On the Zeros, Initial Undershoot, and Relative Degree of Lumped-Mass Structures

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Abstract—This paper considers lumped-mass structures where each mass in the structure has a single degree of freedom. Specifically, we analyze the zeros and relative degree of the single-input single-output (SISO) transfer function from the force applied to an arbitrary mass to the position, velocity, or acceleration of another mass.

1. INTRODUCTION

One of the main impediments to achievable performance in linear time-invariant control systems is the presence of nonminimum phase zeros [1, 2]. The role of nonminimum phase zeros in limiting both achievable performance and robust stability suggests the importance of understanding the mechanisms that give rise to such zeros in flexible structures. This issue is discussed in [3], where it is shown that nonminimum phase zeros arise in noncolocated transfer functions for beam models when multiple mechanisms are involved for energy transfer, for example, bending and torsion. Furthermore, it is shown in [4] that nonminimum phase zeros arise in noncolocated transfer functions for beam models when the dynamics are dispersive, as occurs in bending.

Graph theory can provide a systematic framework for analyzing structures and dynamical systems [5, 6]. In particular, [5] uses graph theory to derive expressions for the component forces at the vertices in Figure 1. In this section, we present definitions and basic results that are useful for analyzing the zeros of lumped-mass structures.

Let \( V = \{v_1, v_2, \ldots, v_N\} \). The \( N \) elements of \( V \) are vertices, and \( V \) is the vertex set. Define \( E \triangleq \{\{v_{n_1}, v_{n_2}\} : v_{n_1}, v_{n_2} \in V, v_{n_1} \neq v_{n_2}\} \), and let \( E \subseteq E \). The elements of \( E \) are edges, and \( E \) is the edge set. Since the elements of \( E \) are sets and thus are unordered, the edges do not have directions. Thus all graphs considered in this paper are undirected graphs. Furthermore, we do not consider multiple lines since the elements of \( E \) are distinct, and we do not consider loops since, for all \( i = 1, \ldots, N \), \( \{v_i, v_i\} \notin E \).

Definition 2.1. \( \mathcal{G} = (V, E) \) is a graph. If, in addition, for all \( \{v_i, v_j\} \in E \), a weight \( w_{i,j} > 0 \) is assigned to the edge \( \{v_i, v_j\} \), then \( \mathcal{G} \) is a weighted graph.

Definition 2.2. Let \( \mathcal{G} = (V, E) \) be a graph, and let \( v_{n_0}, v_{n_1} \in V \) be distinct. A walk of length \( l \) from \( v_{n_0} \) to \( v_{n_l} \) is the \( (l+1) \)-tuple \( (v_{n_0}, v_{n_1}, \ldots, v_{n_l}) \in V \times \cdots \times V \) such that, for all \( i = 1, 2, \ldots, l \), \( \{v_{n_{i-1}}, v_{n_i}\} \in E \).

Definition 2.3. The graph \( \mathcal{G} = (V, E) \) is connected if, for all distinct \( \alpha, \beta \in V \), there exists a walk between \( \alpha \) and \( \beta \).

The weighted adjacency matrix \( A_\mathcal{G} \in \mathbb{R}^{N \times N} \) associated with the weighted graph \( \mathcal{G} = (V, E) \) is defined as

\[
A_\mathcal{G} \triangleq \begin{bmatrix}
0 & w_{1,2} & w_{1,3} & \cdots & w_{1,N} \\
w_{2,1} & 0 & w_{2,3} & \cdots & w_{2,N} \\
w_{3,1} & w_{3,2} & 0 & \cdots & w_{3,N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
w_{N,1} & w_{N,2} & w_{N,3} & \cdots & 0
\end{bmatrix}, \tag{2.1}
\]

where, for all \( \{v_i, v_j\} \in E \), \( w_{i,j} = w_{j,i} > 0 \) is the weight assigned to the edge \( \{v_i, v_j\} \) and, for all \( \{v_i, v_j\} \notin E \), \( w_{i,j} = 0 \).

The Laplacian matrix \( L_\mathcal{G} \in \mathbb{R}^{N \times N} \) associated with the weighted graph \( \mathcal{G} = (V, E) \) is defined as

\[
L_\mathcal{G} \triangleq D_\mathcal{G} - A_\mathcal{G}, \tag{2.2}
\]

where \( D_\mathcal{G} \triangleq \text{diag} \left( \sum_{i=2}^{N} w_{i,1} + \sum_{i=3}^{N} w_{i,2} + \cdots + \sum_{i=1}^{N-1} w_{i,N} \right) \).

