A Viability Approach for the Stabilization of an Underactuated Underwater Vehicle in the Presence of Current Disturbances

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***Abstract*— In this paper we present a viability-based formulation for the stabilization of an underactuated underwater vehicle under the influence of a known, constant current and state constraints. The stabilization problem is described by three problems in terms of viability theory. We present a solution to the first problem which addresses the safety of the system, i.e. guarantees that there exists a control law such that the vehicle always remains into the safe set of state constraints. In order to overcome the computational limitations due to the high dimension of the system we develop a two-stage approach, based on forward reachability and game theory. The control law is thus the safety controller when the system viability is at stake, i.e. close to the boundary of the safe set. The viability kernel and the control law are numerically computed.**

I. Introduction

Control of underactuated underwater vehicles and surface vessels has received great interest over the past fifteen years, motivated by their extensive use in oil industry, scientific explorations etc. The design of stabilizing controllers for this class of vehicles is challenging, since they usually exhibit second-order nonholonomic constraints and therefore can not be stabilized by continuous, time invariant state feedback control laws [1]. Furthermore, their dynamics include nonlinear, complex hydrodynamic terms which should not be neglected during the control design. Environmental disturbances should be also considered so that the closed-loop system performs efficiently in real environmental conditions.

Various control strategies have been proposed for the stabilization of underactuated marine vehicles. In [2] a smooth state-feedback law stabilizes an underactuated ship to an equilibrium manifold. Smooth, time-varying controllers yielding asymptotic stability to the origin are proposed in [3]–[6] whereas discontinuous controllers in [7]–[11]. Hybrid control schemes have been also presented in [12]–[14].

None of the aforementioned studies takes into account the influence of environmental disturbances. To the best of our knowledge, pioneer work in this direction is presented in [15], which considers the dynamic positioning of a ship. The proposed time-varying control law provides semi-global practical asymptotic stability. In [16] the dynamic positioning of an underactuated AUV in the presence of a constant, unknown current is considered. An adaptive controller yields convergence to a desired target point, whereas the final orientation is aligned with the direction of the current. The same philosophy regarding the final orientation is adopted in [17], which addresses the station-keeping for a surface vessel in the presence of wind disturbances. In [18] a switching feedback control law stabilizes an underactuated AUV around a small neighborhood of the origin, yielding input-to-state practical stability in the presence of disturbances and measurement noise. Despite these contributions, it is generally accepted that the stabilization of underactuated underwater vehicles in the presence of disturbances has only been partially addressed and is still open in many respects.

In this paper we consider the motion of an underactuated underwater vehicle on the horizontal plane, in the presence of a constant, known current. Based on the remarks of [16] we would like to stabilize the vehicle within a desired set - the goal set, rather than a single point. This choice is motivated by the fact that both the vehicle's position and orientation are critical for many applications, e.g. during inspection tasks. Thus we prefer not to specify the final orientation to be depended on the current direction. Moreover, we take into account that practical systems are usually subject to constraints that can not be violated by any of the system trajectories. More specifically, we consider the problem of regulating a low-weight underactuated ROV to a desired goal set with respect to a specific target, in the presence of a known, constant current, so that this target is always visible through the camera of the vehicle. This specification is motivated by the fact that the resulting closed-loop system could be used, for example, for ship-hull inspections.

We propose an approach towards the solution of this problem by formulating it within the framework of viability theory [19]. The stabilization problem is described by three viability problems. We address the first one in this paper, known as the safety problem. Considering a safe set of state constraints, resulting from the task specifications and the limited sensing capability of the vehicle, we investigate whether there exists a control law that keeps the vehicle in this safe set, despite the influence of the current. We adopt the theoretical results of [20], which connect viability with optimal control. The resulting ”bang-bang” control law guarantees that the vehicle will remain in the safe set. In order to overcome the computational limitations due to the high dimension of the system, we propose a two-stage analysis, based on forward reachability and game theory.

The remainder of the paper is organized as follows; Section II establishes a framework for the stabilization problem based on viability theory. Section III presents the mathematical modeling of the system and state constraints. Section IV

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gives the viability analysis and Section V the computational results. The conclusion and thoughts for further work are summarized in Section VI.

II. Problem Formulation into Viability Theory

Viability theory [19] describes the evolution of systems under the consideration that for different reasons, not all system evolutions are feasible. The system must obey state constraints, called viability constraints and system solutions should be viable in the sense that they must satisfy, at each instant, these constraints.

The problem of stabilizing an underactuated underwater vehicle in a goal set under state constraints and current disturbances is described by the following viability problems:

1) Consider a control system described by

$$\dot{x}(t) \in F\left(x(t)\right) \text{ with } F\big(x(t)\big) = \big\{ f\big(x(t),u\big) | u \in U \big\}, \quad (1)$$

where $x(\cdot) \subseteq \mathbb{R}^n$ is the state vector, $u \in U \subseteq \mathbb{R}^m$ is the control vector, $U \subseteq \mathbb{R}^m$ is compact, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the bounded, uniformly continuous single-valued map of system dynamics and $F\big(x(t)\big)$ is the set of available velocities. Given a set S of viability constraints, describing that the target must always be in the camera field of view, determine a set of initial states $K \subseteq S$ such that for every initial state $x_0 \in K$ there exists at least one solution to (1) starting at x_0 which remains for ever in S, keeping the target in the camera field of view. We say that K is a viability domain of the system. We would like to determine the maximal viability domain contained in S, known as the viability kernel of S, $Viab_F(S)$.

