State Estimation for Equality-Constrained Nonlinear Systems

B. O. S. Teixeira†, J. Chandrasekar‡, L. A. B. Tôrres*, L. A. Aguirre*, and D. S. Bernstein†

Abstract—This paper addresses the state-estimation problem for nonlinear systems in a context where prior knowledge, in addition to the model and the measurement data, is available in the form of a nonlinear equality constraint. Then three suboptimal algorithms based on the unscented Kalman filter (UKF) are developed, namely, the equality-constrained unscented Kalman filter (ECUKF), the projected unscented Kalman filter (PUKF), and the measurement-augmented unscented Kalman filter (MAUKF). These methods are compared on three examples, including a quaternion-based attitude estimation problem, as well as a mechanical system and an idealized flow model, both involving conserved quantities. In addition to very closely satisfying a nonlinear equality constraint on the system, the proposed methods produce more precise and more informative estimates than the unconstrained estimates.

I. INTRODUCTION

Under Gaussian noise and linear dynamics assumptions, the equality-constrained Kalman filter (ECKF) [25] is presented as the solution to the equality-constrained state-estimation problem. ECKF takes advantage of prior knowledge of the state vector provided by a equality constraint and uses this information to obtain better estimates than would be provided by the Kalman filter (KF) [17] in the absence of such information.

Although it is difficult to make correspondingly precise statements in the case of nonlinear systems, the same principles and objectives apply. For example, in undamped mechanical systems, such as a system with Hamiltonian dynamics, conservation laws hold. In the quaternion-based attitude estimation problem, the attitude vector must have unit norm [6, 7]. Additional examples arise in optimal control [11], parameter estimation [1, 27], and navigation [3, 9, 28]. However, the solution to the equality-constrained state-estimation problem for nonlinear systems is complicated [8] by the fact that the random variables are not completely characterized by its first-order and second-order moments. Thus, suboptimal solutions based on the extended Kalman filter (EKF) [14] are generally used. One of the most popular techniques is based on measurement augmentation, in which a perfect measurement of the constrained quantity is assumed to be available [3, 9, 21, 27, 28]. In addition, the estimate-projection method [24] have also been considered.

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In this context, two contributions are presented in this paper. First, we develop and compare three suboptimal algorithms for equality-constrained state estimation for nonlinear systems, namely, the equality-constrained unscented Kalman filter (ECUKF), the projected unscented Kalman filter (PUKF), and the measurement-augmented unscented Kalman filter (MAUKF). These methods, which extend algorithms for equality-constrained state estimation developed for linear systems [25], are based on the unscented Kalman filter (UKF) [15, 16], a sigma-point Kalman filter (SPKF) [26]. Recent work [12, 16, 19, 20, 26] reports the improved performance of SPKF compared to EKF, which is prone to numerical problems such as initialization sensitivity, bias (divergence), and instability for strongly nonlinear systems [23]. A quaternion-based attitude estimation problem is addressed to illustrate the aforementioned algorithms. Although the process model satisfies the unit norm constraint, the constraint is violated by unconstrained Kalman filtering [7].

Finally, we consider the application of equality-constrained Kalman filtering techniques to improve estimation when an approximate discretized model is used to represent a continuous-time process. According to [22], constraints can also be used to correct model error. The problem of using data-driven discrete-time models to perform state estimation by the unconstrained UKF for continuous-time nonlinear systems is treated in [2]. We illustrate the application of equality-constrained unscented Kalman filter techniques to the aforementioned problem through two examples. We consider a discretized model of an undamped single-degree-of-freedom pendulum without external disturbances. Although energy is conserved in the original, continuous-time system, the discretized model is approximate, and the energy constraint is intended to improve estimates of the discretized states. Then, we consider a one-dimensional compressible hydrodynamic model discretized by means of a finite-volume scheme [4, 5]. The boundary conditions are chosen such that density and energy are conserved, and this knowledge is used to improve the state estimates.

II. STATE ESTIMATION FOR NONLINEAR SYSTEMS

For the nonlinear stochastic discrete-time dynamic system

\[ x_k = f_{k-1}(x_{k-1}, u_{k-1}, w_{k-1}), \quad (2.1) \]
\[ y_k = h_k(x_k) + v_k, \quad (2.2) \]

where \( f_{k-1} : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^n \) and \( h_k : \mathbb{R}^n \to \mathbb{R}^m \) are, respectively, the process and observation models, the state estimation problem can be described as follows. Assume that, for all \( k \geq 1 \), the known data are the measurements \( y_k \in \mathbb{R}^m \), the inputs \( u_{k-1} \in \mathbb{R}^p \), and the probability density functions...
PDFs) \( \rho(x_0), \rho(w_{k-1}) \) and \( \rho(v_k) \), where \( x_0 \in \mathbb{R}^n \) is the initial state vector, \( w_{k-1} \in \mathbb{R}^n \) is the process noise, and \( v_k \in \mathbb{R}^m \) is the measurement noise. Next, define the cost function

\[
J(x_k) \triangleq \rho(x_k | (y_1, \ldots, y_k)),
\]

which is the PDF of the state vector \( x_k \in \mathbb{R}^n \) given the past and present measured data \( y_1, \ldots, y_k \). Under the stated assumptions, the maximization of (2.3) is the state estimation problem.