Next, we present two results concerning the Laplacian matrix. These results can be found in [7, p. 144] and [8, Theorem 3.16], respectively. In this paper, a matrix is positive semidefinite if it is symmetric with all nonnegative eigenvalues. Furthermore, a matrix is positive definite if it is symmetric with all positive eigenvalues.

Lemma 2.1. The Laplacian matrix \( L_\mathcal{G} \) is a singular, positive-semidefinite \( M \)-matrix.

Lemma 2.2. The graph \( \mathcal{G} = (V, E) \) is connected if and only if its Laplacian matrix \( L_\mathcal{G} \) is irreducible.

The following result, which concerns nonsingular \( M \)-matrices, is given by [9, Theorem 2.7].
Lemma 2.3. Let $A_M \in \mathbb{R}^{N \times N}$ be an irreducible Z-matrix. Then $A_M$ is a nonsingular M-matrix if and only if every entry of $A_M^{-1}$ is positive.

The next result of this section, which follows immediately from lemmas 2.1-2.3, is used to analyze the zeros of lumped-mass structures.

Lemma 2.4. Assume the graph $G=(V,E)$ is connected. Then the Laplacian matrix $L_2$ is an irreducible, singular, positive-semidefinite M-matrix. Furthermore, let $D \in \mathbb{R}^{N \times N}$ be positive definite and diagonal. Then $D + L_2$ is an irreducible, nonsingular M-matrix, and thus every entry of $(D + L_2)^{-1}$ is positive.

Now, we present results regarding the weighted adjacency matrices of two different graphs having the same vertex set; these results are used to analyze the relative degree of the transfer functions for lumped-mass structures. Let $E_1 \subseteq E$ and $E_2 \subseteq E$, and consider the weighted graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Let $A_{G_1} \in \mathbb{R}^{N \times N}$ and $A_{G_2} \in \mathbb{R}^{N \times N}$ be the weighted adjacency matrices associated with the weighted graph $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, respectively.

Lemma 2.5. Consider the graph $G_{12} = (V, E_1 \cup E_2)$, and let $v_{n_0}, v_{n_1} \in V$ be distinct. Then there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length $l$ on the graph $G_{12}$ between $v_{n_0}$ and $v_{n_1}$ if and only if

$$e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} > 0,$$

where, for all $i = 1, \ldots, l$, $\Theta_i \in \mathbb{R}^{N \times N}$ satisfies

$$\Theta_i = \begin{cases} A_{G_1}, & \text{if } \{v_{n_{i-1}}, v_{n_i}\} \in E_1 \text{ and } \{v_{n_{i-1}}, v_{n_i}\} \not\in E_2, \\ A_{G_2}, & \text{if } \{v_{n_{i-1}}, v_{n_i}\} \not\in E_2 \text{ and } \{v_{n_{i-1}}, v_{n_i}\} \not\in E_1, \\ \{A_{G_1}, A_{G_2}\}, & \text{otherwise.} \end{cases}$$

Proof. We prove this result by induction on the length $l$ of the walk. First, assume that $l = 1$. It follows from the definition of the adjacency matrix that $e_{n_l}^T A_{G_1} e_{n_0} > 0$ if and only if $\{v_{n_0}, v_{n_1}\} \in E_1$ and $e_{n_l}^T A_{G_2} e_{n_0} > 0$ if and only if $\{v_{n_0}, v_{n_1}\} \not\in E_2$. Therefore, there exists a walk of length 1 between $v_{n_0}$ and $v_{n_1}$ if and only if $e_{n_1}^T \Theta_1 e_{n_0} > 0$.

Now, for induction, assume that the result holds for walks of length $l - 1 \geq 1$.

Next, we prove that the result holds for walks of length $l$. Note that there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length $l$ between $v_{n_0}$ and $v_{n_l}$ if and only if there exists a vertex $v_{n_2} \in V$ such that there exists a walk $(v_{n_1}, v_{n_2}, \ldots, v_{n_l})$ of length $l - 1$ between $v_{n_1}$ and $v_{n_2}$, and a walk $(v_{n_0}, v_{n_1})$ of length 1 between $v_{n_0}$ and $v_{n_1}$.

Since the result holds for walks of length $l - 1$, it follows that there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length $l - 1$ between $v_{n_0}$ and $v_{n_l}$ if and only if $e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_2 e_{n_0} > 0$. Furthermore, there exists a walk of length 1 between $v_{n_0}$ and $v_{n_1}$ if and only if $e_{n_0}^T \Theta_1 e_{n_0} > 0$.

