Finite-Time Stability of Switched and Hybrid Systems

Kunal Garg\textsuperscript{a}, Dimitra Panagou\textsuperscript{a}

\textsuperscript{a}Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, 48109, USA

Abstract

In this paper, we study finite-time stability of switched and hybrid systems. We present sufficient conditions in terms of multiple generalized Lyapunov functions for the origin of the switched system to be finite-time stable. We show that even if the value of the generalized Lyapunov function increases between consecutive switches, finite-time stability can still be guaranteed if the finite-time convergent mode is active long enough. We also present a method of designing a finite-time stabilizing switching law. We use the developed schemes to design a finite-time stable state observer for linear switched systems for the case when only one of the modes is observable. Finally, we extend our results to the general case of hybrid systems and present sufficient conditions in terms of multiple generalized Lyapunov functions for finite-time stability of the origin. In contrast to prior work, where the Lyapunov functions are required to be decreasing during the continuous flow and non-increasing at the discrete jumps, we allow the individual generalized Lyapunov functions to increase both during the continuous flows and the discrete jumps. Numerical examples demonstrate the efficacy of the proposed methods.

Key words: Finite-Time Stability; Hybrid Systems; Switched Systems; Multiple Lyapunov Functions.

1 Introduction

Hybrid systems exhibit continuous state evolution and discrete state jumps; as thus, they are capable of modeling complex dynamical systems. The introductory paper [3] provides an overview and the merits of hybrid systems framework; namely, that it helps in managing complexity since it requires less detailed models at higher levels. Stability and performance of switched and hybrid systems has been studied extensively, see for example [16,30,37,10]. For an overview of the theory of switched and hybrid systems, i.e., on solution concepts and the notion of stability, the interested readers are referred to [22] and [27,12], respectively.

1.1 Stability of switched systems

Stability of switched systems has been studied by many researchers. The survey article [23] and the references therein give a detailed overview of stability results for switched and hybrid systems. Stability of switched systems is typically studied based on a common Lyapunov function and on multiple Lyapunov functions. The book [22] discusses necessity and sufficiency of the existence of a common Lyapunov function for all subsystems of a switched system for asymptotic stability under arbitrary switching. The authors in [16] study linear switched systems with dwell-time using a common quadratic control Lyapunov function (CQLF) and state-space partitioning. In the review article [38], the authors study the stability of switched linear systems and linear differential inclusions. They present sufficient conditions for the existence of CQLFs and discuss converse Lyapunov results for switched systems. In [7], the authors introduce the concept of multiple Lyapunov functions to analyze stability of switched systems; since then, a lot of work has been done on the stability of switched systems using multiple Lyapunov functions [44,47,23]. In [44], the authors relax the non-increasing condition on the Lyapunov functions by introducing the notion of generalized Lyapunov functions. They present necessary and sufficient conditions for stability of switched systems under arbitrary switching.

The authors in [46] use discontinuous multiple Lyapunov functions in order to guarantee stability of slowly switched systems, where the stable subsystems are required to switch slower (i.e., stay active for a longer duration) as compared to unstable subsystems. In [47], the authors introduce the concept of multiple linear copositive Lyapunov functions (ML-CLFs) and give suf-
sufficient conditions for exponential stability of switched positive linear systems (SPLS) in terms of feasibility of linear matrix inequalities (LMIs). In [45], the authors generalize the concept of CLF by introducing copositive polynomial Lyapunov functions (CPLFs) and illustrate that some of the variants of classical Lyapunov functions can be seen as special cases of the proposed CPLF.

1.2 Stability of hybrid systems

Compared to switched systems, where the evolution of the state trajectories is continuous in time, hybrid systems constitute a more general class of dynamical systems in the sense that system states are allowed to undergo discrete jumps as well. The survey paper [40] studies Lyapunov stability (LS), Lagrange stability and asymptotic stability (AS) for stochastic hybrid systems (SHS), and provides Lyapunov conditions for stability in probability. The paper also presents open problems on converse results on the stability in probability of SHS. The authors in [8] study input-to-state stability (ISS) for a class of hybrid systems and show the equivalence of ISS, nonuniform ISS and existence of a smooth Lyapunov function, for the case when the system vector field has a convex property with respect to the inputs to the system.