However, the solution to this problem is complicated [8] by the fact that, for nonlinear systems, \( \rho(x_k | (y_1, \ldots, y_k)) \) is not completely characterized by its mean \( \hat{x}_k \) and covariance \( P_k \). Thus we use an approximation to provide a suboptimal solution. To accomplish that, we consider \( \hat{x}_k \) and \( P_k \) of \( \rho(x_k | (y_1, \ldots, y_k)) \), and the mean \( \bar{x}_0 \) and the covariance \( P_0 \) of \( \rho(x_0 | (y_1, \ldots, y_k)) \).

\[ P_0 \triangleq \mathbb{E}\left((x_k - \bar{x}_0)(x_k - \bar{x}_0)^T\right) \]

of \( \rho(x_0) \). Furthermore, we assume that \( w_{k-1} \) and \( v_k \) are white, and mutually independent with known covariance matrices \( Q_{k-1} \) and \( R_k \), respectively.

A. Unscented Kalman Filter

We consider the unscented Kalman filter (UKF) [15] to provide a suboptimal solution to the state-estimation problem. Instead of analytically linearizing (2.1)-(2.2) in order to use the classical Kalman filter equations [17], UKF employs the unscented transform (UT) [16], which is a numerical procedure for approximating the posterior mean \( \hat{x}_{k|k-1} \) and variance \( P_{x_{k|k-1}} \) of a random vector \( x_{k|k-1} \) obtained from the nonlinear transformation \( x_{k|k-1} = f_{k-1}(x_{k-1}^a) \), where \( x_{k-1}^a \) is a prior random vector whose mean \( \hat{x}_{k-1} \) and variance \( P_{x_{k-1}^a} \) are assumed known. UT guarantees the exact values of \( \hat{x}_{k|k-1} \) and \( P_{x_{k|k-1}} \) if \( f_{k-1} = f_{1,k-1} + f_{2,k-1} \), where \( f_{1,k-1} \) is linear and \( f_{2,k-1} \) is quadratic [16]. The UT is based on a representative group of deterministically chosen vectors known as sigma points. To capture the mean \( \hat{x}_{k|k-1} \) of the augmented prior state vector \( x_{k|k-1}^a \triangleq \begin{bmatrix} x_{k-1}^a & w_{k-1} \end{bmatrix} \), where \( x_{k-1}^a \in \mathbb{R}^{na} \) and \( n_a \triangleq n + q \), and the augmented prior error covariance \( P_{x_{k|k-1}^a} \triangleq \begin{bmatrix} P_{x_{k-1}} & 0_{n \times q} \\ 0_{q \times n} & Q_{k-1} \end{bmatrix} \), the sigma points are chosen as

\[
\begin{align*}
X_{0,k-1} & \triangleq \hat{x}_{k-1} \pm \sqrt{(n_a + \lambda)} \text{col} \left[ (P_{x_{k-1}})^{1/2} \right], \quad i = 1, \ldots, na, \\
X_{i+n,k-1} & \triangleq \hat{x}_{k-1} \pm \sqrt{(n_a + \lambda)} \text{col} \left[ (P_{x_{k-1}})^{1/2} \right], \quad i = 1, \ldots, na,
\end{align*}
\]

with weights

\[
\begin{align*}
\gamma_{0}^{(c)} & \triangleq \frac{\lambda}{n_a + \lambda}, \\
\gamma_{0}^{(m)} & \triangleq \frac{\lambda}{n_a + \lambda}, \\
\gamma_{i}^{(c)} & \triangleq \frac{1 - \alpha^2 + \beta}{2(n_a + \lambda)}, \\
\gamma_{i}^{(m)} & \triangleq \gamma_{i+n}^{(c)} + \gamma_{i+n}^{(m)} - \frac{1}{2(n_a + \lambda)}, \quad i = 1, \ldots, na,
\end{align*}
\]

where \( \text{col} \left[ (\cdot)^{1/2} \right] \) is the ith column of the positive semi-definite Cholesky square root, \( 0 < \alpha \leq 1, \beta \geq 0, \kappa \geq 0 \), and \( \lambda \triangleq \alpha^2(n_a + n_a) - n_a \). We set \( \alpha = 1 \) and \( \kappa = 0 \) [12] such that \( \lambda = 0 \) [15] and set \( \beta = 2 \) [12]. Alternative schemes for choosing sigma points are given in [15].

