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Discrete-Time Trailing Horizon Direct Adaptive Disturbance Rejection

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***Abstract*— In this paper, we develop a discrete-time direct adaptive feedback disturbance rejection algorithm that does not require modeling of the disturbance. This method is applicable to minimum phase and non-minimum phase plants. We provide preliminary stability results and an intuitive argument for the overall stability of the algorithm.**

1. Introduction

Much of the adaptive control literature is devoted to proving stability of the closed-loop system and boundedness of solutions, [1–6]. In many applications, however, the plant is open-loop stable and the overriding concern is performance with respect to disturbance rejection [7–9]. Discrete-time adaptive disturbance rejection has broad engineering and scientific applications. It is most relevant in active noise and vibration control [10, 11]. Disturbance rejection algorithms can be of two types, feedforward or feedback. The latter generally exhibit superior performance since they take into account the effect of the feedback path from control to measurement.

In this paper we develop the trailing horizon method for output feedaback disturbance rejection. This method is valid for minimum phase and nonminimum phase plants. A distinguishing feature of this method is the computation of control over a finite horizon in the *past*, thus the name trailing horizon control. To clarify the differences between conventional one step ahead adaptive control methods [3], multiple step ahead control (generally known as predictive control) methods [12–16] and the method proposed in this paper, we first give a brief description of the predictive control design methodology.

In non-adaptive model predictive control, a model of the plant is used to predict the future outputs as a function of the future inputs over a finite horizon. The control inputs, over the prediction horizon, are then computed by minimizing a quadratic cost function that penalizes the predicted error between the desired and predicted outputs and the control effort required. Only the first control value from the computed control horizon is then used, and the prediction and optimization procedure is repeated at the next step.

In a adaptive model predictive control scheme (APCS) the plant model parameters are identified online and then, using the certainty equivalence principle, are used as if they were the true plant parameters. Since APCS is an indirect adaptive control scheme, this method requires *a priori* assumptions on the parameter estimates.

The proposed trailing horizon adaptive control method does not attempt to predict future outputs. Instead this method uses a window of past inputs and outputs and the closed-loop system is re-parameterized in terms of the controller parameters, thus enabling the use of direct adaptive control.

2. Plant Model

Consider the discrete-time LTI system given by

$$x(k+1) = Ax(k) + Bu(k) + D_1 w(k) \tag{2.1}$$

$$z(k) = E_1 x(k) + E_2 u(k) + E_0 w(k) \tag{2.2}$$

where $k = 0, 1, 2, \ldots$, the state vector $x(k) \in \mathbb{R}^n$, the control vector $u(k) \in \mathbb{R}^m$, the disturbance vector $w(k) \in \mathbb{R}^p$, and the performance vector $z(k) \in \mathbb{R}^l$. We make the following assumptions.

Assumption 2.1: An upper bound on the order n of the plant is known.

Assumption 2.2: The triple (A, B, E_1) is controllable and observable.

Assumption 2.3: The number of inputs is greater than or equal to the number of performance variables, i.e., $m \geq l$.

Assumption 2.4: The disturbance $w(k)$ is generated by

$$x_w(k+1) = A_w x_w(k), \quad w(k) = C_w x_w(k),$$

where $A_w \in \mathbb{R}^{n_w \times n_w}$ is Lyapunov Stable.

Assumption 2.5: For all $\lambda \in \text{spec}\,(A_w)$

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ E_1 & E_2 \end{bmatrix} = n + \min(m, l) = n + l,$$

i.e. the plant transfer function matrix from u to z, denoted by G_{zu}, has no transmission zeros at the disturbance modes.

Assumption 2.6: An upper bound on the order n_w of the disturbance is known.

Assumption 2.7: The performance $z(k)$ is measured.

Propagating the state backwards for $n+\mu-1$ times steps, equations (2.1) and (2.2) can be combined to yield

$$z(k) = E_1 A^{n+\mu-1} x(k-n-\mu+1) + \sum_{j=0}^{n+\mu-1} \Omega_j w(k-j) + \sum_{j=0}^{n+\mu-1} \Lambda_j u(k-j), \tag{2.3}$$

where $\mu \geq 1$ and the Markov parameters from the disturbance $w(k)$ to the performance $z(k)$ are given by $\Omega_0 \triangleq E_0$

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and, for $j = 1, \ldots, n+\mu-1$, $\Omega_j \triangleq E_1 A^{j-1} D_1$. The Markov parameters from the control $u(k)$ to the performance $z(k)$ are given by $\Lambda_0 \triangleq E_2$ and, for $j = 1, \ldots, n+\mu-1$, $\Lambda_j \triangleq E_1 A^{j-1} D_1$. By splitting the control window we can rewrite (2.3) as

$$z(k) = E_1 A^{n+\mu-1} x(k-n-\mu+1) + \sum_{j=0}^{n+\mu-1} \Omega_j w(k-j) + \sum_{j=\mu}^{n+\mu-1} \Lambda_j u(k-j) + \sum_{j=0}^{\mu-1} \Lambda_j u(k-j). \quad (2.4)$$

Assumption 2.8: $\mathcal{B}_{zu} \triangleq \begin{bmatrix} \Lambda_0 & \cdots & \Lambda_{n+\mu-1} \end{bmatrix}$ is known.

