tag{2.8}$$

Proof. By definition, the matrix exponential $e^P \in \mathcal{D}^{n\times n}$ is given by

$$e^P = I + \frac{1}{1!}P + \frac{1}{2!}P^2 + \cdots. \tag{2.9}$$

Necessity is immediate. To prove sufficiency, note that, since P^2 exists and $[P] = [P^2]$, it follows from Fact 2.4 that $[P] = [P^2]$ implies that $[P] = [P^k]$ for all positive integers k. Thus e^P exists. Next, it follows from (2.9) that (2.8) holds. □

Fact 2.7. Let $P \in \mathcal{D}^{n\times n}$ and assume e^P exists. Then, for all $i = 1,\ldots,n$,

$$[P_{i,i}] = [\ ]_{\mathrm{kg,m,s}}. \tag{2.10}$$

Proof. The result follows immediately from facts 2.6 and 2.4. □

For a real scalar q and $P \in \mathcal{D}^{n\times m}$, the *Schur power* $P^{\{q\}} \in \mathcal{D}^{n\times m}$ is defined by

$$(P^{\{q\}})_{i,j} \triangleq (P_{i,j})^q, \tag{2.11}$$

assuming the right hand side exists. The notation $[P]_{\mathbb{C}} \in \mathbb{C}^{n\times m}$ denotes the numerical part of the dimensioned matrix $P \in \mathcal{D}^{n\times m}$. Note that

$$P = [P]_{\mathbb{C}} \circ [P], \tag{2.12}$$

where $\circ$ is the Schur (element-wise) product. We write $[P]_{\mathbb{C}}$ as $[P]_{\mathbb{R}}$ if $[P]_{\mathbb{C}}$ is real. Let $I_{\mathbb{R}}$ denote the identity matrix in $\mathbb{R}^{n\times n}$. Furthermore, let $Q \in \mathcal{D}^{m\times p}$ and assume that PQ exists. Then $[PQ]_{\mathbb{C}} = [P]_{\mathbb{C}}[Q]_{\mathbb{C}}$ and

$$\begin{aligned} PQ &= ([P]_{\mathbb{C}} \circ [P])([Q]_{\mathbb{C}} \circ [Q]) = ([P]_{\mathbb{C}}[Q]_{\mathbb{C}}) \circ ([P][Q]) \\ &= [PQ]_{\mathbb{C}} \circ [PQ]. \end{aligned} \tag{2.13}$$

Fact 2.8. Let $P \in \mathcal{D}^{n\times m}$ and let $y \in \mathcal{D}^n$ and $u \in \mathcal{D}^m$ be such that

$$y = Pu. \tag{2.14}$$

Then

$$[P] = [y][u^{\mathrm{T}}]^{\{-1\}}. \tag{2.15}$$

Proof The i^{th} component equation of (2.15) is

$$[P_{i,1}][u_1] + [P_{i,2}][u_2] + \cdots + [P_{i,m}][u_m] = [y_i].$$

Therefore,

$$[P_{i,1}][u_1] = [P_{i,2}][u_2] = \cdots = [P_{i,m}][u_m] = [y_i],$$

and thus $[P_{i,j}] = [y_i]/[u_j]$. Hence (2.15) holds. □

Next, let $P \in \mathcal{D}^{n\times n}$. Then, the determinant det P of P is defined to be

$$\det P = \sum_{p\in\mathcal{P}_n} \sigma(p) P_{1,p_1} P_{2,p_2} \cdots P_{n,p_n}, \tag{2.16}$$

where $\mathcal{P}_n$ is the set of all permutations $p = (p_1,\ldots,p_n)$ of $(1,\,2,\,\ldots,\,n)$, and $\sigma(p)$ is the signature of the permutation p, which is 1 if p is achieved by applying an even number of transpositions to $(1,\,2,\,\ldots,\,n)$ and -1 if p is reached by applying an odd number of transpositions to $(1,\,2,\,\ldots,\,n)$. Note that if $P \in \mathcal{D}^{n\times n}$ then det P exists if and only if $[P_{1,p_1}P_{2,p_2}\cdots P_{n,p_n}]$ is the same for all $p \in \mathcal{P}_n$. Hence, if det P exists, we have

$$[\det P] = [P_{1,p_1}P_{2,p_2}\cdots P_{n,p_n}] \tag{2.17}$$

for all $p \in \mathcal{P}_n$. Note that

$$\det\, [P]_{\mathbb{C}} = [\det P]_{\mathbb{C}} \tag{2.18}$$

and

$$\det P = (\det\, [P]_{\mathbb{C}})[\det P]. \tag{2.19}$$

The following result presents necessary and sufficient conditions for the existence of det P.