The UKF forecast equations are given by

\[
\begin{align*}
X_{k-1} & = f_{k-1}(X_{k-1}^a, u_{k-1}, w_{k-1}), \\
\hat{x}_{k|k-1} & = \mathbb{E}(X_{k|k-1} - \bar{x}_{k|k-1}) | X_{k|k-1} - \bar{x}_{k|k-1})^T, \\
P_{\hat{x}_{k|k-1}} & = \mathbb{E}(X_{k|k-1} - \bar{x}_{k|k-1}) (X_{k|k-1} - \bar{x}_{k|k-1})^T, \\
\hat{y}_{k|k-1} & = h_k(X_{k|k-1}^a), \\
P_{\hat{y}_{k|k-1}} & = \mathbb{E}(Y_{k|k-1} - \bar{y}_{k|k-1}) (Y_{k|k-1} - \bar{y}_{k|k-1})^T, \\
X_{k|k-1} & = X_{k-1} - C X_{k|k-1}^a, \\
R_k & = \mathbb{E}(Z_{k|k-1} (Z_{k|k-1})^T).
\end{align*}
\]

The data- assimilation equations are given by

\[
\begin{align*}
K_k & = P_{x_{k|k-1}} (P_{x_{k|k-1}})^{-1}, \\
\hat{x}_{k|k} & = \hat{x}_{k|k-1} + K_k (\hat{y}_{k|k-1} - \bar{y}_{k|k-1}), \\
P_{x_{k|k}} & = P_{x_{k|k-1}} - K_k P_{x_{k|k-1}} K_k^T.
\end{align*}
\]

where \( K_k \) is the Kalman gain matrix. The notation \( \hat{x}_{k|k-1} \) indicates an estimate of \( x_k \) at time \( k \) based on information available up to and including time \( k-1 \). Likewise, \( \hat{y}_{k|k-1} \) indicates an estimate of \( z \) at time \( k \) using information available up to and including time \( k \). Model information is used during the forecast step, while measurement data are injected into the estimates during the data-assimilation step.

III. STATE ESTIMATION FOR EQUALITY-CONSTRAINED NONLINEAR SYSTEMS

Assume that, for all \( k \geq 1 \), the state vector \( x_k \) satisfies the equality constraint

\[
g_{k-1}(x_k) = d_{k-1},
\]

where \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and \( d_{k-1} \) are assumed known. Then, the objective of the equality-constrained state-estimation problem is to maximize (2.3) subject to (3.1).

**Theorem 3.1 (ECKF [25]):** Assume that \( f_{k-1}, h_k, \) and \( g_{k-1} \) are linear functions, respectively, given by

\[
\begin{align*}
x_k & = A_k x_{k-1} + B_k w_{k-1} + G_k v_{k-1}, \\
y_k & = C_k x_k + v_k, \\
D_{k-1} e_k & = d_{k-1},
\end{align*}
\]

where \( A_k \in \mathbb{R}^{n \times n}, B_k \in \mathbb{R}^{n \times p}, G_k \in \mathbb{R}^{n \times q}, C_k \in \mathbb{R}^{m \times n}, \) and \( D_{k-1} \in \mathbb{R}^{m \times n} \). Then, the recursive solution \( \hat{x}_k^P \) to the equality-constrained state-estimation problem is
given by the equality-constrained Kalman filter (ECKF), whose forecast step is given by
\[ \hat{x}_{k-1} = A_{k-1}\hat{x}_{k-1}^P + B_{k-1}u_{k-1}, \] (3.5)
\[ P_{xx}^k|_{k-1} = A_{k-1}P_{xx}^P|_{k-1}A_{k-1}^T + G_{k-1}Q_{k-1}G_{k-1}^T, \] (3.6)
\[ \hat{y}_{k-1} = C_k\hat{x}_{k-1}, \] (3.7)
\[ P_{yy}^k|_{k-1} = C_kP_{xx}^P|_{k-1}C_k^T + R_k, \] (3.8)
\[ P_{xk}^k = P_{xx}^k - K_k^P P_{xx}^P|_{k-1}K_k^T, \] (3.9)
whose data-assimilation step is given by (2.14)-(2.16), and whose projection step is given by
\[ \hat{d}_{k-1} = D_{k-1}\hat{x}_k, \] (3.10)
\[ P^d_k = D_{k-1}P_{xx}^P D_{k-1}^T, \] (3.11)
\[ P^d_k = P_{xx}^P D_{k-1}, \] (3.12)
\[ K_k^P = P^d_k(P^d_k + R_k)^{-1}, \] (3.13)
\[ \hat{x}_k = \hat{x}_k + K_k^P(d_{k-1} - \hat{d}_{k-1}), \] (3.14)
\[ P^x_k = P_{xx}^P - K_k^P P^d_k K_k^T. \] (3.15)

IV. EQUALITY-CONSTRAINED UNSCENTED KALMAN FILTERS

In this section, we propose three approaches based on UKF to provide a suboptimal solution to the equality-constrained state-estimation problem for nonlinear systems.

A. Projected and Equality-Constrained Unscented Kalman Filters

The projection step of ECKF given by (3.10)-(3.15) is now extended to the nonlinear case by means of UT.