Assumptions 2.1-2.8 are applicable for the remainder of this paper. Additional assumptions will be made where necessary.

3. Controller Model

Consider the instantaneously linear controller

$$u(k) = -\sum_{i=1}^{\overline{n}_c} \Xi_i(k) u(k-\mu-i+1) + \sum_{i=1}^{\overline{n}_c+\mu-1} \Upsilon_i(k) z(k-i), \quad (3.1)$$

where, for all $k \geq 0$, $\Xi_1(k), \ldots, \Xi_{\overline{n}_c}(k) \in \mathbb{R}^{m\times m}$ and $\Upsilon_1(k), \ldots, \Upsilon_{\overline{n}_c+\mu-1}(k) \in \mathbb{R}^{m\times l}$ are determined by the controller parameter estimator described in Section 5. Next, define $q_1 \triangleq \overline{n}_c m$, $q_2 \triangleq (\overline{n}_c+\mu-1)l$, $q_3 \triangleq q_1+q_2$,

$$U(k) \triangleq \begin{bmatrix} u(k-\mu) \\ \vdots \\ u(k-\overline{n}_c-\mu+1) \end{bmatrix} \in \mathbb{R}^{q_1}, \quad (3.2)$$

$$Z(k) \triangleq \begin{bmatrix} z(k-1) \\ \vdots \\ z(k-\overline{n}_c-\mu+1) \end{bmatrix} \in \mathbb{R}^{q_2}, \quad (3.3)$$

and $\phi(k) \triangleq \begin{bmatrix} U(k) \\ Z(k) \end{bmatrix} \in \mathbb{R}^{q_3}$.

Assumption 2.7 implies that $Z(k)$ and thus $\phi(k)$ are known and can be used for feedback.

Next, define the controller parameter matrix

$$\Theta(k) \triangleq \begin{bmatrix} -\Xi_1(k) & \cdots & -\Xi_{\overline{n}_c}(k) & \Upsilon_1(k) & \cdots & \Upsilon_{\overline{n}_c+\mu-1}(k) \end{bmatrix}.$$

With this notation, (3.1) can be written as

$$u(k) = \Theta(k)\phi(k). \quad (3.4)$$

4. Closed-Loop Plant Model

The closed-loop performance (2.4), (3.4) is given by

$$z(k) = E_1 A^{n+\mu-1} x(k-n-\mu+1) + \sum_{j=0}^{n+\mu-1} \Omega_j w(k-j) + \sum_{j=\mu}^{n+\mu-1} \Lambda_j u(k-j) + \Psi^{\mathrm{T}}(k)\theta(k), \quad (4.1)$$

where,

$$\Psi(k) \triangleq \begin{bmatrix} \phi^{\mathrm{T}}(k)\otimes\Lambda_0 & \cdots & \phi^{\mathrm{T}}(k-\mu+1)\otimes\Lambda_{\mu-1} \end{bmatrix}^{\mathrm{T}}$$

$$\theta(k) \triangleq \begin{bmatrix} \mathrm{vec}\,\Theta(k) \\ \vdots \\ \mathrm{vec}\,\Theta(k-\mu+1) \end{bmatrix} \in \mathbb{R}^{q_4},$$

$q_4 \triangleq m q_3 \mu$, and $\otimes$ denotes the Kronecker product.

Now, we make an assumption regarding the existence of a constant gain controller Θ^* that achieves deadbeat disturbance rejection in μ steps. This assumption is slightly stronger than the proven results on deadbeat disturbance rejection given by [17, Corollary 4.1].

Assumption 4.1: For all $\mu \geq 2(n+n_w l)$ there exists $\Theta^* \in \mathbb{R}^{m\times q_3}$ such that, if $k \geq \mu-1$ and $\Theta(k-\mu+1) = \cdots = \Theta(k) = \Theta^*$, then

$$0 = E_1 A^{n+\mu-1} x(k-n-\mu+1) + \sum_{j=0}^{n+\mu-1} \Omega_j w(k-j) + \sum_{j=\mu}^{n+\mu-1} \Lambda_j u(k-j) + \sum_{j=0}^{\mu-1} \Lambda_j \Theta^* \phi(k-j) \quad (4.2)$$

$$= E_1 A^{n+\mu-1} x(k-n-\mu+1) + \sum_{j=0}^{n+\mu-1} \Omega_j w(k-j) + \sum_{j=\mu}^{n+\mu-1} \Lambda_j u(k-j) + \Psi^{\mathrm{T}}(k)\theta^*, \quad (4.3)$$

where $\theta^* \triangleq \begin{bmatrix} \mathrm{vec}\,\Theta^* \\ \vdots \\ \mathrm{vec}\,\Theta^* \end{bmatrix} \in \mathbb{R}^{q_4}$. Furthermore, $\{u^*(k)\}_{k=0}^{\infty}$ is bounded where $u^*(k) \triangleq \Theta^*\phi(k)$.