Fact 2.9. Let $P \in \mathcal{D}^{n\times n}$. Then det P exists if and only if there exist $z_1,\, z_2 \in G^n$ such that

$$[P] = z_1 z_2^{\mathrm{T}}. \tag{2.20}$$

Proof. Sufficiency is immediate. To prove necessity, first let $n = 2$. Then, since det P exists, it follows that

$$\frac{[P_{1,1}]}{[P_{1,2}]} = \frac{[P_{2,1}]}{[P_{2,2}]}. \tag{2.21}$$

Thus the columns of $[P]$ are dimensioned scalar multiples of each other. Next, let $n = 3$ and assume that det P exists. Then it follows from the cofactor expansion of det P that the determinant of every 2×2 submatrix of P exists. Hence (2.21) holds. Next, it follows that $[P_{1,1}P_{2,3}P_{3,2}] = [P_{1,2}P_{2,3}P_{3,1}]$ and hence

$$\frac{[P_{1,1}]}{[P_{1,2}]} = \frac{[P_{3,1}]}{[P_{3,2}]}. \tag{2.22}$$

Furthermore, using $[P_{1,2}P_{2,3}P_{3,1}] = [P_{1,3}P_{2,2}P_{3,1}]$ and $[P_{1,2}P_{2,1}P_{3,2}] = [P_{1,3}P_{2,1}P_{3,2}]$, it follows that $[P_{1,2}]/[P_{1,3}] = [P_{2,2}]/[P_{2,3}]$ and $[P_{1,2}]/[P_{1,3}] = [P_{3,2}]/[P_{3,3}]$. Thus the columns of $[P]$ are dimensioned scalar multiples of each other. Likewise, for all $n \geq 1$, it can be seen that, since det P exists, the columns of $[P]$ are dimensioned scalar multiples of each other. Thus, defining

$$z_1 \triangleq \begin{bmatrix} [P_{1,1}] \\ [P_{2,1}] \\ \vdots \\ [P_{n,1}] \end{bmatrix}, \quad z_2 \triangleq \begin{bmatrix} [P_{1,1}]/[P_{1,1}] \\ [P_{1,2}]/[P_{1,1}] \\ \vdots \\ [P_{1,n}]/[P_{1,1}] \end{bmatrix},$$

it follows that (2.20) holds. □

Note that if P^2 exists then det P exists. However, the following example shows that the converse does not hold.

Example 2.1. Let $P \in \mathcal{D}^{2\times 2}$ be such that

$$[P] = \begin{bmatrix} \mathrm{m} & \mathrm{m}^2 \\ \mathrm{s} & \mathrm{ms} \end{bmatrix}. \tag{2.23}$$

Then det P exists, but P^2 does not exist.

Let $P \in \mathcal{D}^{n\times n}$. Then $\lambda \in \mathcal{D}$ and $v \in \mathcal{D}^n$ are an *eigenvalue-eigenvector pair* of P if $[v]_\mathbb{C}$ is not zero and λ and v satisfy

$$Pv = \lambda v. \tag{2.24}$$

Fact 2.10. Let $P \in \mathcal{D}^{n\times n}$. Then P has an eigenvalue-eigenvector pair $\lambda \in \mathcal{D}$, $v \in \mathcal{D}^n$ if and only if det P exists and, for all $i = 1, \ldots, n$ and $j = 1, \ldots, n$,

$$[P_{i,i}] = [P_{j,j}]. \tag{2.25}$$

In this case,

$$[P] = [\lambda v][v^\mathrm{T}]^{\{-1\}} \tag{2.26}$$

and, for all $i = 1, \ldots, n$,

$$[P_{i,i}] = [\lambda]. \tag{2.27}$$

Proof. To prove necessity, note that it follows from Fact 2.8 that (2.24) implies (2.26). It thus follows from Fact 2.9 that det P exists. Furthermore, it follows from (2.24) that, for all $i = 1, \ldots, n$,

$$[P_{i,i}][v_i] = [\lambda][v_i].$$

Thus

$$[P_{i,i}] = [\lambda].$$

Hence, for $i = 1, \ldots, n$, $j = 1, \ldots, n$, it follows that $[P_{i,i}] = [P_{j,j}]$.