2) Given the viability kernel K of (1) and a goal set $G \subseteq K$, describing that the target is near to the center of the camera field of view, determine the set of initial states $x \in K$ such that there exists at least one solution to (1) starting at x that reaches the goal set G in finite time, without leaving S. This set is called the capture basin of the goal G in K, $Capt_F^K(G)$.

3) Finally, determine a control law such that the solutions to (1) starting at $x_s \in G$ remain for ever in G, i.e. once the system reaches G, it is then stabilized in it. In that case, G is a viability domain of (1).

In this paper, we consider the first of the three parts, known as the safety problem.

III. Mathematical Modeling

We consider the 3-DOF motion on the horizontal plane of an underwater vehicle with two back thrusters but no side thruster; this is a common configuration for marine vehicles. Roll and pitch angles remain always very close to zero, $\phi \approx 0$ and $\theta \approx 0$ respectively, because of the vehicle's mass configuration. The position and orientation vector of the vehicle with respect to a global coordinate frame G is defined as $\boldsymbol{\eta} = \begin{bmatrix} x & y & \psi \end{bmatrix}^T$ whereas the linear and angular velocity vector is defined in the body-fixed coordinate frame B as $\boldsymbol{\nu} = \begin{bmatrix} u & v & r \end{bmatrix}^T$. Following [21] the 3-DOF kinematic equations are : $\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\psi)\boldsymbol{\nu} \Leftrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ r \end{bmatrix}$, and the 3-DOF dynamic equations of motion are: $\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{D}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau} + \boldsymbol{\tau_E}$, where $\boldsymbol{M} = \boldsymbol{M}^T > \boldsymbol{0} \in \mathbb{R}^{3\times3}$ is the inertia matrix including added mass, $\boldsymbol{C}(\boldsymbol{\nu}) = -\boldsymbol{C}^T(\boldsymbol{\nu}) \in \mathbb{R}^{3\times3}$ is the matrix of Coriolis terms including added mass, $\boldsymbol{D}(\boldsymbol{\nu}) > \boldsymbol{0} \in \mathbb{R}^{3\times3}$ is the damping matrix, $\boldsymbol{\tau} \in \mathbb{R}^3$ is the vector of control inputs and $\boldsymbol{\tau_E} \in \mathbb{R}^3$ is the vector of environmental disturbances due to waves, currents and cable effects.

The vehicle moves under the influence of a known, non-rotational, constant current, with velocity V_c and direction β_c with respect to the global frame G. The effect of current-induced forces and moments is modeled in terms of the body-fixed relative velocity $\boldsymbol{\nu_r} = \boldsymbol{\nu} - \boldsymbol{\nu_c}$ [21], where $\boldsymbol{\nu_c} = \boldsymbol{J}^{-1}(\psi)\boldsymbol{V_C^G}$ and $\boldsymbol{V_C^G} = \begin{bmatrix} V_c\cos\beta_c & V_c\sin\beta_c & 0 \end{bmatrix}^T$. The kinematics are written with respect to $\boldsymbol{\nu_r} = \begin{bmatrix} u_r & v_r & r \end{bmatrix}^T$ as $\dot{\boldsymbol{\eta}} = \boldsymbol{J}(\psi)\boldsymbol{\nu_r} + \boldsymbol{V_C^G} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ v_r \\ r \end{bmatrix} + \begin{bmatrix} V_c\cos\beta_c \\ V_c\sin\beta_c \\ 0 \end{bmatrix}$, whereas the dynamics are [21] $\boldsymbol{M}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} + \boldsymbol{C_A}(\boldsymbol{\nu_r})\boldsymbol{\nu_r} + \boldsymbol{D}\big(|\boldsymbol{\nu_r}|\big)\boldsymbol{\nu_r} = \boldsymbol{\tau}$, where $\boldsymbol{M} = \begin{bmatrix} m-X_{\dot{u}} & 0 & 0 \\ 0 & m-Y_{\dot{v}} & 0 \\ 0 & 0 & I_z-N_{\dot{r}} \end{bmatrix}$, $\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} 0 & 0 & -mv \\ 0 & 0 & mu \\ mv & -mu & 0 \end{bmatrix}$, $\boldsymbol{C_A}(\boldsymbol{\nu_r}) = \begin{bmatrix} 0 & 0 & Y_{\dot{v}}v_r \\ 0 & 0 & -X_{\dot{u}}u_r \\ -Y_{\dot{v}}v_r & X_{\dot{u}}u_r & 0 \end{bmatrix}$, $\boldsymbol{D_L} = \begin{bmatrix} -X_u & 0 & 0 \\ 0 & -Y_v & -Y_r \\ 0 & -N_v & -N_r \end{bmatrix}$, $\boldsymbol{D}_{NL}\big(|\boldsymbol{\nu_r}|\big) = \begin{bmatrix} -X_{u|u|}|u_r| & 0 & 0 \\ 0 & -Y_{v|v|}|v_r| & 0 \\ 0 & 0 & -N_{r|r|}|r| \end{bmatrix}$, $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ 0 \\ \tau_2 \end{bmatrix}$, m is the mass and I_z is the moment of inertia with respect to z axis of the vehicle, $X_{\dot{u}}, Y_{\dot{v}}, N_{\dot{r}}$ are the added mass terms, X_u, Y_v, Y_r, N_v, N_r are linear drag terms, $X_{u|u|}, Y_{v|v|}, N_{r|r|}$ are nonlinear drag terms, τ_1 and τ_2 are control inputs in surge and yaw DOF. Under the substitution $\boldsymbol{\nu} = \boldsymbol{\nu_r} + \boldsymbol{\nu_c}$, the kinematic and dynamic equations are rewritten as:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u}_r \\ \dot{v}_r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_r\cos\psi - v_r\sin\psi + V_c\cos\beta_c \\ u_r\sin\psi + v_r\cos\psi + V_c\sin\beta_c \\ r \\ \frac{1}{m_{11}}\big(m_{22}v_r r + X_u u_r + X_{u|u|}|u_r|u_r + X_{\dot{u}}V_c\sin(\beta_c-\psi)r\big) \\ \frac{1}{m_{22}}\big(-m_{11}u_r r + Y_v v_r + Y_r r + Y_{v|v|}|v_r|v_r - Y_{\dot{v}}V_c\cos(\beta_c-\psi)r\big) \\ \frac{1}{m_{33}}\big((m_{11}-m_{22})u_r v_r + N_v v_r + N_r r + N_{r|r|}|r|r\big) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_{11}} \\ 0 \\ 0 \end{bmatrix}\tau_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{m_{33}} \end{bmatrix}\tau_2 \Rightarrow \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, V_c, \beta_c) + \sum_{i=1,2}\boldsymbol{g}_i\tau_i \quad (2)$$