Assumption 4.1 applies to the remainder of this paper. Using (4.1) and (4.3) we obtain

$$z(k) = \Psi^{\mathrm{T}}(k)\theta(k) - \Psi^{\mathrm{T}}(k)\theta^*. \quad (4.4)$$

We have thus used Lemma 4.1 to express the closed-loop performance (4.4) as the estimation error $\Psi^{\mathrm{T}}(k)\theta(k) - \Psi^{\mathrm{T}}(k)\theta^*$. However, the adaptive controller (3.4) is not claimed to converge to Θ^* or achieve deadbeat disturbance rejection.

5. Estimator

The adaptive controller shown in Figure 5 consists of an instantaneously linear controller $G_{\mathrm{c}}(k)$ given by (3.4) and a parameter update law that modifies the controller parameters at each time step k. To obtain the parameter update law we first define the cost function

$$J(k, \hat{\theta}(k-\mu), \ldots, \hat{\theta}(k+1)) \triangleq \sum_{j=k-\mu-1}^{k} \left[\Psi^{\mathrm{T}}(j)\tilde{\theta}(j+1)\right]^{\mathrm{T}} \left[\Psi^{\mathrm{T}}(j)\tilde{\theta}(j+1)\right], \quad (5.1)$$

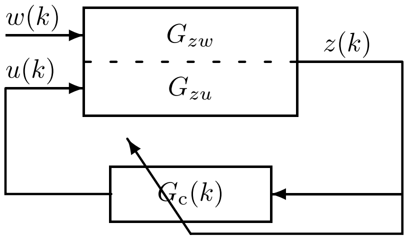

Fig. 1. The Closed-Loop System

where $\hat{\theta}(\cdot) \in \mathbb{R}^{q_4}$ is the estimation variable and $\tilde{\theta}(k) \triangleq \hat{\theta}(k) - \theta^*$.

Then we use a recursive least squares estimate of $\hat{\theta}(k+1)$ to minimize (5.1); for details see, for example, [18]. For all $k \geq \mu - 1$ the RLS estimate for $\hat{\theta}(k+1)$ is given by

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \mathcal{P}(k+1)\Psi(k)z(k), \tag{5.2}$$

$$\mathcal{P}(k+1) = \mathcal{P}(k) - \mathcal{P}(k)\Psi(k)\left(I + \Psi^{\mathrm{T}}(k)\mathcal{P}(k)\Psi(k)\right)^{-1} \times \Psi^{\mathrm{T}}(k)\mathcal{P}(k), \tag{5.3}$$

where $\mathcal{P}(\mu-1) > 0$. Then, the controller is updated according to vec $\Theta(k+1) = \left[\ I_{mq_3} \quad 0\ \right]\hat{\theta}(k+1)$.

6. Preliminary Stability Results

In this section, we provide preliminary stability results and an intuitive argument for the overall stability of the algorithm. Define the state vector

$$X(k) \triangleq \begin{bmatrix} Z(k) \\ \tilde{\theta}(k) \\ \mathrm{vec}\mathcal{P}(k) \end{bmatrix} \in \mathbb{R}. \tag{6.1}$$

Then using (4.4), (5.2) and (5.3) for all $k > \mu - 1$ the closed-loop system can be represented by the state equation

$$Z(k+1) = AZ(k) + B\Psi^{\mathrm{T}}(k)\left[\theta(k) - \theta^*\right], \tag{6.2}$$

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \mathcal{P}(k+1)\Psi(k)\Psi^{\mathrm{T}}(k)\tilde{\theta}(k), \tag{6.3}$$

$$\mathrm{vec}\left[\mathcal{P}(k+1)\right] = \mathrm{vec}\Big[\mathcal{P}(k) - \mathcal{P}(k)\Psi(k) \times \left[1 + \Psi^{\mathrm{T}}(k)\mathcal{P}(k)\Psi(k)\right]^{-1}\Psi^{\mathrm{T}}(k)\mathcal{P}(k)\Big], \tag{6.4}$$

where $A \triangleq \left[\begin{array}{cc} \multicolumn{2}{c}{0_{l\times q_2}} \\ \hline I_{(q_2-l)\times(q_2-l)} & 0_{(q_2-l)\times l} \end{array}\right] \in \mathbb{R}^{q_2\times q_2}$ is nilpotent and thus asymptotically stable, and $B \triangleq \begin{bmatrix} I_{l\times l} \\ 0_{(q_2-l)\times l} \end{bmatrix} \in \mathbb{R}^{q_2\times l}$. Note that every equilibrium of the error system (6.2)-(6.4) is of the form $(0, \tilde{\theta}_e, \mathcal{P}_e)$, where $\mathcal{P}_e$ is positive semi definite.

Now, we provide the preliminary stability results.