The review paper [28] discusses sufficient conditions for stability of three classes of hybrid systems, namely, systems modeled on a time scale, impulsive hereditary systems, and weakly coupled systems defined on Banach spaces. In [25], the authors study hybrid systems exhibiting delay phenomena (i.e., memory). They establish sufficient conditions for AS using Lyapunov-Razumikhin functions and Lyapunov-Krasovskii functionals. More recently, pointwise AS of hybrid systems is studied in [13], where the notion of set-valued Lyapunov functions is used to establish sufficient conditions for AS of a closed set. In [39], the authors impose an average dwell-time for the discrete jumps and devise Lyapunov-based sufficient conditions for exponential stability of closed sets; see [39,13] for details on the notion of stability of closed sets. The authors in [19] study the incremental stability property, i.e., the notion of graphical distance between every pair of maximal solutions, and establish necessary and sufficient conditions for such a property for hybrid systems.

1.3 Finite-time stability

In contrast to AS, which pertains to convergence as time goes to infinity, finite-time stability (FTS) is a concept that requires convergence of the solution in finite time. FTS is a well-studied concept, motivated in part from a practical viewpoint due to properties such as convergence in finite time, as well as robustness with respect to disturbances [36]. In the seminal work [5], the authors introduce necessary and sufficient conditions in terms of Lyapunov functions for continuous, autonomous systems to exhibit FTS. In [6], the authors provide geometric conditions for FTS in terms of homogeneity of the system dynamics. The approach in [9] considers finite-time stabilization using time-varying feedback controllers. More recently in [29], the authors establish necessary and sufficient conditions for FTS of continuous-time, non-autonomous systems using Lyapunov functions, see [14] as well. [15] introduces the concept of semistability for discontinuous dynamical systems and studies sufficient conditions for finite-time semistability.

The authors in [48,24,41,2] use a different definition of FTS, pertaining to the boundedness of the system states over a finite-time interval for given initial conditions (see [1] for an overview of the notion and related results). In this work, we consider FTS as defined in [5] and present the related work in this framework only.

1.4 Related work on FTS

FTS of switched and hybrid systems has gained popularity in the last few years. The authors in [26] consider the problem of designing a controller for a linear switched system under delay and external disturbance with finite- and fixed-time convergence. In [18], the authors design a hybrid observer and show finite-time convergence in the presence of unknown, constant bias. In [31], the authors study FTS of non-linear impulsive dynamical systems and present sufficient conditions to guarantee FTS. The work in [18,31] considers only one continuous flow and one discrete jump dynamics. In this paper, we consider the general case with \( N_f \) continuous flows and \( N_g \) discrete jump dynamics, where \( N_f, N_g \) can be any positive integers.

The authors in [20] present conditions in terms of a common Lyapunov function for FTS of hybrid systems. They require the value of the Lyapunov function to be decreasing during the continuous flow and non-increasing at the discrete jumps. In [4], the authors design an FTS observer for switched systems with unknown inputs. They assume that each linear subsystem is strongly observable, and that the first switching occurs after an a priori known time. In our work, we design FTS observer for a more general case where we assume that only one of the subsystems is observable. The authors in [35] design an FTS state observer for switched systems via sliding mode technique. In [42], the authors consider a switched system whose subsystems possess a homogeneous Lyapunov function and are of constant switching intervals. In [32], the authors introduce the concept of locally homogeneous system and show FTS of switched systems.
with uniformly bounded uncertainties. More recently, [43] studies FTS of homogeneous switched systems by introducing the concept of hybrid homogeneous degree and relating negative homogeneity with FTS. In [11], the authors consider systems in strict-feedback form with positive powers and design a controller as well as a switching law so that the closed-loop system is FTS. In this work, we do not assume that the subsystems of the switched system are homogeneous or are in strict feedback form, and present conditions in terms of multiple generalized Lyapunov functions for FTS of the origin.