Using (2.4)-(2.5) to choose sigma points and associated weights, we have
\[ \hat{x}_k = \hat{x}_k 1 \times n + \sqrt{(n + \lambda)(P_{xx}^x)^{1/2}} \]
\[ \hat{x}_k 1 \times n = \sqrt{(n + \lambda)(P_{xx}^x)^{1/2}}, \] (4.1)
where \( \hat{x}_k \) and \( P_{xx}^x \) are given by (2.15) and (2.16). Then the sigma points \( \hat{x}_{i,k} \in \mathbb{R}^n \), \( i = 0, \ldots, 2n \), are propagated through (3.1) producing
\[ D_{i,k} = g_{k-1}(\hat{x}_{i,k}), \quad i = 1, \ldots, 2n, \] (4.2)
such that \( \hat{d}_{k-1} \), \( P_{dd}^k \), and \( P_{xx}^d \) are given by
\[ \hat{d}_{k-1} = \sum_{i=0}^{2n} \gamma_{i}^{(m)} D_{i,k}, \] (4.3)
\[ P_{dd}^k = \sum_{i=0}^{2n} \gamma_{i}^{(c)}[D_{i,k} - \hat{d}_{k-1}][D_{i,k} - \hat{d}_{k-1}]^T, \] (4.4)
\[ P_{xx}^d = \sum_{i=0}^{2n} \gamma_{i}^{(c)}[\hat{x}_{i,k} - \hat{x}_k][D_{i,k} - \hat{d}_{k-1}]^T, \] (4.5)
and \( P^P_k, \hat{x}_k^P \), and \( P_{xk}^x \) are respectively given by (3.13), (3.14), and (3.15), (4.1)-(4.5),(3.13)-(3.15)

Applying the projection step given by (4.1)-(4.5),(3.13)-(3.15) to UKF equations given by (2.6)-(2.16) without feedback recursion yields the projected unscented Kalman filter (PUKF). PUKF is the nonlinear extension of the projected Kalman filter by estimate projection [24, 25].

Now define \( \hat{x}_{i,k}^{ap} = \left[ \begin{array}{c} \hat{x}_{P,k-1} \\ \hat{u}_{k-1} \end{array} \right] \), and \( P_{k-1}^{ap} = \left[ \begin{array}{c} P_{xx}^P \\ 0_{x \times q} \\ 0_{q \times x} \\ Q_{k-1} \end{array} \right] \), such that the sigma points
\[ \hat{x}_{a,k-1} = \left[ \begin{array}{c} \hat{x}_{i,k}^{ap} \\ \hat{x}_{i,k}^{ap} 1 \times n - \sqrt{(n + \lambda)(P_{xx}^x)^{1/2}} \end{array} \right] \] (4.6)
are chosen based on \( x_{P,k}^P \) (3.14). Then, by appending the projection step given by (4.1)-(4.5),(3.14)-(3.15) to (4.6),(2.7)-(2.16), we obtain the equality-constrained unscented Kalman filter (ECUKF). Note that, unlike PUKF, ECKF connects the projection step to UKF by feedback recursion.

B. The Measurement-Augmentation Unscented Kalman Filter

To extend the measurement-augmentation Kalman filter (MAKF) [3, 9, 21, 27, 28] to the nonlinear case, we replace (2.2) by the augmented observation
\[ \hat{y}_k = \hat{h}_k(x_k) + \hat{v}_k, \] (4.7)
where \( \hat{y}_k \) \( \hat{h}_k(x_k) \) and \( \hat{v}_k \) are given by (4.8)
\[ \hat{y}_k = \gamma_{i}^{(m)} \hat{y}_{i,k-1}, \] (4.9)
\[ \hat{y}_{i,k-1}^{ap} = \gamma_{i}^{(c)}[\hat{y}_{i,k-1}^{ap} 1 \times n - \sqrt{(n + \lambda)(P_{xx}^x)^{1/2}} + \hat{v}_{i,k-1}^T, \] \( \gamma_{i}^{(c)}(\hat{y}_{i,k-1}^{ap})^T, \] (4.10)
where \( \hat{R}_k = \left[ \begin{array}{c} \hat{R}_k \\ 0_{x \times 0} \\ 0_{0 \times x} \end{array} \right] \), and the data-assimilation equations
\[ \tilde{K}_k = \hat{P}_{xx}^{ap} \hat{P}_{xx}^{ap} \hat{y}_k - \tilde{v}_{i,k-1}^T, \] \( \tilde{K}_k = \hat{K}_k \hat{y}_k - \tilde{v}_{i,k-1}^T, \) (4.12)
\[ \tilde{K}_k = \hat{K}_k \hat{y}_k - \tilde{v}_{i,k-1}^T, \] \( \tilde{K}_k = \hat{K}_k \hat{y}_k - \tilde{v}_{i,k-1}^T, \) (4.13)
\[ \tilde{K}_k = \hat{K}_k \hat{y}_k - \tilde{v}_{i,k-1}^T, \] \( \tilde{K}_k = \hat{K}_k \hat{y}_k - \tilde{v}_{i,k-1}^T, \) (4.14)