Lemma 6.1: Define $V_{\mathcal{P}}(\mathcal{P}) \triangleq \mathrm{tr}\,\mathcal{P}^2$, $\triangle V_{\mathcal{P}}(k) \triangleq \mathrm{tr}\left[\mathcal{P}^2(k+1) - \mathcal{P}^2(k)\right]$, $V_{\tilde{\theta}}(\tilde{\theta},\mathcal{P}) \triangleq \tilde{\theta}^{\mathrm{T}}\mathcal{P}^{-1}\tilde{\theta}$, and

$$\triangle V_{\tilde{\theta}}(k) \triangleq \tilde{\theta}^{\mathrm{T}}(k+1)\mathcal{P}^{-1}(k+1)\tilde{\theta}(k+1) - \tilde{\theta}^{\mathrm{T}}(k)\mathcal{P}^{-1}(k)\tilde{\theta}(k). \tag{6.5}$$

Then, for all $k \geq \mu - 1$,

$$\triangle V_{\mathcal{P}}(k) \leq 0, \tag{6.6}$$

$$\triangle V_{\tilde{\theta}}(k) = -\tilde{\theta}^{\mathrm{T}}(k)\Psi(k)\left[I + \Psi^{\mathrm{T}}(k)\mathcal{P}(k)\Psi(k)\right]^{-1} \times \Psi^{\mathrm{T}}(k)\tilde{\theta}(k) \tag{6.7}$$

$$\leq \frac{-\left\|\hat{z}(k)\right\|_2^2}{1 + \tau\gamma\sum_{j=0}^{\mu-1}\left(\left\|\mathcal{U}(k-j)\right\|_2^2 + \left\|Z(k-j)\right\|_2^2\right)}, \tag{6.8}$$

where $\hat{z}(k) \triangleq \Psi^{\mathrm{T}}(k)\tilde{\theta}(k)$, $\gamma \triangleq \lambda_{\max}\left[\mathcal{P}(0)\right]$, $\tau \triangleq \max_{0\leq j\leq\mu-1}\sigma_{\max}\left[\Lambda_j\right]$, and

$$\mathcal{U}(k) \triangleq \begin{bmatrix} u(k-1) \\ \vdots \\ u(k-n_c-\mu+1) \end{bmatrix}, \tag{6.9}$$

Furthermore,

$$\lim_{k\to\infty} \triangle V_{\tilde{\theta}}(k) = 0, \tag{6.10}$$

and $\lim_{k\to\infty}\tilde{\theta}(k)$ and $\lim_{k\to\infty}\mathcal{P}(k)$ exist.

Proof. The results (6.6), (6.7), (6.10), and the convergence of $\{\tilde{\theta}(k)\}_{k=0}^{\infty}$ and $\{\mathcal{P}(k)\}_{k=0}^{\infty}$ follow from standard properties of RLS, see [3, p. 60], [19, p. 22], [20, p. 58] and [14, p. 202].

Since $\mathcal{P}(k) \leq \mathcal{P}(\mu-1)$ for all $k \geq \mu-1$ it follows that

$$\triangle V_{\tilde{\theta}}(k) \leq -\hat{z}^{\mathrm{T}}(k)\left[I + \gamma\Psi^{\mathrm{T}}(k)\Psi(k)\right]^{-1}\hat{z}(k).$$

Furthermore,

$$\begin{aligned}
\Psi^{\mathrm{T}}(k)\Psi(k) = &\left[\ \phi^{\mathrm{T}}(k)\otimes\Lambda_0 \quad \cdots \quad \phi^{\mathrm{T}}(k-\mu+1)\otimes\Lambda_{\mu-1}\right] \times \begin{bmatrix} \phi(k)\otimes\Lambda_0^{\mathrm{T}} \\ \vdots \\ \phi(k-\mu+1)\otimes\Lambda_{\mu-1}^{\mathrm{T}} \end{bmatrix} \\
=&\left[\ \phi^{\mathrm{T}}(k)\phi(k)\otimes\Lambda_0\Lambda_0^{\mathrm{T}} \quad \cdots \quad \phi^{\mathrm{T}}(k-\mu+1)\phi(k-\mu+1)\otimes\Lambda_{\mu-1}\Lambda_{\mu-1}^{\mathrm{T}}\right] \\
=&\left[\ \phi^{\mathrm{T}}(k)\phi(k)\Lambda_0\Lambda_0^{\mathrm{T}} \quad \cdots \quad \phi^{\mathrm{T}}(k-\mu+1)\phi(k-\mu+1)\Lambda_{\mu-1}\Lambda_{\mu-1}^{\mathrm{T}}\right] \\
=&\sum_{j=0}^{\mu-1}\left\|\phi(k-j)\right\|_2^2\Lambda_j\Lambda_j^{\mathrm{T}} \\
\leq&\,\tau\sum_{j=0}^{\mu-1}\left\|\phi(k-j)\right\|_2^2 I_l.
\end{aligned}$$