To prove sufficiency, from (2.20) and (2.25) it follows that

$$[(z_1)_1(z_2)_1] = [(z_1)_2(z_2)_2] = \cdots = [(z_1)_n(z_2)_n], \tag{2.28}$$

where $(z_1)_i$ denotes the i^{th} component of z_1. Thus, $\lambda_G \triangleq z_2^\mathrm{T} z_1$ exists. Note that $\lambda_G z_1 = z_1 z_2^\mathrm{T} z_1 = [P]z_1$. Next, let $\lambda_\mathbb{C} \in \mathbb{C}$ and $v_\mathbb{C} \in \mathbb{C}^n$ be such that

$$[P]_\mathbb{C} v_\mathbb{C} = \lambda_\mathbb{C} v_\mathbb{C}. \tag{2.29}$$

Then defining $\lambda \in \mathcal{D}$ and $v \in \mathcal{D}^n$ by $\lambda \triangleq \lambda_\mathbb{C}\lambda_G$ and $v \triangleq v_\mathbb{C} \circ z_1$ it follows that

$$\begin{aligned} Pv &= ([P]_\mathbb{C} \circ [P])(v_\mathbb{C} \circ [v]) = ([P]_\mathbb{C} v_\mathbb{C}) \circ z_1 z_2^\mathrm{T} z_1 \\ &= (\lambda_\mathbb{C} v_\mathbb{C}) \circ \lambda_G z_1 = \lambda_\mathbb{C}\lambda_G(v_\mathbb{C} \circ z_1) = \lambda v. \end{aligned}$$ □

Next, let $P \in \mathcal{D}^{n\times n}$. Then, if det $[P]_\mathbb{C} \neq 0$, we define the inverse P^{-1} of P by

$$P^{-1} \triangleq \frac{1}{\det P} P^\mathrm{A}, \tag{2.30}$$

where the adjugate P^A is defined by $(P^\mathrm{A})_{i,j} \triangleq (-1)^{i+j}\det P_{[j,i]}$, where $P_{[j,i]}$ denotes the $(n-1)\times(n-1)$ cofactor of $P_{i,i}$. Hence

$$[P^{-1}] = \frac{1}{[\det P]}[P^\mathrm{A}] \tag{2.31}$$

and

$$[P^{-1}]_\mathbb{C} = \frac{1}{[\det P]_\mathbb{C}}[P^\mathrm{A}]_\mathbb{C}. \tag{2.32}$$

The following example shows that for $P \in \mathcal{D}^{n\times n}$ such that P^{-1} exists, in general $[P^{-1}][P] \neq [P][P^{-1}]$.

Example 2.2. Let $P \in \mathcal{D}^{n\times}$ be such that

$$[P] = [\ ]_{\mathrm{m,s}} \begin{bmatrix} \mathrm{m} & 1/\mathrm{s} \\ \mathrm{ms}^2 & \mathrm{s} \end{bmatrix}$$

and assume that P^{-1} exists. Then

$$\begin{aligned} [P^{-1}] &= [\ ]_{\mathrm{m,s}} \begin{bmatrix} 1/\mathrm{m} & 1/\mathrm{ms}^2 \\ \mathrm{s} & 1/\mathrm{s} \end{bmatrix}, \\ [P][P^{-1}] &= [\ ]_{\mathrm{m,s}} \begin{bmatrix} 1 & 1/\mathrm{s}^2 \\ \mathrm{s}^2 & 1 \end{bmatrix}, \end{aligned}$$

and

$$[P^{-1}][P] = [\ ]_{\mathrm{m,s}} \begin{bmatrix} 1 & 1/\mathrm{ms} \\ \mathrm{sm} & 1 \end{bmatrix}.$$

Thus $[P^{-1}][P] \neq [P][P^{-1}]$.