V. NUMERICAL EXAMPLES

A. Attitude Estimation

Consider an attitude estimation problem [6, 7], whose kinematics are modeled as
\[ \dot{e}(t) = -\frac{1}{2} \Omega(t)e(t), \] (5.1)
where the state vector is the quaternion vector \( x(t) = e(t) = [e_0(t) \ e_1(t) \ e_2(t) \ e_3(t)]^T \), the matrix \( \Omega(t) \) is given by
\[ \Omega(t) = \left[ \begin{array}{cccc} 0 & r(t) & -q(t) & p(t) \\ -r(t) & 0 & p(t) & q(t) \\ q(t) & -p(t) & 0 & r(t) \\ -p(t) & q(t) & r(t) & 0 \end{array} \right], \] (5.2)
and the angular velocity vector \( u(t) = [p(t) q(t) r(t)]^T \) is viewed as an input. Given that \( \|x(0)\|_2 = 1 \) and \( \Omega(t) \) is a skew-symmetric matrix, it follows that, for all \( t > 0 \),

\[
\|x(t)\|_2 = 1. \tag{5.3}
\]

We set \( x(0) = [0.9603 \ 0.1387 \ 0.1981 \ 0.1387]^T \), and

\[
u(t) = [0.03 \sin \left( \frac{2 \pi}{3} t \right) 0.03 \sin \left( \frac{2 \pi}{3} t - 300 \right) 0.03 \sin \left( \frac{2 \pi}{3} t - 600 \right)]^T. \tag{5.4}
\]

To perform attitude estimation, we assume that

\[
\tilde{u}_{k-1} = u((k-1)T) + \beta_{k-1} + w_{k-1}^u \tag{5.5}
\]

is measured by rate gyro, where \( T \) is the discretization step, \( w_{k-1}^u \in \mathbb{R}^3 \) is noise and \( \beta_{k-1} \in \mathbb{R}^3 \) is drift error. The discrete-time equivalent of (5.1) augmented by the gyro drift random-walk model [7] is given by

\[
\begin{bmatrix} e_k \\
\beta_k \end{bmatrix} = \begin{bmatrix} A_{k-1} & 0_{4 \times 3} \\
0_{3 \times 4} & I_{3 \times 3} \end{bmatrix} \begin{bmatrix} e_{k-1} \\
\beta_{k-1} \end{bmatrix} + \begin{bmatrix} 0_{4 \times 1} \\
\beta_{k-1} \end{bmatrix}, \tag{5.6}
\]

where \( e_k \triangleq e(kT) \), \( w_{k-1}^\beta \in \mathbb{R}^3 \) is process noise associated to drift-error model, \( x_k \triangleq \begin{bmatrix} e_k \\
\beta_k \end{bmatrix} \in \mathbb{R}^7 \) is the state vector, \( w_{k-1}^u \triangleq \begin{bmatrix} w_{k-1}^u_x \\
0 \end{bmatrix} \in \mathbb{R}^6 \) is the process noise, and

\[
\begin{align*}
A_{k-1} & \triangleq \cos(s_{k-1})I_{4 \times 4} - \frac{1}{2} T \sin(s_{k-1}) \Omega_{k-1}, \\
\Omega_{k-1} & \triangleq \Omega((k-1)T), \\
s_{k-1} & \triangleq \frac{T}{2} \left| \tilde{u}_{k-1} - \beta_{k-1} - w_{k-1}^u \right|_2.
\end{align*} \tag{5.7-5.9}
\]

The constraint (5.3) also holds for \( t = kT \), that is,

\[
x_{1,t}^2 + x_{2,t}^2 + x_{3,t}^2 + x_{4,t}^2 = 1. \tag{5.10}
\]

We set \( T = 0.1 \) s, \( \beta_{k-1} = [0.0001 \ -0.001 \ 0.0005]^T \) rad/s, and \( Q = \text{diag}(10^{-5} I_{3 \times 3}, 10^{-10} I_{3 \times 3}) \). Attitude observations \( y_{k}^i \in \mathbb{R}^3 \) for a single direction sensor (such as star tracker and three-axis magnetometer) are given by [7]

\[
y_{k}^i = C_k r_{k}^i + v_{k}^i, \tag{5.11}
\]

where \( r_{k}^i \in \mathbb{R}^3 \) is a reference direction vector to a known point, and \( C_k \) is the rotation matrix from the reference to the body-fixed frame,

\[
C_k = \begin{bmatrix} x_{1,t}^2 - x_{2,t}^2 - x_{3,t}^2 + x_{4,t}^2 \\
2(x_{1,t} x_{2,t} - x_{3,t} x_{4,t}) \\
-2(x_{1,t} x_{3,t} + x_{2,t} x_{4,t}) \\
2(x_{1,t} x_{4,t} + x_{2,t} x_{3,t}) \\
(-x_{1,t}^2 + x_{2,t}^2 - x_{3,t}^2 + x_{4,t}^2) \\
2(x_{2,t} x_{3,t} - x_{4,t} x_{4,t}) \\
2(x_{2,t} x_{4,t} + x_{3,t} x_{3,t}) \\
(-x_{2,t}^2 + x_{3,t}^2 - x_{4,t}^2 + x_{4,t}^2) \end{bmatrix}. \tag{5.12}
\]

We use the minimum necessary number of direction sensors, that is, two [7, 18] and set the reference vectors to \( r_{k}^{[1]} = [1 \ 0 \ 0]^T \) and \( r_{k}^{[2]} = [0 \ 1 \ 0]^T \). We set \( R_0 = 10^{-4} I_{6 \times 6} \). These direction measurements are assumed to be provided at a lower rate, specifically, at 1 Hz (10T s).