Therefore, there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length $l$ between $v_{n_0}$ and $v_{n_l}$ if and only if there exists a vertex $v_{n_2} \in V$ such that $e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_2 e_{n_0} > 0$. Furthermore, note that

$$e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} = \sum_{k=1}^{N} (e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_k e_k) (e_k^T \Theta_1 e_{n_0}).$$

Thus, $e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_1 e_{n_0} > 0$ if and only if there exists an $k \in \{1, \ldots, N\}$ such that $(e_k^T \Theta_1 e_{n_0}) > 0$.

Therefore, there exists a walk $(v_{n_0}, v_{n_1}, \ldots, v_{n_l})$ of length $l$ between $v_{n_0}$ and $v_{n_l}$ if and only if there exists a vertex $v_{n_2} \in V$ such that $e_{n_l}^T \Theta_l \Theta_{l-1} \cdots \Theta_2 e_{n_0} > 0$. Furthermore, for all $i = 1, \ldots, l$, let $e_i^T \Theta_i \Theta_{i-1} \cdots \Theta_1 e_{n_0} > 0$ such that

$$M \ddot{q}(t) + C \dot{q}(t) + K q(t) = u(t),$$

where $M$, $C$, and $K$ are the mass, damping, and stiffness matrices, respectively. Figure 2 shows a 3-mass structure with all possible spring and dashpot connections. Figure 3 shows the possible spring and dashpot connections to the nth mass.
The lumped-mass structure (3.1)-(3.4) is structurally connected if and only if the force-to-motion transfer functions between every pair of masses is nonzero. Next, we characterize structural connectedness in terms of the damping and stiffness matrices.

**Definition 4.1.** The lumped-mass structure (3.1)-(3.4) is structurally connected if the graph $\mathcal{G}_{\mathcal{L}} \triangleq (\mathcal{V}_L, \mathcal{E}_L)$ is connected.

Definition 4.1 intuitively implies that (3.1)-(3.4) is structurally connected if and only if the force-to-motion transfer functions between every pair of masses is nonzero. Next, we characterize structural connectedness in terms of the damping and stiffness matrices.

**Lemma 4.1.** The lumped-mass structure (3.1)-(3.4) is structurally connected if and only if $K + C$ is irreducible.

**Proof.** Define the weighted graph $\mathcal{G}_{\mathcal{L}K} \triangleq (\mathcal{V}_L, \mathcal{E}_L, K)$, where, for all $\{m_i, m_j\} \in \mathcal{E}_L$, the weight $k_{ij}$ is assigned to the edge $\{m_i, m_j\}$. By examining (3.3) and (3.4), it follows that $L_K + L_L$ is the Laplacian matrix associated with $\mathcal{G}_{\mathcal{L}K}$. Lemma 2.2 implies that $L_K + L_L$ is irreducible if and only if $\mathcal{G}_{\mathcal{L}K}$ is connected. Since $C_L$ and $K_L$ are diagonal positive-semidefinite matrices, we conclude that the damping matrix $C = C_L + L_L$ and the stiffness matrix $K = K_L + L_L$ are positive semidefinite. We have thus proven the known fact that lumped-mass structures of the form (3.1)-(3.4) have positive-semidefinite damping and stiffness matrices.

A notion of structural connectedness is needed to analyze the zeros of lumped-mass structures. Roughly speaking, a lumped-mass structure is structurally connected if it is a single structure rather than two or more disjoint structures. To formalize this idea, define the edge set $E_{\mathcal{L}K} \triangleq \mathcal{E}_L \cup \mathcal{E}_K$.

**Remark 1.** The lumped-mass structure (3.1)-(3.4) is structurally connected if and only if $K + C$ is irreducible.
Since \( z > 0 \), \( M \) is positive definite, and \( C \) and \( K \) are positive semidefinite, it follows that \( zI + \frac{1}{2}M^{-1}(M^2z^2 + Cz + K) \) are nonsingular. Hence, Proposition 2.8.7 of [10] implies that

\[
(sI - A)^{-1} = 
\begin{bmatrix}
\frac{zI}{M^{-1}K} & \frac{-I}{M^{-1}C} \\
\frac{z}{(z^2I + M^{-1}zI + M^{-1}C)^{-1}} & \frac{1}{(z^2I + M^{-1}zI + M^{-1}C)^{-1}}
\end{bmatrix}
\]  