Therefore, we can write

$$\begin{aligned}
\triangle V_{\tilde{\theta}}(k) \leq& \frac{-\hat{z}^{\mathrm{T}}(k)\hat{z}(k)}{1 + \tau\gamma\sum_{j=0}^{\mu-1}\left\|\phi(k-j)\right\|_2^2} \\
\leq& \frac{-\left\|\hat{z}(k)\right\|_2^2}{1 + \tau\gamma\sum_{j=0}^{\mu-1}\left(\left\|\mathcal{U}(k-j)\right\|_2^2 + \left\|Z(k-j)\right\|_2^2\right)}.
\end{aligned}$$

□

Lemma 6.2: Let $P, R \in \mathbb{R}^{q_2\times q_2}$ be positive-definite matrices that satisfy

$$P = A^{\mathrm{T}}PA + R + I. \tag{6.11}$$

Furthermore, let $\varsigma > 0$ and define

$$V_Z(Z) \triangleq \ln(1+\varsigma Z^{\mathrm{T}}PZ),$$
$$\triangle V_Z(k) \triangleq V_Z(Z(k+1)) - V_Z(Z(k)).$$

Then

$$\triangle V_Z(k) \le -\varsigma\frac{Z^{\mathrm{T}}(k)RZ(k)}{1+\rho\varsigma\|Z(k)\|_2^2} + \varsigma\frac{(\sigma^2+1)\beta\|z(k)\|_2^2}{1+\rho\varsigma\|Z(k)\|_2^2},$$

where $\sigma \triangleq \lambda_{\max}^{1/2}(A^{\mathrm{T}}PA)$, $\rho \triangleq \lambda_{\min}(P)$, and $\beta \triangleq \lambda_{\max}\left(B^{\mathrm{T}}PB\right)$.

Proof. Define $F \triangleq \frac{1}{\sigma}P^{1/2}A$, $G \triangleq \sigma P^{1/2}B$, and $\mathcal{J}(Z) \triangleq Z^{\mathrm{T}}PZ$. Then,

$$\begin{aligned}\triangle\mathcal{J}_Z(k) &\triangleq Z^{\mathrm{T}}(k+1)PZ(k+1) - Z^{\mathrm{T}}(k)PZ(k)\\ &= Z^{\mathrm{T}}(k)A^{\mathrm{T}}PAZ(k) + Z^{\mathrm{T}}(k)A^{\mathrm{T}}PBz(k)\\ &\quad + z^{\mathrm{T}}(k)B^{\mathrm{T}}PAZ(k)\\ &\quad + z^{\mathrm{T}}(k)(k)B^{\mathrm{T}}PBz(k) - Z^{\mathrm{T}}(k)PZ(k).\end{aligned}$$

Adding and subtracting $Z^{\mathrm{T}}F^{\mathrm{T}}FZ$ and $z^{\mathrm{T}}G^{\mathrm{T}}Gz$, and omitting the explicit dependence on k we have

$$\begin{aligned}\triangle\mathcal{J}_Z(k) =& Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z\\ &- \begin{bmatrix} Z^{\mathrm{T}} & -z^{\mathrm{T}}\end{bmatrix}\begin{bmatrix} F^{\mathrm{T}}F & F^{\mathrm{T}}G\\ G^{\mathrm{T}}F & G^{\mathrm{T}}G\end{bmatrix}\begin{bmatrix} Z\\ -z\end{bmatrix}\\ &+ z^{\mathrm{T}}\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z\\ \le& Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z\\ &+ z^{\mathrm{T}}\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z.\end{aligned}$$

Noting that

$$F^{\mathrm{T}}F = \frac{A^{\mathrm{T}}PA}{\sigma^2} = \frac{A^{\mathrm{T}}PA}{\lambda_{\max}(A^{\mathrm{T}}PA)} \le \frac{\lambda_{\max}\left(A^{\mathrm{T}}PA\right)I_n}{\lambda_{\max}(A^{\mathrm{T}}PA)} = I_n,$$

it follows from (6.11) that $A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F \le A^{\mathrm{T}}PA - P + I = -R$. Therefore, $Z^{\mathrm{T}}\left(A^{\mathrm{T}}PA - P + F^{\mathrm{T}}F\right)Z \le -Z^{\mathrm{T}}RZ$, which implies that

$$\triangle\mathcal{J}_Z(k) \le -Z^{\mathrm{T}}(k)RZ(k) + z^{\mathrm{T}}(k)\left(B^{\mathrm{T}}PB + G^{\mathrm{T}}G\right)z(k). \tag{6.12}$$

Since $G^{\mathrm{T}}G = \sigma^2 B^{\mathrm{T}}PB$, it follows from (6.12) that

$$\triangle\mathcal{J}_Z(k) \le -Z^{\mathrm{T}}(k)RZ(k) + (\sigma^2+1)\beta\left[z^{\mathrm{T}}(k)z(k)\right].$$

Now, since $\ln x \le x - 1$ for all $x > 0$,

$$\begin{aligned}\triangle V_Z(k) =& \ln\left(1 + \varsigma\frac{\triangle\mathcal{J}_Z(k)}{1+\varsigma Z^{\mathrm{T}}(k)PZ(k)}\right)\\ \le& -\varsigma\frac{Z^{\mathrm{T}}(k)RZ(k)}{1+\rho\varsigma\|Z(k)\|_2^2} + \varsigma\frac{(\sigma^2+1)\beta\|z(k)\|_2^2}{1+\rho\varsigma\|Z(k)\|_2^2}.\end{aligned}$$

□

Now, we make an additional assumption.