where $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{\eta}^T & \boldsymbol{\nu_r}^T \end{bmatrix}^T$ is the state vector, $\boldsymbol{f}(\boldsymbol{x}, V_c, \beta_c)$ is the drift vector field, $\boldsymbol{g}_1, \boldsymbol{g}_2$ are the control vector fields, $m_{11} = m - X_{\dot{u}}$, $m_{22} = m - Y_{\dot{v}}$, $m_{33} = I_z - N_{\dot{r}}$. Moreover, the thrust allocation implies that $\tau_1 = F_P + F_{ST}$ and $\tau_2 = D\left(F_P - F_{ST}\right)$, where $F_P \in [-F_p, F_p]$, $F_{ST} \in [-F_{st}, F_{st}]$ are the port and starboard thrust forces and $2D$ is the distance between the two thrusters. Thus, $\boldsymbol{u} = \begin{bmatrix} F_P & F_{ST} \end{bmatrix}^T \in U \subset \mathbb{R}^2$ is the vector of control inputs for (2), where $U = [-F_p, F_p] \times [-F_{st}, F_{st}]$.

A. Modeling of Viability Constraints

We consider the set of state constraints that result from a vision-based sensor system, which employs the onboard camera and two laser pointers mounted on the ROV [22]. The sensor system provides the vehicle's pose vector $\boldsymbol{\eta}$ with respect to the global frame G on the center of a target, which is assumed to lay on a vertical surface A, see Fig. 1. The target and the two laser dots projected on the

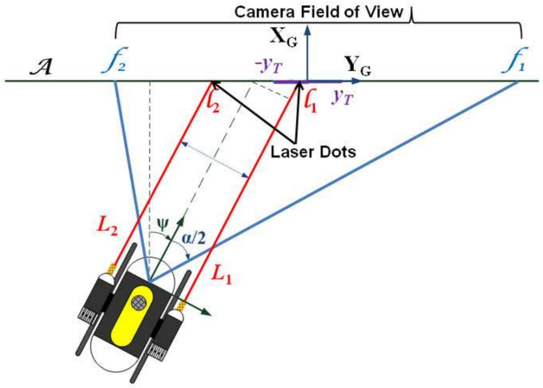

Fig. 1. Modeling of the State Constraints imposed by the Sensor System

surface are tracked using computer vision algorithms and this information is used to estimate the pose vector $\boldsymbol{\eta}$.

We define the safe set of the system as the set $\boldsymbol{S}$ such that
1. The target and the laser dots must always be in the camera field of view, i.e. $[-y_T, y_T] \subseteq [f_2, f_1]$ and $[l_2, l_1] \subseteq [f_2, f_1]$.
2. The ranges L_1, L_2 must be less than a critical range L.
3. The distance between the laser dots on the image plane must be greater than a minimum distance ε, so that they do not overlap and are effectively detected.
4. The width of the target on the image plane must be greater than a critical value δ, so that the target is sufficiently visible.

These specifications impose k nonlinear inequality constraints $c_j(x,y,\psi) \leq 0,\ j = 1,\ldots,k$ determining the safe set $\boldsymbol{S}$. The vector $\boldsymbol{\eta}$ must always remain in $\boldsymbol{S}$ for the sensor system to be effective. The analytical expression of $c_j(x,y,\psi) \leq 0$, $j = 1,\ldots,k$ is omitted here in the interest of space.

IV. Viability Analysis

We are interested in determining the viability kernel of $\boldsymbol{S}$, $Viab(\boldsymbol{S})$ under (2) and a control law which guarantees that the system trajectories starting in the kernel will remain for ever in it. We adopt the approach presented in [20] which relates viability with minimum-cost optimal control, coding the viability kernel as the level set of the value function of an appropriate optimal control problem.