(4.2)

where \( \frac{z}{(z^2I + M^{-1}zI + M^{-1}C)^{-1}} \) denotes an inconsequential entry. Combining (4.1) and (4.2) yields

\[
G_{i,j}(z) = e_i^T(Mz^2 + Cz + K)^{-1}e_j
= e_i^T[(Mz^2 + Cwz + Kw) + (L_{Gz} + L_K)]^{-1}e_j,
\]  

(4.3)

Next, it follows from (3.3) and (3.4) that \( L_{Gz} + L_K \) is the Laplacian matrix of the weighted graph \( S_{C,K} \hat{=} (V_M,E_{C,K}) \), where, for all \( \{m_i,m_j\} \in E \), the weight \( c_{i,j}+k_{i,j} \) is associated with the edge \( \{m_i,m_j\} \). Furthermore, \( Mz^2 + Cwz + Kw \) is a diagonal positive-definite matrix. Since \( S_{C,K} \) is connected and \( Mz^2 + Cwz + Kw \) is a diagonal positive-definite matrix, Lemma 2.4 implies that \( (Mz^2 + Cwz + Kw) + (L_{Gz} + L_K) \) is an irreducible, nonsingular M-matrix and every entry of \( (Mz^2 + Cz + K)^{-1} = [(Mz^2 + Cwz + Kw) + (L_{Gz} + L_K)]^{-1} \) is positive. Therefore, for all \( i,j=1,\ldots,N \), it follows from (4.3) that \( G_{i,j}(z) > 0 \), and thus \( z \) is not a zero of \( G_{i,j}(s) \).

Now consider the case \( z = 0 \). It follows from (4.1) that

\[
G_{i,j}(0) = [e_i^T 0] \begin{bmatrix} 0 & -I \end{bmatrix} M^{-1}K M^{-1}C \begin{bmatrix} 0 \end{bmatrix} e_j = e_i^T(Kw + L_K)^{-1}e_j.
\]  

(4.4)

Since \( M > 0 \) and \( K > 0 \) it follows that \( M^{-1}K \) is nonsingular. Hence, Fact 2.15.2 of [10] implies that

\[
\begin{bmatrix} 0 & -I \end{bmatrix} M^{-1}K M^{-1}C \begin{bmatrix} 0 \end{bmatrix} e_j = \begin{bmatrix} -I \end{bmatrix} M^{-1}K^{-1}M e_j.
\]  

(4.5)

Combining (4.4) and (4.5) yields \( G_{i,j}(0) = e_i^T(Kw + L_K)^{-1}e_j \). Since \( S_{C,K} \) is connected, it follows from Lemma 2.1 and Lemma 2.2 that \( L_K \) is an irreducible, singular, M-matrix. Since \( K_{Gz} \) is diagonal, it follows that \( K = K_{Gz} + L_K \) is an irreducible M-matrix. Furthermore, since \( K \) is positive definite, it follows that \( K \) is an irreducible, nonsingular, M-matrix. It then follows from Lemma 2.3 that every entry of \( K^{-1} \) is positive. Therefore, \( G_{i,j}(0) = e_i^T(Kw + L_K)^{-1}e_j \) is positive and \( z = 0 \) is not a zero of \( G_{i,j}(s) \).

**Corollary 4.1.** Assume that the system (3.5)-(3.7) is structurally connected. Then, for all \( i,j=1,\ldots,N \), the transfer functions from \( u_i(t) \) to \( q_i(t) \) and from \( u_j(t) \) to \( q_j(t) \) have no positive zeros.

**5. THE COMPLEX NONMINIMUM PHASE ZEROS OF LUMPED-MASS STRUCTURES**

Theorem 4.1 and Corollary 4.1 guarantee that every force-to-motion transfer function of a structurally connected lumped-mass structure has no positive zeros. However, these results do not guarantee that every force-to-motion transfer function is minimum phase; the force-to-motion transfer functions can have complex zeros in the open right half plane. In fact, for \( N = 3 \), there exists a lumped-mass structure (3.1)-(3.4) that is structurally connected and has a nonminimum phase force-to-motion transfer function. Specifically, consider the 3-mass lumped-mass structure in Figure 2, where \( m_1 = m_2 = m_3 = 1 \) kg, \( k_1 = k_3 = 0 \) kg/s², \( k_2 = k_{1,2} = k_{2,3} = 5 \) kg/s², \( c_1 = c_2 = c_3 = c_{1,2} = c_{2,3} = 0 \) kg/s, and \( c_{1,3} = 5 \) kg/s. This system is structurally connected since the graph \( G_{C,K} \) is connected. Furthermore, the transfer function from \( u_1(t) \) to \( q_3(t) \), given by

\[
G_{1,3}(s) = \frac{5x^3 + 5s^2 + 75s + 100}{s^6 + 10s^5 + 35s^4 + 200s^3 + 325s^2 + 250s + 375},
\]

is nonminimum phase. The zeros of \( G_{1,3}(s) \) are approximately 0.150 ± 3.92 and -1.30. In fact, for all \( N \geq 3 \), there exists a lumped-mass structure (3.1)-(3.4) that is structurally connected and has a nonminimum phase force-to-motion transfer function.

