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Hysteresis Analysis and Feedback Stabilization of Snap-through Buckling*

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Abstract— **We study a preloaded two-bar linkage that exhibits hysteresis due to the presence of multiple attracting equilibria. The dynamics at the unstable equilibrium, through which a snap-through buckle occurs, are not linearizable due to a singularity that is solution-dependent. We stabilize the unstable equilibrium using two distinct nonlinear controllers. The feedback linearization controller requires knowledge of the linkage parameters, whereas the potential shaping PD-type controller requires only an upper bound on the stiffness.**

I. INTRODUCTION

The phenomenon of hysteresis is widespread and extensively studied [1], [2]. Although hysteresis arises in diverse applications, all hysteretic phenomena have a common origin. Specifically, hysteresis is the low-frequency limit of the dynamic input-output response of a system when this limit is a nondegenerate loop. A necessary condition for a system to exhibit hysteresis is the existence of multiple attracting equilibria for a given constant input [3], [4]. This statement is the *principle of multistability*.

The principle of multistability implies that any system with multiple equilibria can potentially be hysteretic. In structural mechanics, the phenomenon of buckling is closely associated with multiple equilibria, which arise when the axial load applied to a structure counteracts the stiffness of the structure [5]. Consequently, hysteresis can potentially arise when a structure passes through buckling, and this possibility is the motivation for the present paper. Although hysteresis is closely associated with energy dissipation [6], the hysteresis we consider is not a consequence of energy dissipation, which occurs in all real structures under deformation, but rather is due to the multiple equilibria arising from buckling. In fact, the damping model we assume is linear and viscous, and thus is not inherently hysteretic.

In many applications, buckling can lead to structural failure, and thus the usual objective is to avoid conditions under which buckling might occur [7]. There are, however, useful aspects of buckling. For example, buckled elements have been considered for vibration isolation, where the axial and transverse motions have widely different stiffnesses [8], [9]. Another application is in mechanical actuators, where the structural dynamics near buckling provide significant mechanical advantage [10], [11].

In the present paper we study the preloaded two-bar linkage shown in Figure 1, which serves as a lumped analogue of a structure that can undergo snap-through buckling. The word preloaded refers to the force provided by the stiffness k when the bars are horizontal in the horizontal equilibrium. The preloaded two-bar linkage exhibits the essential features of snap-through buckling, in which a perturbation from the horizontal equilibrium results in a sudden, fast response toward a stable equilibrium.

Under the approximating assumption that the bars are inertialess, the two-bar linkage has the property that the inertia is singular at the unstable (horizontal) equilibrium, thereby combining instability with singular dynamics. Mass singularities arise in linearized vibration theory when certain modes are viewed as inertialess [12], [13]. If the system is nearly inertialess, then classical singular perturbation techniques can be used to approximate the solution in the vicinity of the singularity. More generally, singularities that depend on a fixed, small parameter have been widely studied in the control literature [14]. Furthermore, singular coefficients that multiply the highest-order derivative and that are functions of the *independent* (time or spatial) variable have been extensively studied in classical dynamical systems literature (see, for example, [15, Chapter V]). The connection between singularities and hysteresis is explored in [16]. In contrast, the singularity in the linkage dynamics (16) is solution dependent rather than independent-variable dependent, and thus is not addressed by the classical singular perturbation literature.

A slightly different problem arises in the case of kinematically redundant mechanisms [17]–[21]. The kinematic redundancy entails additional degrees of freedom that have no direct impact on meeting task-space objectives but provide advantages in terms of constraint and limit avoidance. In this case, the coefficient of the highest-order derivative has a nonsquare Jacobian leading to nonunique motions for realizing a given task; in fact, joint motion within the Jacobian's (nontrivial) null space does not affect the trajectory of the end-effector, and thus the inverse kinematics are not unique. The nonunique motions can be chosen to satisfy a subtask or can be specified in terms of the generalized inverse of the Jacobian [22]. The preloaded two-bar linkage possesses a kinematic singularity since, for $\theta = 0$, zero velocity in task space (that is, $\dot{q} = 0$) does not correspond to a unique velocity $\dot{\theta}$ in joint space.