3. Dimensions of Matrices in State-Space Models

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3.1}$$
$$y(t) = Cx(t) + Du(t), \tag{3.2}$$

where $[t] = \mathrm{s}$, $x(t) \in \mathcal{D}^n$, $y(t) \in \mathcal{D}^l$, $u(t) \in \mathcal{D}^m$, $A \in \mathcal{D}^{n\times n}$, $B \in \mathcal{D}^{n\times m}$, $C \in \mathcal{D}^{l\times n}$, and $D \in \mathcal{D}^{l\times m}$. Every component of $x(t)$, $y(t)$, $u(t)$, and thus every entry of A, B, C, D, is a dimensioned scalar. Taking units on both sides of (3.1) yields

$$[\dot{x}(t)] = [A][x(t)] = [B][u(t)], \tag{3.3}$$
$$[y(t)] = [C][x(t)] = [D][u(t)]. \tag{3.4}$$

The following result is given on page 150 of [6].

Fact 3.1.

$$[A] = \frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}, \tag{3.5}$$
$$[B] = \frac{1}{\mathrm{s}}[x(t)][u^{\mathrm{T}}(t)]^{\{-1\}}, \tag{3.6}$$
$$[C] = [y(t)][x^{\mathrm{T}}(t)]^{\{-1\}}, \tag{3.7}$$

and

$$[D] = [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}}. \tag{3.8}$$

Proof. The result follows from $[\dot{x}(t)] = \frac{1}{\mathrm{s}}[x(t)]$ and Fact 2.8. □

Next, define the transfer function matrix $H(s) \in \mathcal{D}^{l\times m}$ by

$$H(s) \triangleq C(sI_s - A)^{-1}B + D, \tag{3.9}$$

where $s \in \mathcal{D}$ is the Laplace variable, $[s] = 1/\mathrm{s}$, and $I_s \triangleq I_{\mathbb{R}} \circ \mathrm{s}[A]$.

Fact 3.2.

$$[H(s)] = [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}}. \tag{3.10}$$

Proof. Note that

$$\begin{aligned}&[C(sI - A)^{-1}B]\\ &\quad = [y(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[x(t)][u^{\mathrm{T}}(t)]^{\{-1\}},\\ &\quad = [y(t)][u^{\mathrm{T}}(t)]^{\{-1\}} = [D].\end{aligned}$$
□

Fact 3.3. For all $i = 1,\ldots,n$,

$$[A_{i,i}] = [\ ]_{\mathrm{kg,m}}\mathrm{s}^{-1}. \tag{3.11}$$

Furthermore, det A exists and satisfies

$$[\det A] = [\ ]_{\mathrm{kg,m}}\mathrm{s}^{-n}. \tag{3.12}$$

Proof. It follows from (3.5) that

$$[A_{i,i}] = \frac{1}{\mathrm{s}}\frac{[x_i(t)]}{[x_i(t)]} = \frac{[\ ]_{\mathrm{kg,m}}}{\mathrm{s}}.$$

Next, note that

$$[A_{i,p_i}] = \frac{1}{\mathrm{s}}\frac{[x_i(t)]}{[x_{p_i}(t)]}.$$

Thus, for all $p \in \mathcal{P}_n$,

$$[A_{1,p_1}\cdots A_{n,p_n}] = \frac{1}{\mathrm{s}^n}\frac{[x_1(t)]\cdots[x_n(t)]}{[x_{p_1}(t)]\cdots[x_{p_n}(t)]} = \frac{[\ ]_{\mathrm{kg,m}}}{\mathrm{s}^n}.$$

Since $[A_{1,p_1}A_{2,p_2}\cdots A_{n,p_n}]$ is the same for all $p \in \mathcal{P}_n$, det A exists. Finally, since $[\det A] = \prod_{i=1}^n [A_{1,p_i}]$ for all $p \in \mathcal{P}_n$, it follows that

$$[\det A] = [A_{1,p_1}A_{2,p_2}\cdots A_{n,p_n}] = \frac{[\ ]_{\mathrm{kg,m}}}{\mathrm{s}^n}.$$
□

Fact 3.4. Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then

$$\det\,[At] = [\ ]_{\mathrm{kg,m,s}}. \tag{3.13}$$

4. Matrix Exponential

Lemma 4.1. Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then the following statements hold:

i) For all positive integers k, A^k exists.
ii) For all $k \geq 1$, $[A^k] = \frac{1}{\mathrm{s}^{k-1}}[A]$.
iii) For all $k \geq 1$, $[A^k] = \frac{1}{\mathrm{s}}[A^{k-1}]$.
iv) For all $k \geq 1$, $[A^k t^k] = [At]$.
v) $[A]^{\{-1\}} = \frac{1}{\mathrm{s}^2}[A]$.