Assumption 6.1: One of the following statements is true

(a) All $\lambda \in \mathbb{C}$ such that

$$\operatorname{rank}\begin{bmatrix} A-\lambda I & B\\ E_1 & E_2\end{bmatrix} < n + l, \tag{6.13}$$

$|\lambda| < 1$ i.e. the transmission zeros of G_{zu}, lie inside the unit circle.

(b) The matrix A is asymptotically stable and the matrix transfer functions G_{zu} and $G_{z\tilde{u}}$ have have no common unstable blocking zeros, where $\tilde{u}(k) \triangleq u(k) - u^*(k)$ and $G_{z\tilde{u}}$ denotes the matrix transfer functions from $\tilde{u}$ to z.

Lemma 6.3: Let Assumption 6.1 hold. Then that there exist $a_1 \ge 0$ and $a_2 \ge 0$ such that

$$\|\mathcal{U}(k)\|_2^2 + \|W(k)\|_2^2 \le a_1 + a_2\|Z(k)\|_2^2. \tag{6.14}$$

where

$$W(k) \triangleq \begin{bmatrix} w(k-1)\\ \vdots\\ w(k-n_c-\mu+1)\end{bmatrix}. \tag{6.15}$$

Proof. The performance (4.4) is

$$\begin{aligned}z(k) &= \Psi^{\mathrm{T}}(k)\hat{\theta}(k) - \Psi^{\mathrm{T}}(k)\theta^*\\ &= \sum_{j=0}^{\mu-1}\Lambda_j\hat{\Theta}(k-j)\phi(k-j) - \sum_{j=0}^{\mu-1}\Lambda_j\Theta^*\phi(k-j)\\ &= \sum_{j=0}^{\mu-1}\Lambda_j\left[u(k-j) - u^*(k-j)\right]\\ &= B_{zu}\begin{bmatrix} u(k)\\ \vdots\\ u(k-\mu+1)\end{bmatrix} - B_{zu}\begin{bmatrix} u^*(k)\\ \vdots\\ u^*(k-\mu+1)\end{bmatrix}\\ &= B_{zu}\begin{bmatrix} \tilde{u}(k)\\ \vdots\\ \tilde{u}(k-\mu+1)\end{bmatrix},\end{aligned} \tag{6.16}$$

where $B_{zu} \triangleq \begin{bmatrix} \Lambda_0 & \cdots & \Lambda_{\mu-1}\end{bmatrix} \in \mathbb{R}^{l\times m\mu}$.

From (6.16) it follows that $G_{z\tilde{u}}$ is FIR and thus asymptotically stable. Since $\{u^*(k)\}_{k=0}^{\infty}$ is bounded by Lemma 4.1, an unbounded $\{u(k)\}_{k=0}^{\infty}$ implies an unbounded $\{z(k)\}_{k=0}^{\infty}$ unless $u(k)$ is blocked by the blocking zeros of $G_{z\tilde{u}}$ (see Figure 2). Also an asymptotically stable A implies that an unbounded $\{u(k)\}_{k=0}^{\infty}$ results in an unbounded $\{z(k)\}_{k=0}^{\infty}$ unless $u(k)$ is blocked by the blocking zeros of G_{zu} (see Figure 5). If Assumption 6.1(b) holds then an unbounded mode of $u(k)$ cannot be blocked by both G_{zu} and $G_{z\tilde{u}}$. Therefore all unbounded modes are present in $z(k)$. Hence $z(k)$ grows unbounded at the same rate as $u(k)$ and there exist $a_2 \ge 0$ and $a_3 \ge 0$ such that $\|\mathcal{U}(k)\|_2^2 \le a_3 + a_2\|Z(k)\|_2^2$. Now since $\{w(k)\}_{k=0}^{\infty}$ is bounded by Assumption 2.4 we have $\|\mathcal{U}(k)\|_2^2 + \|W(k)\|_2^2 \le$

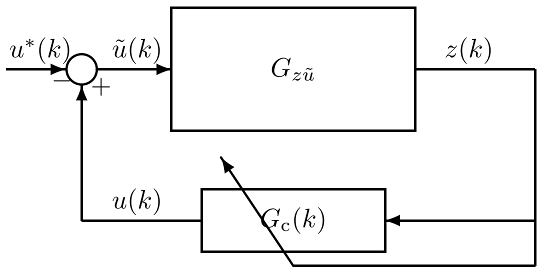

Fig. 2. Equivalent Closed-Loop System Representation

$a_1 + a_2 \|Z(k)\|_2^2$ where $a_1 = a_3 + \sup\{u(k)\}_{k=0}^{\infty}$. If G_{zu} is minimum phase then (6.14) follows from [3, p. 487]. □