The contents of the paper are as follows. In Section II, we present the two-bar preloaded linkage and determine its equilibria through static analysis. We also derive the static

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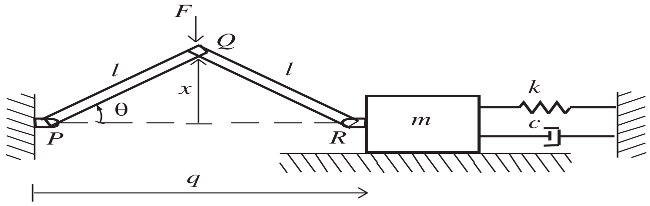


Fig. 1. The preloaded two-bar linkage. The word 'preloaded' refers to the presence of stiffness k in the system.

equilibria using energy methods, which help in determining the stability of the equilibria. Next, in Section III, we derive the equations of motion with a force input. The resulting equation, whose multiple equilibria coincide with the equilibria obtained from static analysis, possesses a state-dependent inertia singularity at the unstable, horizontal equilibrium. In Section IV we show the hysteresis in the preloaded two-bar linkage.

In Section V we consider control strategies for stabilizing the unstable equilibrium. We apply a feedback linearizing controller that renders the equilibrium asymptotically stable. Finally, to reduce dependence on the linkage parameters, in Section VI we apply the almost-global potential-shaping controller of [23], which stabilizes the equilibrium by exploiting the structure of the dynamics. A preliminary version of the results in this paper appeared in [24].

II. Static Analysis of a Preloaded Two-bar Linkage

In this section we analyze the statics of the preloaded two-bar linkage with joints P, Q, and R and preloaded by a stiffness k as shown in Figure 1. A constant vertical force F is applied at Q, where the two bars are joined by a frictionless pin. Let θ denote the counterclockwise angle that the left bar makes with the horizontal, and let q denote the distance between the joints P and R. When $F = 0$, the linkage has three equilibrium configurations given by $q = \pm q_0$ and $q = 0$. For the equilibria $q = \pm q_0$, the spring k is relaxed. Let $q_0 = 2l\cos\theta_0$ where θ_0 corresponds to the equilibrium angle θ. For the third equilibrium, both bars are horizontal with $\theta = 0$.

For a constant nonzero value of the force F, the two-bar linkage has two equilibrium configurations. For one of the equilibria, the equilibrium angle θ_{01} is positive and the spring is compressed, whereas for the other equilibria, the equilibrium angle θ_{02} is negative and the spring is extended.

The static equilibria can be obtained using energy methods. The potential energy associated with the system, which is the difference between the spring energy and the work done by the external force, is given by

$$P \triangleq 2kl^2(\cos\theta - \cos\theta_0)^2 - Fl(\sin\theta_0 - \sin\theta). \quad (1)$$

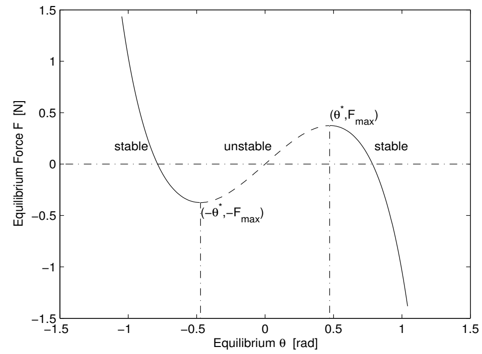


Fig. 2. Dependence of the static equilibrium force F on θ given by (2). Chosen parameter values are $\theta_0 = \frac{\pi}{4}$ rad, $k = 1$ N-m, and $l = 1$ m. For the given parameters, $\theta^* = -0.4715$ rad and $F_{\max} = 0.3747$ N.