A. An Optimal Control Problem related to Viability

Consider the control system (1) and let $\mathscr{U}_{[0,T]}$ denote the set of Lebesgue measurable functions $u(\cdot) : [0,T] \to U$, with $T > 0$ an arbitrary time horizon. Given a set of state constraints S, the control input $u(\cdot) \in \mathscr{U}_{[0,T]}$ should be selected so that the viability constraints are met at each time instant $t \in [0,T]$. Let us define a cost function $\ell(\cdot) : \mathbb{R}^n \to \mathbb{R}$ of the state x, over the time horizon $[0,T]$, such that $\ell(x) > 0$ for $x \in S$ and $\ell(x) \leq 0$ for $x \notin S$. Then, the objective for the control input $u(\cdot)$ is to maximize the minimum value attained by the cost function $\ell(\cdot)$ along the state trajectory $x(t)$ over the horizon $[0,T]$. The value function of this optimal control problem (SUPMIN problem) is defined as $V(x,t) = \sup_{u(\cdot)\in\mathscr{U}_{[0,T]}} \min_{t\in[0,T]} \ell(x(t))$. One can show [20] that the set $\{x \in \mathbb{R}^n \,|\, V(x,t) > 0\}$ is precisely the set of states for which there exists a control input $u(\cdot) \in \mathscr{U}_{[0,T]}$ such that $x(t) \in S$ for all $t \in [0,T]$, i.e. the viability kernel $Viab(S)$. Moreover, using dynamic programming, it can be also shown [20] that $V(x,t)$ is the unique, bounded and uniformly continuous viscosity solution to $\frac{\partial V}{\partial t}(x,t) + \min\left\{0, \sup_{u\in U} \frac{\partial V}{\partial x}(x,t) f(x,u)\right\} = 0$, with $V(x,T) = \ell(x)$ over $(x,t) \in \mathbb{R}^n \times [0,T]$, where the Hamiltonian function is defined as $\mathscr{H}_1 = \sup_{u\in U} p^T f(x,u)$. Thus, in our case we encode the viability constraints as the cost function $\boldsymbol{\ell}(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ such that $\boldsymbol{\ell}(\boldsymbol{\eta}) > 0$ for $\boldsymbol{\eta} \in \boldsymbol{S}$ and $\boldsymbol{\ell}(\boldsymbol{\eta}) \leq 0$ for $\boldsymbol{\eta} \notin \boldsymbol{S}$. An illustration of $\boldsymbol{\ell}(\boldsymbol{\eta})$ for orientation angle $\psi = 0$ is given in Fig. 2. The region in black color is where $\boldsymbol{\ell}(x,y) > 0$, i.e. the safe state space on the xy plane for $\psi = 0$. Substituting (2) into the Hamiltonian yields:

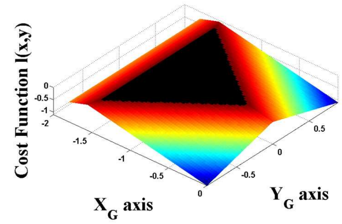

Fig. 2. Cost Function $\boldsymbol{\ell}(x,y)$

$$\begin{aligned}
\mathscr{H}_1 = \sup_{\boldsymbol{u}\in U} \Big(& p_1 \left(u_r \cos\psi - v_r \sin\psi + V_c \cos\beta_c \right) + \\
& + p_2 \left(u_r \sin\psi + v_r \cos\psi + V_c \sin\beta_c \right) + p_3 r + \\
& + p_4 \left(\tfrac{1}{m_{11}} \left(m_{22} v_r r + X_u u_r + X_{u|u|} |u_r| u_r + X_{\dot{u}} V_c \sin(\beta_c - \psi) r \right) \right) + \\
& + p_5 \left(\tfrac{1}{m_{22}} \left(-m_{11} u_r r + Y_v v_r + Y_r r + Y_{v|v|} |v_r| v_r - Y_{\dot{v}} V_c \cos(\beta_c - \psi) r \right) \right) + \\
& + p_6 \left(\tfrac{1}{m_{33}} \left((m_{11} - m_{22}) u_r v_r + N_v v_r + N_r r + N_{r|r|} |r| r \right) \right) + \\
& + \left(p_4 \tfrac{1}{m_{11}} + p_6 \tfrac{D}{m_{33}} \right) \hat{u}_1 + \left(p_4 \tfrac{1}{m_{11}} - p_6 \tfrac{D}{m_{33}} \right) \hat{u}_2 \Big)
\end{aligned}$$

where $p_i = \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}_i}$, $i = 1,\ldots,6$, $u_1 = F_P$ and $u_2 = F_{ST}$. From this, we can conclude that the optimal controls which ensure that the viability constraints are met whenever possible are:

$$\hat{u}_1 = \begin{cases} F_p & \text{if } \frac{p_4}{m_{11}} + \frac{p_6 D}{m_{33}} \geq 0 \\ -F_p & \text{if } \frac{p_4}{m_{11}} + \frac{p_6 D}{m_{33}} < 0 \end{cases}, \quad \hat{u}_2 = \begin{cases} F_{st} & \text{if } \frac{p_4}{m_{11}} - \frac{p_6 D}{m_{33}} \geq 0 \\ -F_{st} & \text{if } \frac{p_4}{m_{11}} - \frac{p_6 D}{m_{33}} < 0 \end{cases} \tag{3}$$

whereas the viability kernel $Viab(\boldsymbol{S})$ is given as the set of states for which $\boldsymbol{V}(\boldsymbol{\eta},t) > 0$. However, the existing computational tools for time-dependent Hamilton-Jacobi PDEs are effective for low-dimensional problems (1-4 dimensions).