Assumption 6.1(b) is often true since the matrix polynomial that determines the zeros of $G_{z\tilde{u}}$, i.e. $\Lambda_0 + \Lambda_1 \mathbf{q}^{-1} + \cdots + \Lambda_{\mu-1}\mathbf{q}^{-\mu+1}$ is obtained by truncating the last n coefficients of the matrix polynomial that determines the zeros of G_{zu}, i.e. $\Lambda_0 + \Lambda_1\mathbf{q}^{-1} + \cdots + \Lambda_{\overline{n}+\mu-1}\mathbf{q}^{-\overline{n}-\mu+1}$. Furthermore, since by Assumption 2.8 the matrices $\Lambda_0, \Lambda_1, \ldots, \Lambda_{\overline{n}+\mu-1}$ are known, Assumption 6.1(b) is verifiable.

Now, we present an intuitive argument for the overall stability of the closed-loop system (6.2)-(6.4)). If we assume that the performance $z(k)$ and the variable $\hat{z}(k)$ are equal, that is, $z(k) \equiv \hat{z}(k)$, then the following stability argument is valid.

From (6.8) it follows that

$$\triangle V_{\tilde{\theta}}(k) \le \frac{-\|\hat{z}(k)\|_2^2}{1 + \tau\gamma \sum_{j=0}^{\mu-1}\left(\|\mathcal{U}(k-j)\|_2^2 + \|Z(k-j)\|_2^2\right)} \le \frac{-\|\hat{z}(k)\|_2^2}{1 + \tau\gamma\sum_{j=0}^{\mu-1}\mathcal{M}(k-j)}, \tag{6.17}$$

where $\mathcal{M}(k-j) \triangleq \|\mathcal{U}(k-j)\|_2^2 + \|Z(k-j)\|_2^2 + \|W(k-j)\|_2^2$. Next, Assumption 6.1 implies

$$\|\mathcal{U}(k)\|_2^2 + \|W(k)\|_2^2 \le a_1 + a_2\|Z(k)\|_2^2. \tag{6.18}$$

Using (6.18) in (6.17) we have

$$\triangle V_{\tilde{\theta}}(k) \le \frac{-\|\hat{z}(k)\|_2^2}{1 + \mu a_1 \tau\gamma + \tau\gamma(1+a_2)\sum_{j=0}^{\mu-1}\|Z(k-j)\|_2^2} \le \frac{-\|\hat{z}(k)\|_2^2}{a_3 + a_4\sum_{j=0}^{\mu-1}\|Z(k-j)\|_2^2}, \tag{6.19}$$

where $a_3 \triangleq 1 + \mu a_1\tau\gamma$ and $a_4 \triangleq \tau\gamma(1+a_2)$. Now suppose that $\{\|z(k)\|\}_{k=0}^{\infty}$ is unbounded. Then it follows from Lemma 1.1 that there exist $a_5 > 0$ and $a_6 > 0$ such that the maximal monotonically increasing subsequence $\{\|Z(k_i)\|\}_{i=0}^{\infty}$ satisfies

$$\triangle V_{\tilde{\theta}}(k_i) \le \frac{-\|\hat{z}(k_i)\|_2^2}{a_5 + a_6\|Z(k_i)\|_2^2} \tag{6.20}$$

for all $i = 1, 2, \ldots$. Define

$$V(X) \triangleq V_{\mathcal{P}}(\mathcal{P}) + V_{\tilde{\theta}}(\tilde{\theta}, \mathcal{P}) + aV_Z(Z). \tag{6.21}$$

Then using lemmas 6.1 and 6.2 it follows that

$$\begin{aligned}\triangle V(k_i) &\triangleq V(X(k_{i+1})) - V(X(k_i)) \\ &\le \frac{-\|\hat{z}(k_i)\|_2^2}{a_5 + a_6\|Z(k_i)\|_2^2} - a\varsigma\frac{Z^{\mathrm{T}}(k_i)RZ(k_i)}{1 + \rho\varsigma\|Z(k_i)\|_2^2} \\ &\quad + a\varsigma\frac{(\sigma^2+1)\beta\|z(k_i)\|_2^2}{1+\rho\varsigma\|Z(k_i)\|_2^2}.\end{aligned} \tag{6.22}$$

Next choose $\varsigma = a_6/a_5\rho$ and $a = \rho/a_6\beta(\sigma^2+1)$ and use the assumption that $z(k) \equiv \hat{z}(k)$. Then

$$\triangle V(k_i) \le -a\varsigma\frac{Z^{\mathrm{T}}(k_i)RZ(k_i)}{1+\rho\varsigma\|Z(k_i)\|_2^2}. \tag{6.23}$$

Therefore $\{Z(k_i)\}_{i=0}^{\infty}$ is bounded, and we have

$$\triangle V(k_i) \le -a\varsigma\frac{Z^{\mathrm{T}}(k_i)RZ(k_i)}{1+\rho\varsigma\sup\{\|Z(k_i)\|_2^2\}}. \tag{6.24}$$

Now using (6.10) we have

$$\lim_{i\to\infty} a\varsigma\frac{Z^{\mathrm{T}}(k_i)RZ(k_i)}{1+\rho\varsigma\sup\{\|Z(k_i)\|_2^2\}} = 0. \tag{6.25}$$

which implies that $Z(k_i) \to 0$ as $i \to \infty$. Therefore $Z(k) \to 0$ as $k \to \infty$ and consequently $z(k) \to 0$ as $k \to \infty$. Finally, from (6.14) we conclude that $\{u(k)\}_{k=0}^{\infty}$ is bounded.