Then the static equilibria of the system are given by

$$\frac{\partial P}{\partial \theta} = 0,$$

which yields

$$(\sin\theta)\left(1 - \frac{\cos\theta_0}{\cos\theta}\right) = \frac{F}{4kl}. \quad (2)$$

Figure 2 shows F as a function of θ. Let $F_{\max}$ be the local maximum value of F and let $\theta = \theta^*$ at $F = F_{\max}$. Solving $\frac{dF}{d\theta} = 0$ where F is given by (2), yields

$$\theta^* = \cos^{-1}(\cos\theta_0)^{1/3}. \quad (3)$$

To analyze the stability of the static equilibria, we evaluate the second derivative of the potential energy P, which is given by

$$\frac{1}{4kl^2}\frac{\partial^2 P}{\partial \theta^2} = -\cos\theta(\cos\theta - \cos\theta_0) + \sin^2\theta - \frac{F}{4kl}\sin\theta. \quad (4)$$

Substituting (2) into (4) yields

$$\frac{1}{4kl^2}\frac{\partial^2 P}{\partial \theta^2} = -\cos\theta(\cos\theta - \cos\theta_0) + \sin^2\theta - (\sin^2\theta)\left(1 - \frac{\cos\theta_0}{\cos\theta}\right),$$

which can be rewritten as

$$\frac{1}{4kl^2}\frac{\partial^2 P}{\partial \theta^2} = (\cos\theta_0 - \cos^3\theta)\frac{1}{\cos\theta}. \quad (5)$$

Using the potential energy theorem [25], p. 56, the condition $\frac{\partial^2 P}{\partial \theta^2} > 0$ implies that the equilibrium θ is stable for $\theta < -\theta^*$ and $\theta > \theta^*$, whereas $\frac{\partial^2 P}{\partial \theta^2} < 0$ implies that the equilibrium θ is unstable for $-\theta^* < \theta < \theta^*$.

Now, consider the preloaded two-bar linkage with a torsional spring $k_{\rm t} > 0$ as shown in Figure 3. To find the static equilibria for this system, we set the net torque about joint P to zero, which yields

$$Fl\cos\theta + k_{\rm t}\theta = C_1 l\sin(2\theta).$$

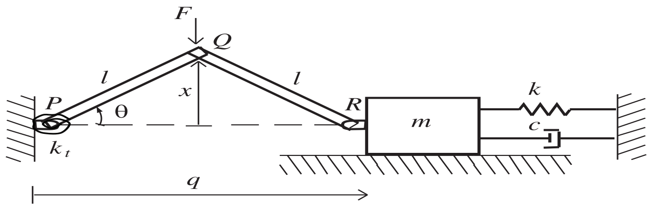


Fig. 3. The two-bar linkage model with a torsional spring $k_{\rm t}$.

Using (**??**) and substituting $q = 2l\cos\theta$, we obtain

$$Fl\cos\theta + k_{\rm t}\theta = 4kl^2(\cos\theta - \cos\theta_0)\sin\theta. \tag{6}$$

Assuming $F = 0$, the static equilibria satisfy

$$k_{\rm t}\theta = 4kl^2(\cos\theta - \cos\theta_0)\sin\theta. \tag{7}$$

Equation (7) is satisfied by $\theta = 0$ as well as all θ satisfying

$$\cos\theta = \frac{k_{\rm t}}{4kl^2} + \cos\theta_0. \tag{8}$$

Note that (8) has a non-zero solution θ if and only if $0 \le k_{\rm t} \le 4kl^2(1-\cos\theta_0)$. Furthermore, if $k_{\rm t} \ge 4kl^2(1-\cos\theta_0)$ then the preloaded two-bar linkage has exactly one equilibrium, namely, $\theta = 0$.