We implement Kalman filtering using UKF, ECUKF, PUKF, and MAUKF with equations (5.6), (5.11), and (5.10). We initialize these algorithms with \( \bar{x}_0 = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T \) and \( P_{0x} = \text{diag}(0.5 I_{4 \times 4}, 0.01 I_{3 \times 3}) \). Table I shows the results obtained from a 100-run Monte Carlo simulation. With the usage of prior knowledge, more informative (smaller trace of error covariance) estimates are produced compared to unconstrained estimates. However, a slight increase in the RMS error is observed for algorithms ECUKF and MAUKF implying loss of accuracy compared to UKF. The equality constraint is more closely tracked whenever a constrained filter is employed; see Figure 1. Percent errors around \( 10^{-4} \) are obtained and ECUKF exhibits the smallest error.

### Table I: Average of percent RMS constraint error, trace of error covariance matrix, and RMS estimation error for 100-run Monte Carlo simulation for attitude estimation using UKF, ECUKF, PUKF, and MAUKF.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>UKF</th>
<th>ECUKF</th>
<th>PUKF</th>
<th>MAUKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Percent RMS constraint error (( \times 10^{-4} ))</td>
<td>382.1</td>
<td>6.23</td>
<td>8.34</td>
<td>6.26</td>
</tr>
<tr>
<td>Trace of error covariance matrix for attitude (( \times 10^{-5} ))</td>
<td>8.21</td>
<td>3.42</td>
<td>8.17</td>
<td>3.43</td>
</tr>
<tr>
<td>Trace of error covariance matrix for drift (( \times 10^{-5} ))</td>
<td>9.214</td>
<td>9.204</td>
<td>9.213</td>
<td>9.208</td>
</tr>
</tbody>
</table>

![Fig. 1. Estimates of the quaternion vector norm using UKF, ECUKF, PUKF, and MAUKF algorithms. ECUKF, PUKF, and MAUKF estimates almost coincide.](image-url)
By conservation of energy in (5.13) we have
\[
-mgL \cos(x_1(t)) + \frac{mL^2}{2} \dot{x}_2^2(t) = E(0),
\]
(5.16)
where \(E(0)\) is the total mechanical energy and \(m\) is the pendulum mass. Next, we define the approximate energy \(E_k\) of the discrete-time model by
\[
E_k \triangleq -mgL \cos(x_{1,k}) + \frac{mL^2}{2} \dot{x}_{2,k}^2.
\]
(5.17)
We set \(L = 1\) m, \(T = 10\) ms, and initial conditions \(\theta(0) = \frac{\pi}{4}, \dot{\theta}(0) = 0\). We implement equality-constrained state estimation imposing \(E_k = E(0)\) for all \(k \geq 1\). The state estimation is initialized with \(Q_k = \sigma_w^2 I_2 \times 2, R_{k-1} = \sigma_v^2 I_2 \times 2, \dot{x}_0 = [1 \ 1]^T\), and \(P_{0}^{xx} = I_2 \times 2\), where three different values of observation noise are tested, namely, \(\sigma_v = 0.1, 0.25, \) and 0.5, and process noise with \(\sigma_w = 0.007\) is chosen to ensure convergence and consistency of estimates. A 100-run Monte Carlo simulation is performed for each \(\sigma_v\).

Table II shows the average percent RMS errors related to the equality constraint (5.17). Figure 2 compares the performance of the algorithms with relation to (5.17). It can be noticed, in this example, that the free-run simulation of the discretized model results in an unrealistic increasing energy \(E_k\). The UKF is not able to closely track the constraint. For higher observation noise levels, that is, \(\sigma_v = 0.5\), RMS constraint errors between 4% – 7% are observed. Whenever equality-constrained filters are employed, these indices are substantially reduced. In addition to the improvement in the constraint estimate, the usage of prior knowledge also results in smaller RMS estimate errors and associated covariances.

**TABLE II**: Average of percent RMS constraint error, trace of error covariance matrix, and RMS estimation error for 100-run Monte Carlo simulation for single pendulum, concerning different levels of observation noise \(\sigma_v = 0.1, 0.25, \) and 0.5, and algorithms, namely, UKF, ECUKF, PUKF, and MAUKF.