Thus, under the assumption that $z(k) \equiv \hat{z}(k)$, every equilibrium of (6.2)-(6.4)) is Lyapunov stable, $\{u(k)\}_{k=0}^{\infty}$ is bounded, and $z(k) \to 0$ as $k \to \infty$.

7. Acoustic Duct Example

Example 7.1: Consider the acoustic duct shown in Figure 3. We treat the duct as a one-dimensional waveguide

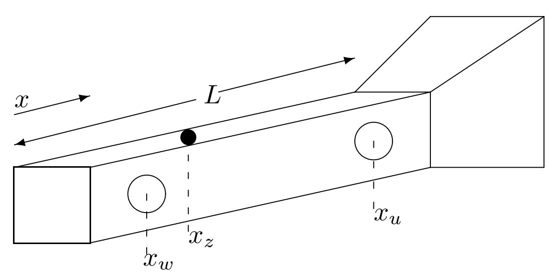

Fig. 3. Acoustic Duct

with spatial coordinate x, where $0 \le x \le L$. We use the mathematical model for the acoustic duct derived in [21], where the speed of acoustic waves is 343 m/s, the density of air is 1.21 kg/m^3, and the duct model includes five modes.

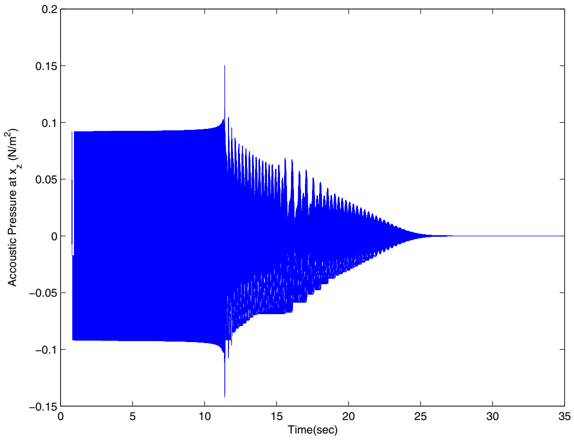

Fig. 4. Closed loop response with Trailing Horizon Controller

Let the disturbance speaker be placed at x_w, the control speaker at x_u, and the performance microphone at x_z. For $L = 6$ m, $x_w = 0.1$ m, $x_z = 0.3$ m, and $x_u = 5.95$ m the state equations are given in [6].

The modal frequencies of the duct are 85.4167 Hz, 170.8333 Hz, 256.25 Hz, 341.6667 Hz and 427.0833 Hz. The sample rate used is 2000 Hz and B_{zu} is identified using the method described in [22]. The disturbance speaker is excited at the modal frequency 427.0833 Hz. The closed-loop response with the trailing horizon algorithm is shown in Figure 4.

Appendix

Lemma 1.1: Let $\{\alpha(k)\}_{k=0}^{\infty}$ be a sequence of positive scalars. Let N be a positive integer, let $g_1 > 0$, $g_2 > 0$, and define

$$L(k) \triangleq g_1 + g_2 \sum_{j=0}^{N} \alpha(k-j). \tag{1.1}$$

Also, define the maximal monotonically increasing subsequence $\{\alpha(k_i)\}_{i=0}^{\infty}$ such that $\alpha(k) < \alpha(k_i)$ for all $k < k_i$. Then the following statements hold

1) If $\{\alpha(k)\}_{k=0}^{\infty}$ is bounded, then there exist $g_3 > 0$, $g_4 > 0$ such that, for all $k \geq 0$

$$L(k) \leq g_3 + g_4 \alpha(k). \tag{1.2}$$

2) If $\{\alpha(k)\}_{k=0}^{\infty}$ is unbounded then there exist $g_3 > 0$, $g_4 > 0$ such that for all $i = 1, 2, \ldots$ the maximal monotonically increasing subsequence $\{\alpha(k_i)\}_{i=0}^{\infty}$, satisfies

$$L(k_i) \leq g_3 + g_4 \alpha(k_i). \tag{1.3}$$

Proof. If $\{\alpha(k)\}_{k=0}^{\infty}$ is bounded, then (1.2) is satisfied with $g_3 = g_1 + (N+1) g_2 \sup_{k \geq 0} \alpha(k)$ and $g_4 > 0$. Now suppose that $\{\alpha(k)\}_{k=0}^{\infty}$ is unbounded, then (1.3) is satisfied with $g_3 = g_1$ and $g_4 = N + 1$. □

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