III. Dynamics of the Preloaded Two-bar Linkage

We now derive the equations of motion for the preloaded two-bar linkage. The system has one degree of freedom given by the angle θ, which can be viewed as the joint-space variable. Let $m_{\rm bar}$ be the inertia of each bar. Ignoring gravity, the kinetic and potential energies of the system are given by

$$T = \frac{1}{2}m\dot{q}^2 + T_{\rm bars}, \quad V = \frac{1}{2}k(q-q_0)^2,$$

where $T_{\rm bars}$ is the kinetic energy of the bars. Substituting $q = 2l\cos\theta$ and $T_{\rm bars} = (\frac{9}{8}m_{\rm bar}l^2\sin^2\theta + \frac{5}{24}m_{\rm bar}l^2)\dot{\theta}^2$ we obtain

$$T = \Big((2ml^2 + \frac{9}{8}m_{\rm bar}l^2)\sin^2\theta + \frac{5}{24}m_{\rm bar}l^2\Big)\dot{\theta}^2, \tag{9}$$

$$V = 2kl^2(\cos\theta - \cos\theta_0)^2. \tag{10}$$

The generalized nonconservative force $Q_{\rm nc}$ is given by

$$Q_{\rm nc} = -Fl\cos\theta - 4cl^2\dot{\theta}\sin^2\theta. \tag{11}$$

Now, Lagrange's equation $\frac{d}{dt}\big(\frac{\partial L}{\partial\dot{\theta}}\big) - \frac{\partial L}{\partial\theta} = Q_{\rm nc}$, where $L = T - V$, yields

$$\begin{aligned}&\Big((2ml^2 + \frac{9}{8}m_{\rm bar}l^2)\sin^2\theta + \frac{5}{24}m_{\rm bar}l^2\Big)\ddot{\theta}\\ &+ (2ml^2 + \frac{9}{8}m_{\rm bar}l^2)(\sin\theta)(\cos\theta)\dot{\theta}^2\\ &+ 2cl^2(\sin^2\theta)\dot{\theta} + 2kl^2(\cos\theta_0 - \cos\theta)(\sin\theta) = -\frac{l\cos\theta}{2}F.\end{aligned} \tag{12}$$

Note that, if $m_{\rm bar} > 0$, then (12) does not have a mass singularity. However, if $m_{\rm bar} = 0$, then (12) has a mass singularity for $\theta = 0$.

Next, note that the transformation

$$q = 2l\cos\theta, \tag{13}$$

is invertible for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. However, the velocity relation

$$\dot{q} = -2l(\sin\theta)\dot{\theta}, \tag{14}$$

is singular for $\theta = 0$. Using $q = 2l\cos\theta$ the dynamics (12) can be expressed in terms of the displacement q as

$$\begin{aligned}&\Big((m+\frac{9}{16}m_{\rm bar})(4l^2-q^2) + \frac{5}{12}m_{\rm bar}l^2\Big)(4l^2-q^2)\ddot{q}\\ &+ \frac{5}{12}m_{\rm bar}l^2q\dot{q}^2 + c\dot{q}(4l^2-q^2)^2\\ &+ k(q-q_0)(4l^2-q^2)^2 = \frac{1}{2}q(4l^2-q^2)^{\frac{3}{2}}F.\end{aligned} \tag{15}$$

Note that (15) has an inertia singularity at $q = 2l$, that is, for $\theta = 0$, whether or not $m_{\rm bar} = 0$. Note also that if $m_{\rm bar} \neq 0$ and $q = 2l$, then $\dot{q} = 0$, that is, the links always come to rest when the links are stretched out.

Now assume that $m_{\rm bar} = 0$. For this case, the dynamics (12) become

$$\begin{aligned}&2ml^2(\sin^2\theta)\ddot{\theta} + 2ml^2(\sin\theta)(\cos\theta)\dot{\theta}^2 + 2cl^2(\sin^2\theta)\dot{\theta}\\ &+2kl^2(\cos\theta_0 - \cos\theta)(\sin\theta) = -\frac{l\cos\theta}{2}F.\end{aligned} \tag{16}$$

Defining $x_1 = \theta$ and $x_2 = \dot{\theta}$, (16) can be written as

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix} x_2 \\ -\frac{\cos x_1}{\sin x_1}x_2^2 - \frac{c}{m}x_2 - \frac{k}{m}\frac{\cos\theta_0 - \cos x_1}{\sin x_1}\end{bmatrix}\\ &\quad - \begin{bmatrix} 0 \\ \frac{\cos x_1}{4ml\sin^2 x_1}\end{bmatrix}F.\end{aligned} \tag{17}$$