**6. INITIAL UNDERSHOOT IN LUMPED-MASS STRUCTURES**

Initial undershoot describes the qualitative behavior of the step response of a transfer function. A system has initial undershoot if the step response initially moves in the direction that is opposite its asymptotic value. We now define initial undershoot and state a result classifying the existence of initial undershoot. The definition and result are given in [12-14].

**Definition 6.1.** Let \( H(s) \) be a single-input single-output asymptotically stable transfer function with relative degree \( r > 0 \). Let \( y(t) \) be the step response of \( H(s) \). Assume that \( H(0) \neq 0 \). Then the step response of \( H(s) \) has initial undershoot if \( y(0) \lim_{s \to \infty} y(t) < 0 \).

**Lemma 6.1.** Let \( H(s) \) be a single-input single-output asymptotically stable transfer function with relative degree \( r > 0 \). Assume that \( H(0) \neq 0 \). Then the step response of \( H(s) \) has initial undershoot if and only if \( H(s) \) has an odd number of positive zeros.

The main result of this section addresses the existence of initial undershoot in a force-to-position transfer function of an asymptotically stable lumped-mass structure.

**Theorem 6.1.** Assume that the system (3.5)-(3.7) is structurally connected. Furthermore, assume that \( A \) is asymptotically stable and the graph \( G_{C,K} \) is connected. Then, for all \( i,j=1,\ldots,N \), the step response of the transfer function from \( u_i(t) \) to \( q_j(t) \) does not exhibit initial undershoot.

**Proof.** Let \( G_{i,j}(s) \) be the transfer function from the force input on mass \( m_j \) to the position of mass \( m_i \). Since \( A \) is asymptotically stable, Lemma 3.2 implies \( K > 0 \). It follows from Theorem 4.1 that \( G_{i,j}(s) \) has no nonnegative zeros. Therefore, \( G_{i,j}(0) \neq 0 \) and \( G_{i,j}(s) \) has no positive zeros. Since \( A \) is asymptotically stable, \( G_{i,j}(s) \) is asymptotically stable. Since \( G_{i,j}(s) \) is an asymptotically stable transfer function, \( G_{i,j}(0) \neq 0 \), and \( G_{i,j}(s) \) has no positive zeros, it follows from Lemma 6.1 that \( G_{i,j}(s) \) does not exhibit initial undershoot.

**7. EXAMPLE: 3-MASS LUMPED-MASS STRUCTURE**

Consider the structurally connected 3-mass structure shown in Figure 2 whose dynamics are given by (3.1)-(3.4), where \( N = 3 \). For this example, the masses are \( m_1 = m_2 = m_3 = 5 \) kg; the spring stiffnesses are \( k_1 = 1 \) kg/s², \( k_2 = 2 \) kg/s², \( k_3 = 3 \) kg/s², \( k_{1,2} = 12 \) kg/s², \( k_{2,3} = 13 \) kg/s², and \( k_{3,1} = 23 \) kg/s²; and the damping coefficients are \( c_1 = 10 \) kg/s, \( c_2 = 20 \) kg/s, \( c_3 = 30 \) kg/s, \( c_{1,2} = 120 \) kg/s, \( c_{1,3} = 130 \) kg/s, and \( c_{2,3} = 230 \) kg/s.
The transfer functions from $u_1$ to $q_1$, from $u_1$ to $q_2$, and from $u_1$ to $q_3$ are
\[
G_{1,1}(s) = \frac{0.2s^4 + 30.4s^3 + 734.24s^2 + 146.24s + 7.312}{p(s)},
\]
\[
G_{2,1}(s) = \frac{4.8s^3 + 614.08s^2 + 122.72s + 6.136}{p(s)},
\]
\[
G_{3,1}(s) = \frac{5.2s^3 + 606.12s^2 + 121.12s + 6.056}{p(s)},
\]
respectively, where $p(s) = s^6 + 204s^5 + 10328s^4 + 39813.6s^3 + 114286.6s^2 + 113256s + 37752$. The zeros of $G_{1,1}(s)$ are approximately -121.9, -29.86, -0.1001, and -0.1003. The zeros of $G_{2,1}(s)$ are approximately -127.7, -0.1000, and -0.1001. The zeros of $G_{3,1}(s)$ are approximately -116.4, -0.1000, and -0.1001. Therefore, $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ have no nonnegative zeros as guaranteed by Theorem 4.1. Furthermore, Theorem 6.1 implies that the step responses of $G_{1,1}(s)$, $G_{2,1}(s)$, and $G_{3,1}(s)$ do not have initial undershoot. Figure 4 verifies that the step responses do not have initial undershoot.