<table>
<thead>
<tr>
<th>(\sigma_v)</th>
<th>UKF</th>
<th>ECUKF</th>
<th>PUKF</th>
<th>MAUKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.6 \times 10^{-2}</td>
<td>1.33</td>
<td>2.3</td>
<td>1.34</td>
</tr>
<tr>
<td>0.25</td>
<td>4.1 \times 10^{-2}</td>
<td>2.86</td>
<td>5.7</td>
<td>2.86</td>
</tr>
<tr>
<td>0.5</td>
<td>5.7 \times 10^{-2}</td>
<td>5.2</td>
<td>11.4</td>
<td>5.19</td>
</tr>
</tbody>
</table>

Trace of error covariance matrix (\(10^{-2}\))

<table>
<thead>
<tr>
<th>(\sigma_v)</th>
<th>UKF</th>
<th>ECUKF</th>
<th>PUKF</th>
<th>MAUKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.24</td>
<td>0.697</td>
<td>0.35</td>
<td>0.697</td>
</tr>
<tr>
<td>0.25</td>
<td>4.37</td>
<td>1.836</td>
<td>2.22</td>
<td>1.837</td>
</tr>
<tr>
<td>0.5</td>
<td>11.0</td>
<td>3.778</td>
<td>5.59</td>
<td>3.779</td>
</tr>
</tbody>
</table>

RMS estimation error for \(x_1\) and \(x_2\) (\(10^{-2}\))

<table>
<thead>
<tr>
<th>(\sigma_v)</th>
<th>UKF</th>
<th>ECUKF</th>
<th>PUKF</th>
<th>MAUKF</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>1.92, 2.48</td>
<td>0.848, 1.794</td>
<td>0.89, 1.90</td>
<td>0.849, 1.794</td>
</tr>
<tr>
<td>0.25</td>
<td>3.08, 4.96</td>
<td>1.265, 2.832</td>
<td>1.49, 3.44</td>
<td>1.266, 2.833</td>
</tr>
<tr>
<td>0.5</td>
<td>4.81, 8.69</td>
<td>1.738, 3.874</td>
<td>2.51, 5.50</td>
<td>1.739, 3.876</td>
</tr>
</tbody>
</table>

According to Table III, the same comparative analysis is applicable when the true continuous-time model (5.13) or when a different linear observation model is used. The usage of a better discretized model (4-th Runge-Kutta integration method replacing Euler discretization) implies in a better performance, but with an approximately 6 times greater computational time.

This numerical example suggests that the ECUKF and MAUKF are better suited to equality-constrained nonlinear filtering than the PUKF. In this nonlinear example, the ECUKF and MAUKF performances are similar. Figure 3 illustrates how the estimate error for angular position \(x_{1,k}\) and associated covariance evolve with time. It is interesting to remark that, whenever \(x_{1,k} = 0\), a more informative estimate is observed for \(x_{2,k}\).
C. One-Dimensional Hydrodynamics

We consider state estimation for one-dimensional hydrodynamic flow based on a finite volume model [4, 5]. The flow of an inviscid, compressible fluid along a one-dimensional channel is governed by Euler’s equations [10]

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} \rho v, \quad (5.18)$$

$$\frac{d}{dt} \left( \frac{p}{\rho} \right) = 0, \quad (5.19)$$

$$\frac{\partial v}{\partial t} = \frac{\rho}{\rho} \frac{\partial v}{\partial x} - \frac{\partial p}{\partial x}, \quad (5.20)$$

where \( \rho \in \mathbb{R} \) is the density, \( v \in \mathbb{R} \) is the velocity, \( p \in \mathbb{R} \) is the pressure of the fluid, and \( \gamma = \frac{5}{3} \) is the ratio of specific heats of the fluid. A discrete-time model of hydrodynamic flow is obtained by using a finite-volume based spatial and temporal discretization.

Assume that the channel consists of \( l \) identical cells (see Figure 4). For all \( i = 1, \ldots, l \), let \( \rho[i], v[i], \) and \( p[i] \) be the density, velocity, and pressure in the \( i \)th cell, and define \( U[i] \in \mathbb{R}^3 \) by

$$U[i] = \left[ \begin{array}{c} \rho[i] \\ m[i] \\ E[i] \end{array} \right]^T, \quad (5.21)$$

where the momentum \( m[i] \) and energy \( E[i] \) in the \( i \)th cell are given by

$$m[i] = \rho[i] v[i], \quad E[i] = \frac{1}{2} \rho[i] (v[i])^2 + \frac{p[i]}{\gamma - 1}. \quad (5.22)$$

We use a second-order Rusanov scheme [13] to discretize (5.18)-(5.20) and obtain a discrete-time model that enables us to update the flow variables at the center of each cell.

![Fig. 4. One-dimensional grid used in the finite volume scheme.](image)

The discrete-time state update equation [13] is given by

$$U[k] = U[k-1] - \frac{T}{\Delta x} \left[ F[0]_{\text{Rus}, k-1} - F[0]_{\text{Rus}, k-1} \right], \quad (5.23)$$

where \( T > 0 \) is the sampling time and \( \Delta x \) is the width of each cell, and \( F[0]_{\text{Rus}, k-1} \) depends on \( U[k-1], \ldots, U[k+1] \). Hence, \( U[k] \) depends on \( U[k-1], \ldots, U[k+1] \), as expected for a second-order scheme.