To linearize (17), we evaluate the Jacobian as

$$J = \begin{bmatrix} 0 & 1 \\ \frac{x_2^2}{\sin^2 x_1} + \frac{k\cos\theta_0}{(\sin x_1)(\cos x_1)} - \frac{k}{\sin^2 x_1} & \frac{-2x_2^2}{\tan x_1} - \frac{c}{m}\end{bmatrix}. \tag{18}$$

However, J does not exist at $x_1 = 0,\ x_2 = 0$, and thus the system does not have a linearization at the horizontal equilibrium. Hence, we design nonlinear controllers for the two-bar linkage system in the later sections. In the next section we analyze the hysteretic behavior of the system.

IV. Hysteresis in the Two-bar Linkage

The hysteresis map of a system is the response of the system in the limit of DC operation, that is, the response under periodic inputs with frequency approaching zero [4]. It is shown in [27] that a system that exhibits hysteresis has a multi-valued equilibrium map and that the hysteresis map is a subset of the equilibrium map. The existence of multiple equilibria for a two-bar linkage suggests that the system may be hysteretic. We simulate the linkage dynamics given by (12) under the periodic external force $F = \sin(\omega t)$ N using

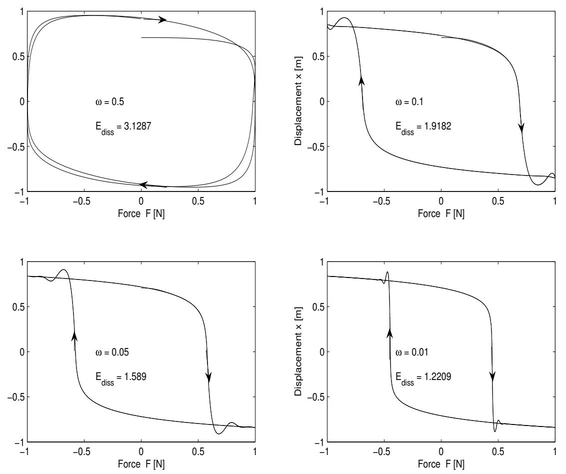


Fig. 4. Input-output maps between the vertical force F and the vertical displacement x for the two-bar linkage model (12) for various values of frequency ω in rad/s. The nonvanishing clockwise displacement-force loop at asymptotically low frequencies is the hysteresis map. $E_{\rm diss}$ is the energy dissipated by the dashpot in one complete cycle. The parameters used are $k = 1$ N/m, $m = 1$ kg, $c = 1$ N-s/m, $m_{\rm bar} = 0.5$ kg, $l = 1$ m, and $F(t) = \sin(\omega t)$ N.

the parameter values $k = 1$ N/m, $m = 1$ kg, $c = 1$ N-s/m, $m_{\rm bar} = 0.5$ kg, and $l = 1$ m. As shown in Figure 4 there exists a nontrivial clockwise hysteresis map from the vertical force F to the vertical displacement x (equivalent to a counterclockwise map from the the vertical displacement x to the vertical force F) at low frequencies. The vertical displacement is $x = l\sin\theta$. The presence of a nontrivial loop at asymptotically low frequencies constitutes hysteresis. For details see [4].

V. Feedback Linearization Control of the Two-bar Linkage

The linkage dynamic model (16) has an unstable equilibrium at $\theta = 0$ leading to snap-through behavior. What makes the model challenging to stabilize is that it does not have a linearization at $\theta = 0$, which in turn is due to the presence of $\sin\theta$ in the coefficients of $\ddot{\theta}$ and $\dot{\theta}$.