If, in addition, A^{-1} exists, then

vi) $[A^{-1}] = [A^{\mathrm{T}}]^{\{-1\}}$.
vii) $[A^{-1}] = \mathrm{s}^2[A]^{\mathrm{T}}$.

Proof. Statements $i) - iv)$ follow from Fact 2.4. Next, we prove $vi)$. Since $(A^{-1})_{i,i} = \det A_{[i,i]}/\det A$, it follows that $[(A^{-1})_{i,i}] = \det\,[A_{[i,i]}]/\det\,[A] = 1/[A_{i,i}]$. Thus the diagonal entries of $[A][A^{-1}]$ satisfy

$$([A][A^{-1}])_{i,\,i} = [\ ]_{\mathrm{kg,m,s}}, \quad i = 1,\ldots,n.$$

Therefore,

$$([A][A^{-1}])_{i,i} = [A_{i,1}][A^{-1}_{1,i}] + \cdots + [A_{i,n}][A^{-1}_{n,i}] = [\ ]_{\mathrm{kg,m,s}},$$

which implies that

$$[(A^{-1})_{i,j}] = [A_{j,i}]^{-1}. \tag{4.1}$$

Thus, $vi)$ is satisfied.

To prove $vii)$, note that

$$([A]^{\mathrm{T}})_{i,j} = \frac{1}{\mathrm{s}}\frac{[x_j(t)]}{[x_i(t)]}. \tag{4.2}$$

Next, from (4.1) it follows that

$$[(A^{-1})_{i,j}] = [A_{j,i}]^{-1} = \mathrm{s}\frac{[x_j(t)]}{[x_i(t)]}. \tag{4.3}$$

Thus from (4.2) and (4.3), it follows that

$$[A]^{\mathrm{T}} = \frac{1}{\mathrm{s}^2}[A^{-1}]. \tag{4.4}$$

To prove $v)$, using $vi)$ in (4.4), we have

$$[A]^{\mathrm{T}} = \frac{1}{\mathrm{s}^2}[A^{\mathrm{T}}]^{\{-1\}}. \tag{4.5}$$

Taking transposes yields v). □

Fact 4.1.

$$[A^{-1}] = \mathrm{s}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}. \tag{4.6}$$

Furthermore,

$$[A^{-1}][A] = [A][A^{-1}]. \tag{4.7}$$

Proof. Note that

$$[A^{-1}] = [A^{\mathrm{T}}]^{\{-1\}} = \mathrm{s}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}.$$

Hence $[A^{-1}][A] = [A][A^{-1}] = [x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}$. □

Fact 4.2. Let $t \in \mathcal{D}$ be such that $[t] = \mathrm{s}$. Then

$$[e^{At}] = [At] = [x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}. \tag{4.8}$$

5. Eigenvalues and Eigenvectors of A

Fact 5.1. Let $\lambda \in \mathcal{D}$ be an eigenvalue of A, and let $v \in \mathcal{D}^n$ be an associated eigenvector. Then, for $i = 1, \ldots, n$,

$$[\lambda] = [A_{i,i}] \tag{5.1}$$

and

$$[v] = [x^{\mathrm{T}}(t)]^{\{-1\}}[v][x(t)]. \tag{5.2}$$

Proof. Since $Av = \lambda v$, it follows that, for all $i = 1, \ldots, n$, $[A_{i,i}][v_i] = [\lambda][v_i]$, and thus $[\lambda] = [A_{i,i}]$. Next, since $Av = \lambda v$, it follows that

$$\frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}}[v] = \frac{1}{\mathrm{s}}[v],$$

which implies (5.2). □

6. Example

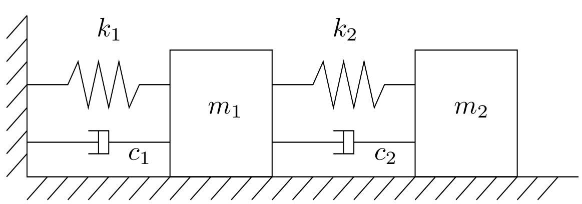

Fig. 1. Two-mass spring damper system.