B. Reachability Analysis

In order to overcome the computational limitations due to the high dimension of the system, we split (2) into:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} V_c \cos\beta_c \\ V_c \sin\beta_c \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\psi \\ \sin\psi \\ 0 \end{bmatrix} u_r + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} -\sin\psi \\ \cos\psi \\ 0 \end{bmatrix} v_r \tag{4}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{u}_r \\ \dot{v}_r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} r \\ \frac{1}{m_{11}} \left(m_{22} v_r r + X_u u_r + X_{u|u|} |u_r| u_r + X_{\dot{u}} V_c \sin(\beta_c - \psi) r \right) \\ \frac{1}{m_{22}} \left(-m_{11} u_r r + Y_v v_r + Y_r r + Y_{v|v|} |v_r| v_r - Y_{\dot{v}} V_c \cos(\beta_c - \psi) r \right) \\ \frac{1}{m_{33}} \left((m_{11} - m_{22}) u_r v_r + N_v v_r + N_r r + N_{r|r|} |r| r \right) \end{bmatrix} + \\ + \begin{bmatrix} 0 \\ 1/m_{11} \\ 0 \\ D/m_{33} \end{bmatrix} F_P + \begin{bmatrix} 0 \\ 1/m_{11} \\ 0 \\ -D/m_{33} \end{bmatrix} F_{ST} \tag{5}$$

At this point, the consideration of (4) and (5) inspires us

1. to investigate the forward reachability of subsystem (5) over the time horizon $[0,T]$, i.e. to compute the set $\mathscr{F}$ of states $\boldsymbol{x_2} = [\psi, u_r, v_r, r]^T$ which the system trajectories can reach starting from an initial set $\mathscr{N}$. In this way, we acquire an estimation for the bounds of u_r, v_r, r that can be reached from an initial set during the system evolution, so that

2. further on, to investigate the viability of subsystem (4), considering the relative velocities u_r, r as the control inputs along the two actuated DOF and the relative velocity v_r as a disturbance along the unactuated DOF.

The concept of reachability is mostly used for the safety analysis of continuous and hybrid systems. Given an initial set of states $\mathscr{N}$, the forward reachable set $\mathscr{F}$ is the set of states that can be reached at time $t \in [0,T]$ by the system trajectories starting from $\mathscr{N}$, whereas the backward reachable set $\mathscr{B}$ is the set of states from which start the system trajectories that can reach the set $\mathscr{N}$ at time $t \in [-T,0]$, $T > 0$ is an arbitrary time horizon.

We consider the relation between reachability and minimum-cost optimal control [20]. Given the control system (1) and a set of states $\mathscr{N}$, the reachable set is $Reach(t,\mathscr{N}) = \left\{x \in \mathbb{R}^n \middle| \exists u(\cdot) \in \mathscr{U}_{[0,T]}\ \exists t \in [0,T]\ x(t) \in \mathscr{N}\right\}$. This definition coincides which the one for the backward reachable set $\mathscr{B}$, taking into account that $t \in [-T,0]$. Furthermore, one can show the connection between the reachability problem and the invariance problem. The invariant set of (1) is defined as $Inv(t,\mathscr{N}) = \left\{x \in \mathbb{R}^n \middle| \forall u(\cdot) \in \mathscr{U}_{[0,T]}\ \forall t \in [0,T]\ x(t) \in \mathscr{N}\right\}$, i.e. as the set of initial states from which *all* the system trajectories remain for ever in $\mathscr{N}$. It is easily verified that $Reach(t,\mathscr{N}) = (Inv(t,\mathscr{N}^c))^c$, where $\mathscr{N}^c$ is the complement of $\mathscr{N}$. The invariance problem is formulated as an optimal control problem (INFMIN problem) [20]. The control objective is to minimize the minimum value of the cost function $\ell(\cdot)$ defined such that $\ell(x) \geq 0$ for $x \in \mathscr{N}$ and $\boldsymbol{\ell}(x) < 0$ for $x \notin \mathscr{N}$, over the time horizon $[0,T]$. The value function $V_2(x,t) = \inf_{u(\cdot)\in\mathscr{U}_{[0,T]}} \min_{t\in[0,T]} \ell(x(t))$ is proven to be the unique, bounded and uniformly continuous viscosity solution to $\frac{\partial V_2}{\partial t}(x,t) + \min\left\{0, \inf_{u\in U} \frac{\partial V_2}{\partial x}(x,t) f(x,u)\right\} = 0$, with $V_2(x,T) = \ell(x)$ over $(x,t) \in \mathbb{R}^n \times [0,T]$, where the Hamiltonian function is defined as $\mathscr{H}_2 = \inf_{u\in U} p^T f(x,u)$. The invariant set of (1) is $Inv(t,\mathscr{N}) = \left\{x \in \mathbb{R}^n \middle| V_2(x,t) \geq 0\right\}$. Consequently, if the cost function of the INFMIN problem is defined as $\ell(x) \geq 0$ for $x \in \mathscr{N}^c$ and $\ell(x) < 0$ for $x \notin \mathscr{N}^c$, the solution of the above PDE yields the invariant set $Inv(t,\mathscr{N}^c) = \left\{x \in \mathbb{R}^n \middle| V_2(x,t) \geq 0\right\}$. Then, the (backward) reachable set is $Reach(t,\mathscr{N}) = \left\{x \in \mathbb{R}^n \middle| V_2(x,t) < 0\right\}$.