To stabilize the linkage at $\theta = 0$, we use feedback linearization to generate a control signal that makes θ decay according to the second-order system

$$\ddot{\theta} = -a\dot{\theta} - b\theta, \tag{19}$$

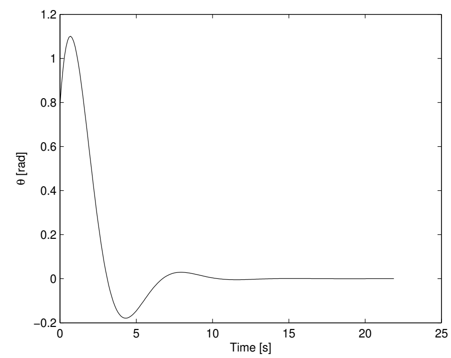
where $a > 0$ and $b > 0$. Substituting (19) into (16) we obtain

$$m\big[(\sin^2\theta)(-a\dot{\theta} - b\theta) + (\sin\theta)(\cos\theta)\dot{\theta}^2\big] + c(\sin^2\theta)\dot{\theta} + k(\sin\theta)(\cos\theta_0 - \cos\theta) = -\frac{\cos\theta}{4l}F.$$

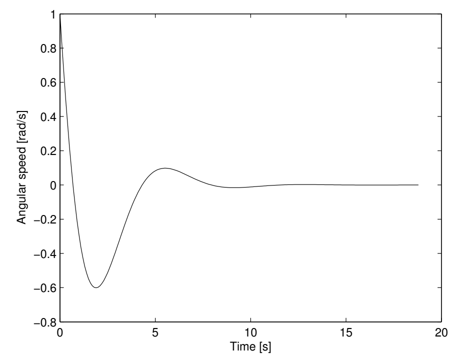
Solving for F yields

$$F = -\frac{4l}{\cos\theta}\big[m(\sin\theta)(\cos\theta)\dot{\theta}^2 - (ma - c)(\sin^2\theta)\dot{\theta} - mb(\sin^2\theta)\theta + k\sin\theta(\cos\theta_0 - \cos\theta)\big]. \tag{20}$$

To illustrate (20) we choose the parameter values $a = 1$ s^{-1}, $b = 1$ s^{-2}, $m = 1$ kg, $c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, and $\theta_0 = \frac{\pi}{4}$ rad. For the initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s, the time histories of θ, $\dot{\theta}$, and control input F are shown in figures 5 and 6, respectively.



(a)



(b)

Fig. 5. Time histories of θ and $\dot{\theta}$ under the action of the feedback linearization controller (20) with initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s. The controller stabilizes the horizontal equilibrium position. The parameter values used are $a = 1$ s^{-1}, $b = 1$ s^{-2}, $m = 1$ kg, $c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, and $\theta_0 = \frac{\pi}{4}$ rad.

VI. Intrinsic Nonlinear PD (INPD) and Robust-INPD Controller

Since the feedback-linearization controller in Section V requires knowledge of all of the linkage parameters, we now develop an intrinsic nonlinear PD controller based on the theory given in [23], [28]. In [23], a robust nonlinear controller is given for fully actuated mechanical systems. Since the two-bar linkage is fully actuated, the theory given in [23], [28] is applicable.

The control law given in [23] requires the construction of an error function as well as functions Φ and Ψ. First, note that the configuration space of the preloaded two-bar linkage has the topology of S^1. Let $\theta \in [-\pi, \pi)$ parameterize S^1, and let $\theta_s \in [-\pi, \pi)$ denote the desired setpoint. Then, one choice [28] of an error function $e : S^1 \to \mathbb{R}$ is given by $e(\theta) = 1 - \cos(\theta_s - \theta)$.

To stabilize $\theta_s = 0$, the error function is given by $e(\theta) = 1 - \cos\theta$. We next choose the functions $\Phi(\theta) = k_1 l\theta$ and

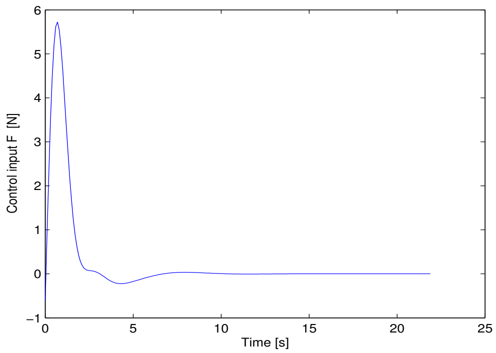


Fig. 6. Control input F given by (20) with initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s. The parameter values used are $a = 1$ s^{-1}, $b = 1$ s^{-2}, $m = 1$ kg, $c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, and $\theta_0 = \frac{\pi}{4}$ rad.