Next, define the state vector \( x_k \in \mathbb{R}^{3(l-4)} \) by

$$x_k \triangleq \left[ (U[k])^T \cdots (U[k+2])^T \right]^T. \quad (5.24)$$

Furthermore, we assume reflective boundary conditions at cells with indices 1, 2, \( l - 1 \) and \( l \) so that the cells 1, 2, \( l - 1 \) and \( l \) mimic walls reflecting any waves incident on them. It follows from (5.23) that the second-order Rusanov scheme yields a nonlinear discrete-time update model of the form \( x_k = f(x_{k-1}) \). Let \( l = 54 \) so that \( n = 150 \) and \( x_k \in \mathbb{R}^n \). We assume that the truth model is given by \( x_k = f(x_{k-1}) + G w_{k-1} \) (2.1), where \( w_{k-1} = \eta \times 1, \) \( w_{k-1} \in \mathbb{R} \), \( Q_{k-1} = I_{6 \times 6} \), and \( G \in \mathbb{R}^{150 \times 1} \), where

$$G_j = \begin{cases} 0.7, & \text{if } j = 7, 22, 37, 67, 97, 127, \\
0, & \text{otherwise.} \end{cases} \quad (5.25)$$

It follows from (2.1) and (5.25) that although the flow variables in the 5th, 10th, 15th, 25th, 35th, and 45th cell are directly affected by \( w_{k-1} \), the total density and the energy of the cells \( 3, \ldots, l - 2 \), is not altered by \( w_{k-1} \), that is,

$$\sum_{i=3}^{l-2} \rho_k[i] = \sum_{i=3}^{l-2} \rho_0[i], \quad \sum_{i=3}^{l-2} E_k[i] = \sum_{i=3}^{l-2} E_0[i]. \quad (5.26)$$

So, the linear equality constraint in (5.26) can be expressed as (3.4) with

$$D = \left[ \begin{array}{cccccc} 1 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{array} \right], \quad d = \left[ \begin{array}{c} \sum_{i=3}^{l-2} \rho_k[i] \\ \sum_{i=3}^{l-2} E_k[i] \end{array} \right]. \quad (5.27)$$

Next, for \( i = 3, \ldots, l - 2 \), define \( C[i] \in \mathbb{R}^{3 \times 3(l-4)} \)

$$C[i] \triangleq \left[ \begin{array}{ccc} 0_{3 \times \delta(i-4-i)} & I_{3 \times 3} & 0_{3 \times 3(l-i-1)} \end{array} \right]. \quad (5.28)$$

So that the linear measurement model given by (3.3), for \( y_k \in \mathbb{R}^m \), \( m = 6 \), corresponding to density, momentum and energy at cells with indices 24 and 26, has \( C = \)
[\begin{bmatrix} C^{(24)} \end{bmatrix}^T \begin{bmatrix} C^{(26)} \end{bmatrix}^T]^T \text{ and } R = 0.01I_{5 \times 5}. \text{ That is, we have a non-linear process model with additive noise, a linear observation model, and a linear equality constraint.}

Next, we compare the performance of UKF, ECUKF, PUKF, and MAUKF. Note that since the constraint in (5.27) and the observation model are linear, ECUKF and MAUKF estimates are equal [25]. The RMS error in estimates at each cell \( i = 1, \ldots, 54 \) is shown in Figure 5. The error in the energy estimates when no data assimilation is performed is also shown in the same figure for comparison. Note that the performance of the ECUKF, PUKF, and MAUKF are better than the performance of UKF because of the enforcement of the constant total density and energy constraint. Figure 6 shows the total density estimated by UKF, ECUKF, PUKF, and MAUKF. The actual total density of the truth model is also plotted for comparison. The total density of the estimates of ECUKF, PUKF, and MAUKF is very close to the truth model, but UKF does not conserve total density or the total energy.

VI. CONCLUDING REMARKS

We have addressed the equality-constrained state-estimation problem for nonlinear systems. Three suboptimal algorithms based on the unscented Kalman filter (UKF) were derived, namely, the equality-constrained unscented Kalman filter (ECUKF), the projected unscented Kalman filter (PUKF), and the measurement-augmented unscented Kalman filter (MAUKF). These methods were compared on three examples, including a quaternion-based attitude estimation problem, as well as a mechanical system and an idealized flow model, both involving conserved quantities.

Numerical results suggest that, in addition to very closely satisfying a known nonlinear equality constraint of the system, the proposed methods can be used to produce more accurate and more informative estimates. Moreover, ECUKF and MAUKF seem to be the most suitable for equality-constrained filtering applications.

We have also addressed the case where an approximate discretized model is used to represent a continuous-time process in state estimation. Improved estimates were obtained when equality-constrained filtering algorithms were employed to enforce conserved quantities of the original continuous-time model, but without the higher computational burden required by integration schemes.

REFERENCES