8. RELATIVE DEGREE OF LUMPED-MASS STRUCTURES

In this section, we analyze the relative degree of the lumped-mass structure (3.5)-(3.7). If we assume that (3.5)-(3.7) is structurally connected, then the $i$th and $j$th masses are connected by means of a sequence of springs and dashpots. To calculate the relative degree of the transfer function from $u_i(t)$ to $q_i(t)$, we consider all sequences of springs and dashpots that connect $m_j$ to $m_i$. More precisely, we consider all walks on the graph $\mathcal{G}_{\text{CK}} = (\mathcal{V}_M, E_{\text{CK}})$ from $m_j$ to $m_i$. For all $i, j = 1, \ldots, N$ such that $i \neq j$, define
\[
\Omega_{i,j} = \{ \omega: \omega \text{ is a walk on } \mathcal{G}_{\text{CK}} \text{ from } m_j \text{ to } m_i \}, \quad (8.1)
\]
and, for all $i = j = 1, \ldots, N$, define $\Omega_{i,i} = \emptyset$.

Let $\omega = (m_{n_0}, m_{n_1}, \ldots, m_{n_l})$ be a walk of length $l$ on $\mathcal{G}_{\text{CK}}$ from $m_{n_0}$ to $m_{n_l}$. Now, count the number of edges $n_C(\omega)$ in the walk $\omega$ that are dashpots only, the number of edges $n_K(\omega)$ in the walk $\omega$ that are springs only, and the number of edges $n_{CK}(\omega)$ in the walk $\omega$ that are springs and dashpots, that is,
\[
n_C(\omega) = \sum_{i=1}^{l} \alpha_i, \quad n_K(\omega) = \sum_{i=1}^{l} \beta_i, \quad n_{CK}(\omega) = \sum_{i=1}^{l} \gamma_i,
\]
where, for $i = 1, \ldots, l$,
\[
\alpha_i = \begin{cases} 1, & {m_{n_{i-1}}, m_{n_i}} \in E_C \text{ and } {m_{n_{i-1}}, m_{n_i}} \notin E_K, \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\beta_i = \begin{cases} 1, & {m_{n_{i-1}}, m_{n_i}} \in E_K \text{ and } {m_{n_{i-1}}, m_{n_i}} \notin E_C, \\ 0, & \text{otherwise}, \end{cases}
\]
\[
\gamma_i = \begin{cases} 1, & {m_{n_{i-1}}, m_{n_i}} \in E_K \cap E_C, \\ 0, & \text{otherwise}. \end{cases}
\]
The next result provides an expression for the relative degree of the transfer function from $u_i(t)$ to $q_i(t)$ and characterizes the sign of the first non-zero Markov parameter (often called the high-frequency gain).