$\Psi(\theta, \dot{\theta}) = k_2 l\dot{\theta}$, where $k_1, k_2 > 0$. Then, the control law in [23] for the preloaded two-bar linkage is given by

$$F = -\frac{1}{\cos\theta}u, \tag{21}$$

where

$$u = -\frac{\Phi'(e(\theta))}{l}\frac{\partial e}{\partial\theta}(\theta) + \frac{1}{l}\frac{\partial V}{\partial\theta}(\theta) - \frac{\Psi(\theta,\dot{\theta})}{l} + \frac{u_f}{l}, \tag{22}$$

$V \triangleq 2kl^2(\cos\theta - \cos\theta_0)^2$ is the potential energy and $u_f \triangleq 4cl^2\sin^2(\theta)\dot{\theta}$ is the energy dissipated by the damping force. Using (22), we obtain the INPD controller

$$u = -\big[k_1 + 4kl(\cos\theta - \cos\theta_0)\big]\sin\theta - \big(k_2 - 4cl\sin^2\theta\big). \tag{23}$$

The controller (23) almost globally asymptotically stabilizes the desired equilibrium with local exponential convergence. Since we do not require knowledge of the mass m or the linkage mass $m_{\rm bar}$ of the system, (23) is unconditionally robust with respect to the inertia parameters of the system.

Although the INPD controller presented in (23) is unconditionally robust with respect to inertia parameters, it requires complete knowledge of the potential function $V(\theta)$ given in (10). We now remove this limitation by presenting a more robust version of the INPD controller (R-INPD) that requires less modeling information than the controller (23).

Theorem 6.1: Consider the two-bar linkage model (12), and choose F as in (21) and

$$u = -k_p\sin\theta - k_d\dot{\theta}, \tag{24}$$

where $k_p > 4kl$ and $k_d \geq 0$. Then there exists a positive-definite function $\mathcal{V} : S^1 \times \mathbb{R} \to \mathbb{R}$ such that the equilibrium $(\theta, \dot{\theta}) = (0, 0)$ is asymptotically stable with a domain of attraction that contains a sublevel set of $\mathcal{V}$. Furthermore, the set $\{(\theta, \dot{\theta}) \in (-\pi/2, \pi/2) \times \mathbb{R} : \dot{\theta} = 0\}$ is contained in the domain of attraction. Finally, if $k_d > 0$, then the closed-loop solutions converge to the equilibrium $(\theta, \dot{\theta}) = (0, 0)$ locally exponentially fast. □

The R-INPD controller (24) requires no knowledge of the system parameters other than an upper bound on kl. The gains k_p and k_d in the controller (24) are analogous to linear PD gains, and hence the closed-loop response can be modified by adjusting the values of these gains appropriately.

Next, consider the closed-loop dynamics obtained by substituting (24) into (16) yielding

$$m\big[(\sin^2\theta)\ddot{\theta} + (\sin\theta)(\cos\theta)\dot{\theta}^2\big] + c(\sin^2\theta)\dot{\theta} + k(\sin\theta)(\cos\theta_0 - \cos\theta) = -\frac{\cos\theta}{4l}\frac{k_p\sin\theta + k_d\dot{\theta}}{\cos\theta}.$$

To determine the static equilibria, we set all derivatives to zero, yielding

$$k(\sin\theta)\Big(\cos\theta_0 + \frac{k_p}{4kl} - \cos\theta\Big) = 0. \tag{25}$$

Note that (25) holds for $\theta = 0$ as well as for θ that satisfies

$$\cos\theta = \frac{lk_p}{4kl^2} + \cos\theta_0, \tag{26}$$

which is identical to (8), where lk_p plays the role of the torsional stiffness k_t. There exists $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ satisfying (26) if and only if $lk_p < 4kl^2(1 - \cos\theta_0)$, that is, $k_p < 4kl(1 - \cos\theta_0)$. Since $k_p > 4kl$ in the controller (24), it follows that (26) has no solution and thus the linkage has exactly one asymptotically stable equilibrium at $\theta = 0$ under the action of the controller.