Therefore, by considering the INFMIN problem for (5) we determine the backward reachable set for an initial set of states $\boldsymbol{x_2} \in \mathscr{N}$. Finally, we consider the connection between forward and backward reachability [23], which states that the forward reachable set of a control system H is the same with the backward reachable set of the system $\overleftarrow{H}$ with inverse dynamics. Thus, by substituting (5) with inverse dynamics in the Hamiltonian, $\mathscr{H}_2 = \inf_{\boldsymbol{u}\in U}\left(-p^T \boldsymbol{f_2}(\boldsymbol{x_2},\boldsymbol{u})\right)$, where $p_i = \frac{\partial V_2}{\partial \boldsymbol{x_2}_i}$, $i = 1,\ldots,4$ and $\boldsymbol{f_2}$ the drift vector field of (5), we can conclude that the optimal control inputs for the forward reachability computation are:

$$\hat{u}_1 = \begin{cases} F_p & \text{if } \frac{p_2}{m_{11}} + \frac{p_4 D}{m_{33}} \geq 0 \\ -F_p & \text{if } \frac{p_2}{m_{11}} + \frac{p_4 D}{m_{33}} < 0 \end{cases},\ \hat{u}_2 = \begin{cases} F_{st} & \text{if } \frac{p_2}{m_{11}} - \frac{p_4 D}{m_{33}} \geq 0 \\ -F_{st} & \text{if } \frac{p_2}{m_{11}} - \frac{p_4 D}{m_{33}} < 0 \end{cases}$$

The computational results are given in Section IV.

C. Viability Analysis using a Differential Game Formulation

Given the estimation for the bounds of u_r, v_r, r as $u_r = [u_{rm}, u_{rM}]$, $v_r = [v_{rm}, v_{rM}]$, $r = [r_m, r_M]$ we investigate the viability of the subsystem (4) in the safe set $\boldsymbol{S}$, where $\boldsymbol{x_1} = \begin{bmatrix} x & y & \psi \end{bmatrix}^T \in \mathbb{R}^3$ is the state vector, $\boldsymbol{u_1} = \begin{bmatrix} u_r & r \end{bmatrix}^T \in U_1 \subset \mathbb{R}^2$ are considered as the bounded control inputs, $v_r \in V_r \subset \mathbb{R}$ is considered as a bounded disturbance in the unactuated DOF, $\mathscr{U}_{1[0,T]}$ is the set of Lebesgue measurable functions $\boldsymbol{u_1}(\cdot) : [0,T] \to U_1$ and $\mathscr{V}_{r[0,T]}$ is the set of Lebesgue measurable functions $v_r(\cdot) : [0,T] \to V_r$.

We follow the formulation of a differential game with two players [24]. The control input $\boldsymbol{u_1}(\cdot)$ is the first player who tries to keep the vehicle into the safe set $\boldsymbol{S}$, whereas the disturbance $v_r(\cdot)$ is the second player who tries to drive the vehicle out of $\boldsymbol{S}$. Furthermore, it is important to define what information the players know about each others decisions. A state feedback strategy, i.e. allowing both players to choose their actions based on the current state, is the most appropriate for the problem considered here. However, state feedback is not easily formulated into Hamilton-Jacobi PDEs [25]. Besides, it is preferable to underapproximate the viability kernel rather than overapproximate it. Therefore we give the advantage to the disturbance $v_r(\cdot)$, which tries to make the viable set larger, by allowing the control input $\boldsymbol{u_1}(\cdot)$ to use only non-anticipative strategies, as presented in [24]. Consequently, computing the viability kernel for (4) is equivalent with computing the set of initial states for which the control input $\boldsymbol{u_1}(\cdot)$ wins the game. This set is called the discriminating kernel of $\boldsymbol{S}$, $Disc(t,\boldsymbol{S}) = \{\boldsymbol{x_1} \in \mathbb{R}^3 \big| \exists \text{ nonant/ve } \gamma(\cdot)\ \forall v_r \in \mathscr{V}_{r[t,T]}\ \forall t_1 \in [t,T]\ \boldsymbol{x_1}(t_1) \in \boldsymbol{S}\}$. One can show [24] that $Disc(t,\boldsymbol{S}) = \left\{\boldsymbol{x_1} \in \mathbb{R}^3 \middle| \boldsymbol{V_1}(\boldsymbol{x_1},t) > 0\right\}$ where $\boldsymbol{V_1}(\boldsymbol{x_1},t)$ is the value function $\boldsymbol{V_1}(\boldsymbol{x_1},t) = \sup_{\text{nonant/ve } \boldsymbol{u_1}(\cdot)} \inf_{v_r(\cdot)\in\mathscr{V}_{r[t,T]}} \min_{t_1\in[t,T]} \boldsymbol{\ell}(\boldsymbol{x_1}(t_1))$ of a SUPMIN problem with cost function $\boldsymbol{\ell}(\cdot) : \mathbb{R}^3 \to \mathbb{R}$ defined in Section IV-A. Moreover, $\boldsymbol{V_1}(\boldsymbol{x_1},t)$ is shown to be the unique, bounded and uniformly continuous viscosity solution to $\frac{\partial \boldsymbol{V_1}}{\partial t}(\boldsymbol{x_1},t) + \min\{0, \sup_{\boldsymbol{u_1}\in U_1} \inf_{v_r\in V_r} \frac{\partial \boldsymbol{V_1}}{\partial \boldsymbol{x_1}}(\boldsymbol{x_1},t)\, \boldsymbol{f_1}(\boldsymbol{x_1},\boldsymbol{u_1},v_r)\} = 0$ over $t \in [0,T]$ with $\boldsymbol{V_1}(\boldsymbol{x_1},T) = \boldsymbol{\ell}(\boldsymbol{x_1})$. Thus, the solution of this PDE yields the discriminating kernel of (4) and an optimal control law which guarantees that the trajectories of (4) starting in $Disc(t,\boldsymbol{S})$ will remain for ever in $\boldsymbol{S}$, despite the effect of the current disturbance. In order to derive the optimal control law we consider the Hamiltonian:

$$\mathscr{H}_3 = \sup_{\boldsymbol{u_1}\in U_1} \inf_{v_r\in V_r} \begin{pmatrix} p_1 V_c \cos\beta_c + p_2 V_c \sin\beta_c + p_3 \hat{r} + \\ +(p_1\cos\psi + p_2\sin\psi)\hat{u_r} + (-p_1\sin\psi + p_2\cos\psi)\hat{v_r} \end{pmatrix}$$

where $p_i = \frac{\partial \boldsymbol{V_1}}{\partial \boldsymbol{x}_{1_i}}$, $i = 1 \ldots 3$. The optimal control inputs are:

$$\hat{u}_r = \begin{cases} u_{rM} & \text{if } (p_1 \cos\psi + p_2 \sin\psi) \geq 0 \\ u_{rm} & \text{if } (p_1 \cos\psi + p_2 \sin\psi) < 0 \end{cases} \quad \hat{r} = \begin{cases} r_M & \text{if } p_3 \geq 0 \\ r_m & \text{if } p_3 < 0 \end{cases} \tag{6}$$

whereas the disturbance input is selected such that it has the worst possible impact on the system, as $\hat{v}_r = v_{rm}$ if $(-p_1 \sin\psi + p_2 \cos\psi) \geq 0$ and $\hat{v}_r = v_{rM}$ if $(-p_1 \sin\psi + p_2 \cos\psi) < 0$. Thus we have a robust estimation of the discriminating kernel $Disc(\boldsymbol{S})$, since at each time instance t we consider the effect of the worst-case disturbance v_r, i.e. of the worst-case linear velocity in the unactuated sway DOF. The computational results are given in Section V.

So far we have assumed that the current has known, constant direction β_c. In order to determine viability in a more robust manner, we would like to characterize the discriminating kernel of (4) which is irrelevant to the current direction β_c. Thus, we consider the angle β_c as an additional disturbance input, which is trying to minimize the Hamiltonian $\mathscr{H}_3$, i.e. minimize the term $h_3(\beta_c) = p_1 V_c \cos\beta_c + p_2 V_c \sin\beta_c$. The minimum value of $h_3(\beta_c)$ is attained for $\beta_c = \arctan2(p_2, p_1) + \pi$ if $p_2 < 0$, and for $\beta_c = \arctan2(p_2, p_1) - \pi$ if $p_2 \geq 0$. Thus, for the computation of $Disc(t, \boldsymbol{S})$ we consider the worst-case current direction $\beta_c(p_1, p_2)$ at each iteration. The computational results are given in Section V.

V. Computational Results

The forward reachability computation for the system (5) was performed on a $26 \times 26 \times 26 \times 26$ grid of the state space using the Level Set Methods Toolbox [26]. The initial set $\mathscr{N}$ was defined as a cube centered at the origin. The velocity and direction of the current were selected as $V_c = 0.5$ m/sec and $\beta_c = \pi/2$. The dynamic parameters in (5) were chosen to resemble the vehicle properties. The computations were performed for different values of the time horizon T, see Fig. 3. We found out that for each time horizon and for all values of angle ψ, the resulting reachable sets of u_r, v_r, r were practically the same. This is justified since the ψ-dependent terms are negligible compared to the other dynamic terms. Furthermore, after a time horizon the state vector saturates and the reachable set does not expand any more, since the damping forces counterbalance thrust. Since the reachable sets for $T = 5$ sec and $T = 8$ sec are practically the same, it is safe to choose the bounds of u_r, v_r, r. To further justify this, we performed computations for various angles β_c, which verified that the reachable sets do not differ at $T = 5$ sec.

The viability computation for system (4) was performed on a $51 \times 51 \times 51$ grid of the state space with $V_c = 0.5$ m/sec and $\beta_c = \pi/2$ rad. The safe set $\boldsymbol{S}$ is given in Fig. 4 on the left side. As it was expected, $\boldsymbol{S}$ is shrinking as t increases until a time horizon $T \approx 2.5$ sec. The discriminating kernel $Disc(\boldsymbol{S})$ at time $t = 3$ sec is given in Fig. 4, on the right. Their projections on the xy plane are given in Fig. 5. The shape of $Disc(\boldsymbol{S})$ is consistent with physical intuition, i.e. $Disc(\boldsymbol{S})$ depends on the current direction β_c. The lack of symmetry means that there is no control input (6) that can prevent the current to drive the states on the right side out of the $Disc(\boldsymbol{S})$. Moreover, the $Disc(\boldsymbol{S})$ for current direction