**Theorem 8.1.** Assume that the system (3.5)-(3.7) is structurally connected. Then, for all $i, j = 1, \ldots, N$, the relative degree of the transfer function from $u_i(t)$ to $q_i(t)$ is
\[
r_{i,j} \triangleq \min_{\omega \in \Omega_{i,j}} \text{Re} \left( 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2 \right). \quad (8.2)
\]
Furthermore, for all $i, j = 1, \ldots, N$, the first non-zero Markov parameter of the transfer function from $u_i(t)$ to $q_i(t)$ is
\[
H_{i,j} \triangleq e_1^T C_p A_p^{n_{i,j}-1} B_{e_j} > 0. \quad (8.3)
\]
Proof. Let $i$ and $j$ be positive integers between 1 and $N$, and let $\omega \in \Omega_{i,j}$ be the minimizer in (8.2) so that $r_{i,j} = 2n_K(\omega) + n_C(\omega) + n_{CK}(\omega) + 2$. For all $n = 1, \ldots, N$,
\[
H_{i,j} \triangleq e_1^T C_p A_p^{n-1} B_{e_j} = \begin{bmatrix} e_1^T & 0 \end{bmatrix} A_{n-1}^{T} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix}. \quad (8.4)
\]
To prove (8.2) and (8.3), it suffices to show that $H_0, H_1, \ldots, H_{r_{i,j}-1} = 0$ and $H_{r_{i,j}} > 0$. Performing the matrix multiplications in (8.4) implies
\[
H_n = \begin{bmatrix} e_1^T & 0 \end{bmatrix} \begin{bmatrix} I & \Gamma_n \end{bmatrix} \begin{bmatrix} 0 \\ M^{-1} e_j \end{bmatrix} = e_1^T \Gamma_n M^{-1} e_j, \quad (8.5)
\]
where $\Gamma_n$ denotes an inconsequential entry, $\Gamma_1 \triangleq 0, \Gamma_2 \triangleq I$, and, for all $n = 3, \ldots, N$, $\Gamma_n \triangleq -M^{-1} C_{\Gamma_{n-1}} M^{-1} \Gamma_{n-2}$. By manipulating the terms of (8.5), it follows that
\[
H_n = e_1^T M^{-\frac{1}{2}} \Gamma_n M^{-\frac{1}{2}} e_j = \frac{1}{\sqrt{\text{det}(\Delta_{i,j})}} e_1^T \Gamma_n e_j, \quad (8.6)
\]
where $\Delta_1 \triangleq 0, \Delta_2 \triangleq I$, and, for all $n = 3, \ldots, N$,
\[
\Delta_n \triangleq \text{det} \Gamma_{n-1} + \text{det} \Gamma_{n-2}, \quad (8.7)
\]
where $P \triangleq -M^{-\frac{1}{2}} C_{\Gamma} M^{-\frac{1}{2}}$ and $Q \triangleq -M^{-\frac{1}{2}} K M^{-\frac{1}{2}}$.

Since $\Gamma_1 = 0$, it follows that $H_1 = 0$. Since $\Gamma_2 = I$, it follows that $H_2 > 0$ if and only if $r_{i,j} = 1$, which is equivalent to $H_2 > 0$ if and only if $\Omega_{i,j} = \emptyset$. Thus, $H_2 > 0$ if and only if $r_{i,j} = 2$.

Now, consider the case where $\Omega_{i,j} \neq \emptyset$ and thus $r_{i,j} > 2$. Note that $P$ and $Q$ each can be expressed as the sum of a weighted adjacency matrix and a diagonal negative semidefinite matrix, that is,
\[
P = A_{D_C} + D_C, \quad Q = A_{D_K} + D_K, \quad (8.8)
\]
where $D_C \triangleq -\text{diag}(\frac{\epsilon_{m_1}}{m_1} + \sum_{j=1}^{N} \frac{\epsilon_{m_{j+1}}}{m_{j+1}}, \frac{\epsilon_{m_2}}{m_2} + \sum_{j=1, j \neq 2}^{N} \frac{\epsilon_{m_{j+1}}}{m_{j+1}}, \ldots, \frac{\epsilon_{m_N}}{m_N} + \sum_{j=1, j \neq N}^{N} \frac{\epsilon_{m_{j+1}}}{m_{j+1}})$ and $D_K \triangleq -\text{diag}(\frac{k_1}{m_1} + \sum_{j=2}^{N} \frac{k_j}{m_j} - \sum_{j=1, j \neq 2}^{N} \frac{k_j}{m_j}, \ldots, \frac{k_N}{m_N} + \sum_{j=1, j \neq N}^{N} \frac{k_j}{m_j})$ are diagonal negative semidefinite, and $A_{D_C}$ is the weighted adjacency.
which implies that 

\[ e^T \bar{G} e \geq 0 \]

is assigned to the edge \( \{n_p, n_q\} \). \( A_{G_K} \) is the weighted adjacency matrix associated with \( G_K \), where, for all \( \{n_p, n_q\} \in E_K \), the weight \( w_{n_p, n_q} \) is assigned to the edge \( \{n_p, n_q\} \).