To illustrate the R-INPD controller (24) we use the same parameters and initial conditions as in Section V. Let $m = 1, c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, $k_p = 10$ N-m, $k_d = 2$ N-m-s, and $\theta_0 = \frac{\pi}{4}$. For the initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s, the time histories of θ, $\dot{\theta}$, and control input F are shown in figures 7 and 8, respectively. The closed-loop system for the R-INPD controller converges to the equilibrium $(0, 0)$ in 5 seconds.. These may be compared with results for the feedback linearization based controller presented in figures 5 and 6 for the same initial conditions.

VII. Conclusion

We studied a preloaded two-bar linkage, which serves as a lumped analogue of a structure that can undergo snap-through buckling. We showed that the linkage exhibits hysteresis between the force actuation and the vertical displacement. We also showed that the two-bar linkage with massless bars has an inertia singularity and thus is not linearizable at the horizontal (unstable) equilibrium. Finally, we presented an INPD and a R-INPD controller that stabilizes the unstable equilibrium with less parametric knowledge of the system.

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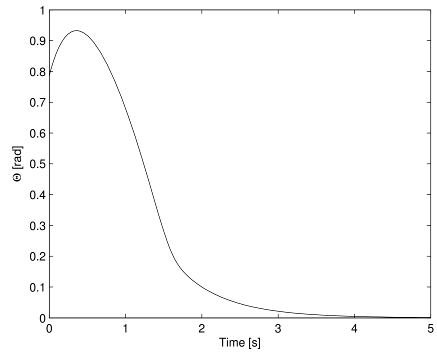

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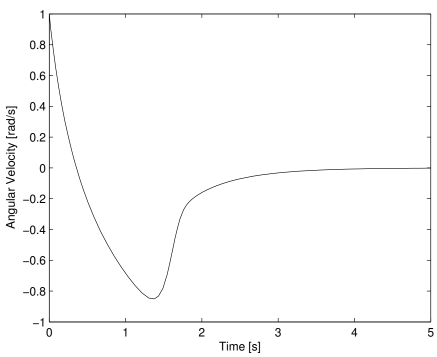

(b)

Fig. 7. Time histories of θ and $\dot{\theta}$ under the action of the nonlinear controller (24) with initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s. The controller stabilizes the horizontal equilibrium position. The parameter values used are $k_p = 5$ N-m, $k_d = 2.5$ N-m-s, $m = 1$ kg, $c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, and $\theta_0 = \frac{\pi}{4}$ rad.

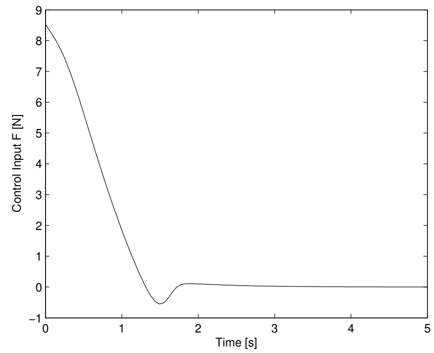

Fig. 8. Control input F given by (24) with initial conditions $\theta(0) = \frac{\pi}{4}$ rad and $\dot{\theta}(0) = 1$ rad/s. The parameter values used are $k_p = 5$ N-m, $k_d = 2.5$ N-m-s, $m = 1$ kg, $c = 1$ N-s/m, $k = 1$ N/m, $l = 1$ m, and $\theta_0 = \frac{\pi}{4}$ rad.

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