Consider the two-spring-mass system shown in Figure 1. By defining the state $x(t) \triangleq \begin{bmatrix} q_1 & \dot{q}_1 & q_2 & \dot{q}_2 \end{bmatrix}^{\mathrm{T}}$, where q_i and $\dot{q}_i$ are the displacement and velocity of the i^{th} mass, respectively, we have

$$A \triangleq \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_1+k_2)}{m_1} & -\frac{(c_1+c_2)}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_2} & \frac{c_1}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \tag{6.1}$$

Taking units yields

$$[x(t)] = \begin{bmatrix} \mathrm{m} & \mathrm{m/s} & \mathrm{m} & \mathrm{m/s} \end{bmatrix}^{\mathrm{T}}. \tag{6.2}$$

Thus

$$\begin{aligned} [A] &= \frac{1}{\mathrm{s}}[x(t)][x^{\mathrm{T}}(t)]^{\{-1\}} \\ &= [\;]_{\mathrm{m}} \begin{bmatrix} 1/\mathrm{s} & 1 & 1/\mathrm{s} & 1 \\ 1/\mathrm{s}^2 & 1/\mathrm{s} & 1/\mathrm{s}^2 & 1/\mathrm{s} \\ 1/\mathrm{s} & 1 & 1/\mathrm{s} & 1 \\ 1/\mathrm{s}^2 & 1/\mathrm{s} & 1/\mathrm{s}^2 & 1/\mathrm{s} \end{bmatrix}. \end{aligned} \tag{6.3}$$

Hence $[\det A] = [\;]_{\mathrm{m}}/\mathrm{s}^4$. Furthermore,

$$[\det A]_{\mathbb{C}} = \det\,[A]_{\mathbb{R}} = \left[\frac{k_2}{m_1 m_2}\right]_{\mathbb{R}}. \tag{6.4}$$

Thus

$$\det A = \left[\frac{k_2}{m_1 m_2}\right]_{\mathbb{R}} \frac{[\;]_{\mathrm{m}}}{\mathrm{s}^4}. \tag{6.5}$$

Next, if $[k_2]_{\mathbb{R}} \neq 0$ then $\det\,[A]_{\mathbb{R}} \neq 0$ and $[A^{-1}]$ is given by

$$[A^{-1}] = [\;]_{\mathrm{m}} \begin{bmatrix} \mathrm{s} & \mathrm{s}^2 & \mathrm{s} & \mathrm{s}^2 \\ 1 & \mathrm{s} & 1 & \mathrm{s} \\ \mathrm{s} & \mathrm{s}^2 & \mathrm{s} & \mathrm{s}^2 \\ 1 & \mathrm{s} & 1 & \mathrm{s} \end{bmatrix}. \tag{6.6}$$

Finally,

$$[e^{At}] = [\;]_{\mathrm{m}} \begin{bmatrix} [\;]_{\mathrm{s}} & \mathrm{s} & [\;]_{\mathrm{s}} & \mathrm{s} \\ 1/\mathrm{s} & [\;]_{\mathrm{s}} & 1/\mathrm{s} & [\;]_{\mathrm{s}} \\ [\;]_{\mathrm{s}} & \mathrm{s} & [\;]_{\mathrm{s}} & \mathrm{s} \\ 1/\mathrm{s} & [\;]_{\mathrm{s}} & 1/\mathrm{s} & [\;]_{\mathrm{s}} \end{bmatrix}. \tag{6.7}$$

7. Conclusions

Physical dimensions are the link between mathematical models and the real world. In this article we extended results of [6] by determining the dimensional structure of a matrix under which standard operations involving the inverse, powers, exponential, and eigenvalues are valid. These results were applied to state space models. We also distinguished between different types of dimensionless units.

8. Acknowledgments

We wish to thank Jan Willems for his insightful suggestions and comments.

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