It follows from the regression (8.7) that, for all \( n = 3, \ldots, N \),

\[ \bar{G}_n = \sum p^n Q^{n-1} \cdots Q^{n-3} Q^{n-2} p^{n-2}, \]  

(8.9)

where \( \sum \) denotes the sum over all distinct products such that \( p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-3} \in \{0, 1\} \) and \( \sum j=1 p_j + 2 \sum j=1 q_j + 2 = n \). Combining (8.8)-(8.9) and performing the multiplications yields

\[ \bar{G}_n = \sum (A_{G_C} + D_{C})^{p_1} (A_{G_K} + D_{K})^{p_2} \cdots (A_{G_C} + D_{C})^{p_{n-3}} \times (A_{G_K} + D_{K})^{p_{n-2}} = \sum A^{(p_1)}_{G_C} A^{(p_2)}_{G_K} \cdots A^{(n-3)}_{G_C} A^{(n-2)}_{G_K}, \]  

(8.10)

where, for all \( n = 3, \ldots, N \),

\[ \Lambda_n = \sum A^{(p_1)}_{G_C} A^{(p_2)}_{G_K} \cdots A^{(n-3)}_{G_C} A^{(n-2)}_{G_K} \]  

(8.11)

Since the \( \omega \) is the minimizer of (8.2), then there does not exist a walk \( \bar{w} \in D_{\bar{G}} \) such that \( 2n_K(\omega) + n_C(\omega) + n_K(\omega) < 2n(\omega) + n_C(\omega) + n_K(\omega) \). Thus, Corollary 2.1 implies that, for all \( n = 3, \ldots, n \),

\[ e^T_i A^{(p_1)}_{G_C} A^{(p_2)}_{G_K} \cdots A^{(n-3)}_{G_C} A^{(n-2)}_{G_K} e_j = 0, \]  

(8.12)

where \( p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-3} \in \{0, 1\} \) satisfy \( \sum j=1 p_j + 2 \sum j=1 q_j + 2 = n \). By combining (8.10) and (8.12), it follows that, for all \( n = 3, \ldots, n \),

\[ e^T_i \Lambda_n e_j = 0. \]

Next, note that every term of \( e^T_i \Lambda_n e_j \) of the form \( e^T D_{C}^p D_{K}^q D_{C}^r \cdots D_{C}^s D_{K}^t D_{C}^u D_{K}^v e \) is zero because \( D_C \) and \( D_K \) are diagonal and \( i \neq j \). The remaining terms of \( e^T_i \Lambda_n e_j \) have the form of (8.12) where a negative semidefinite matrix \( D_C \) or \( D_K \) may appear between the matrices \( A_{G_C} \) and \( A_{G_K} \). Therefore, Lemma A.1 and (8.12) imply that for all \( n = 3, \ldots, n \),

\[ e^T_i \Lambda_n e_j = 0. \]

Now, it suffices to show that \( H_{r_i, ej} > 0 \). Again, note that every term of \( e^T_i \Lambda_n D_{C}^p D_{K}^q D_{C}^r \cdots D_{C}^s D_{K}^t D_{C}^u D_{K}^v e_j \) is zero because \( D_C \) and \( D_K \) are diagonal and \( i \neq j \). The remaining terms of \( e^T_i \Lambda_n D_{C}^p D_{K}^q D_{C}^r \cdots D_{C}^s D_{K}^t D_{C}^u D_{K}^v e_j \) have the form of (8.12) where a negative semidefinite matrix \( D_C \) or \( D_K \) may appear between the matrices \( A_{G_C} \) and \( A_{G_K} \). Therefore, Lemma A.1 and (8.12) imply that \( e^T_i \Lambda_n D_{C}^p D_{K}^q D_{C}^r \cdots D_{C}^s D_{K}^t D_{C}^u D_{K}^v e_j = 0. \)

(8.13)

Furthermore, note that each product \( e^T_i A^{(p_1)}_{G_C} A^{(p_2)}_{G_K} \cdots A^{(n-3)}_{G_C} A^{(n-2)}_{G_K} e_j \) is nonnegative.

Since there exists a walk \( \omega \) such that \( \bar{r}_i, j = 2nK(\omega) + n_C(\omega) + n_K(\omega) = 2 \), it follows from Lemma 2.5 that there exist \( p_1, \ldots, p_{n-2}, q_1, \ldots, q_{n-3} \in \{0, 1\} \) such that \( \sum j=1 p_j + 2 \sum j=1 q_j + 2 = n \) and \( e^T_i A^{(p_1)}_{G_C} A^{(p_2)}_{G_K} \cdots A^{(n-3)}_{G_C} A^{(n-2)}_{G_K} e_j > 0 \). Therefore, at least one term in the summation (8.13) is positive. Thus,

\[ e^T_i \bar{G}_{r_i, ej} > 0, \]

which implies that \( H_{r_i, ej} > 0 \).