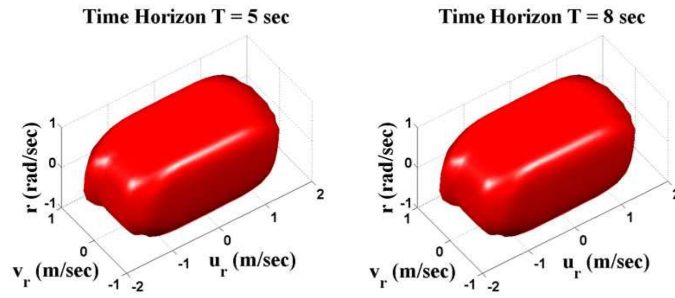

Fig. 3. Forward Reachable Sets for T=5 and T=8 sec and $\beta_c = \pi/2$

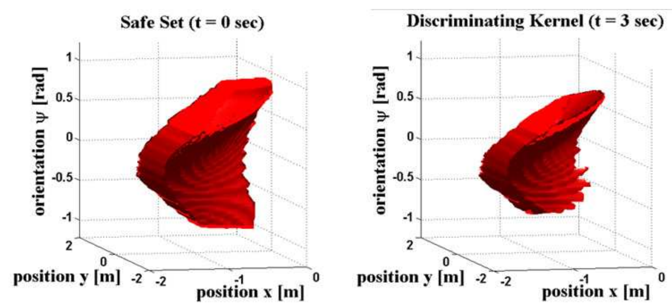

Fig. 4. Safe Set $\boldsymbol{S}$ and Discriminating Kernel $Disc(\boldsymbol{S})$ at t=3 sec

$\beta_c \in [-\pi, \pi]$ and velocity $V_c = 0.5$ m/sec is depicted in Fig. 6. This is the set of initial states for which the control law (6) ensures that the viability constraints are met for all possible current directions. As one would expect, it is smaller than the one computed for fixed angle β_c. The vector field of system (4) under (6) for $\beta_c = \pi/2$ and $\psi = 0$ is given in Fig. (7). It verifies that the system is forced into the $Disc(\boldsymbol{S})$ when the state is close to its boundary, see the velocity vectors on the bound parallel to y axis. Moreover, the velocity vectors close to the other two sides of $Disc(\boldsymbol{S})$, along with the corresponding control input r, see Fig. 8, imply that the

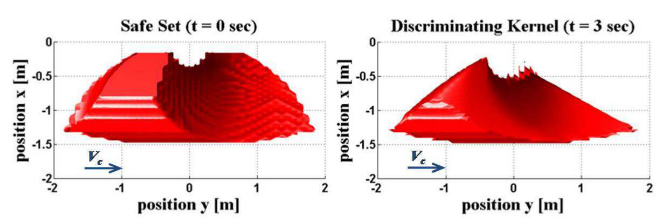

Fig. 5. Projection of $\boldsymbol{S}$ and $Disc(\boldsymbol{S})$ on the $x-y$ plane for $\beta_c = \pi/2$

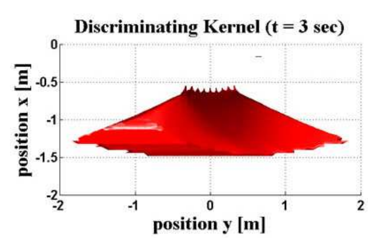

Fig. 6. Discriminating Kernel $Disc(\boldsymbol{S})$ at t=3 sec for $\beta_c \in [-\pi, \pi]$

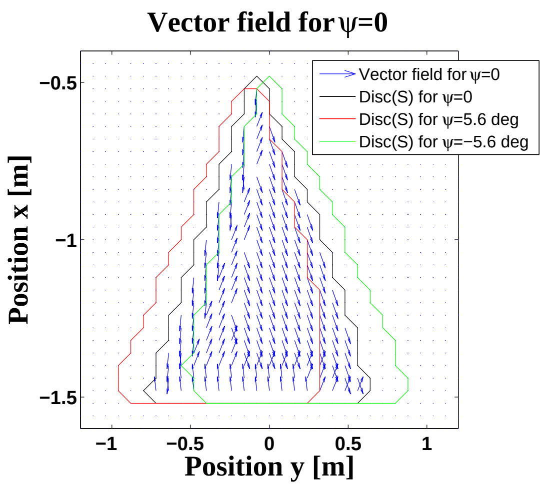

Fig. 7. Vector field of closed-loop system for $\psi = 0$

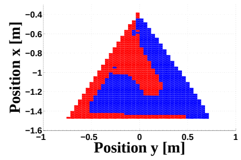

Fig. 8. Control input $r \geq 0$ in Red Area and $r < 0$ in Blue Area

state remains into $Disc(\boldsymbol{S})$ with $\psi \neq 0$, since $Disc(\boldsymbol{S})$ either expands to the left with $\dot{\psi} > 0$ (red boundary, $\psi > 0$) or to the right with $\dot{\psi} < 0$ (green boundary, $\psi < 0$).

VI. Conclusions

In this paper, we present a viability formulation for the problem of controlling an underactuated underwater vehicle with respect to a target, in the presence of a known, constant current disturbance and under state constraints. Considering a safe set of state constraints resulting from the task specifications and sensor limitations, we investigated whether there exists a control law such that the vehicle remains for ever in this set, despite the influence of the current. This analysis, based on an approach connecting viability and optimal control, yields the viability kernel and an optimal control law that maintains viability. To overcome the computational limitations due to the high dimension of the system, we presented a two-stage analysis, based on forward reachability and game theory. The computation of the viability kernel is necessary so to further proceed to the design of control laws that steer the vehicle into a goal set. The derivation of the safety controller is important, since this control law can be used when viability is at stake, i.e. close to the boundary of the safe set. Future work will be towards the solution of the overall stabilization problem in terms of viability.

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