1.5 Contributions of the Paper

In this paper, we consider a general class of continuous-time, autonomous switched and hybrid systems under arbitrary switching, and develop sufficient conditions for FTS of its equilibrium. We define the notion of FTS for switched systems so that it does not restrict each subsystem of the switched system to be FTS. The contributions of the paper are as follows:

(1) We present sufficient conditions in terms of multiple generalized Lyapunov functions for FTS of switched systems. To the best of authors' knowledge, this is the first work considering multiple generalized Lyapunov functions for FTS of switched or hybrid systems. Under some mild conditions, we show that if one of the subsystems of the switched system is FTS and is active for a minimum required time, then the switched system is FTS;
(2) In contrast to earlier work, e.g. [44], we allow the generalized Lyapunov functions to increase when each subsystem is active, as well as at switching instants;
(3) We also present a method for designing a switching law so that the origin of the resulting switched system is FTS. As an application, we employ the proposed method to design an FTS observer for switched linear systems when only one of the subsystems is observable;
(4) Finally, we develop sufficient conditions for FTS of hybrid systems in terms of multiple generalized Lyapunov functions. We relax the requirement in [20] that the Lyapunov function is non-increasing at the discrete jumps, and strictly decreasing during the continuous flow. More specifically, we allow the generalized Lyapunov functions to increase both during the continuous flow and at the discrete jumps, and only require that these increments are bounded.

1.6 Organization

The paper is organized as follows: Section 2 presents an overview of the theory of FTS. In Section 3, we present conditions for FTS of switched systems under arbitrary switching. Next, we present a method of designing a finite-time stabilizing switching law and apply it to the design of an FTS observer for a class of switched linear systems. Finally, we establish conditions for FTS of hybrid system in Section 4. Section 5 evaluates the performance of the proposed finite-time controllers via simulation results. Our conclusions and thoughts on future work are summarized in Section 6.

1.7 Notations

We denote by $\| \cdot \|$ the Euclidean norm of vector $(\cdot)$, $| \cdot |$ the absolute value if $(\cdot)$ is scalar and the length if $(\cdot)$ is a time interval. The set of non-negative reals is denoted by $\mathbb{R}_+ = [0, \infty)$, set of non-negative integers by $\mathbb{Z}_+$ and set of positive integers by $\mathbb{N}$. We denote by $\text{int}(S)$ the interior of the set $S$, and by $t^{-}$ and $t^{+}$ the time just before and after the time instant $t$, respectively.

2 Overview of FTS

Consider the system:

$$\dot{y}(t) = f(y(t)), \quad (1)$$

where $y \in \mathbb{R}^n$, $f : D \rightarrow \mathbb{R}^n$ is continuous on an open neighborhood $D \subseteq \mathbb{R}^n$ of the origin and $f(0) = 0$. Let $\psi^x : [0, T(x)] \rightarrow D$ denote the unique right maximal solution of system (1) with initial condition $y(0) = x$. In [5], the authors define FTS of (1) as follows:

**Definition 1** The origin is said to be an FTS equilibrium of (1) if there exist an open neighborhood $\mathcal{N} \subset D$ of the origin and a function $T : \mathcal{N} \setminus \{0\} \rightarrow (0, \infty)$, called the settling-time function, such that the following statements hold:

(1) Finite-time convergence: For every $x \in \mathcal{N} \setminus \{0\}$, $\psi^x(t)$ is defined on $[0, T(x)]$, $\psi^x(t) \in \mathcal{N} \setminus \{0\}$ for all $t \in [0, T(x)]$, and $\lim_{t \rightarrow T(x)} \psi^x(t) = 0$.
(2) Lyapunov stability: For every open neighborhood $U_e$ of 0, there exists an open subset $U_\delta$ of $\mathcal{N}$ containing 0 such that, for every $x \in U_\delta \setminus \{0\}$, $V^t(t) \in U_e$, for all $t \in [0, T(x)]$.

The origin is said to be a globally FTS (or GFTS) equilibrium if it is FTS with $D = \mathcal{N} = \mathbb{R}^n$. The authors also presented Lyapunov conditions for FTS of (1):

**Theorem 1** ([5]) Suppose there exists a continuous function $V : D \rightarrow \mathbb{R}$ such that the following hold:
(i) $V$ is positive definite
(ii) There exist real numbers $c > 0$ and $\alpha \in (0, 1)$, and an open neighborhood $\mathcal{V} \subseteq D$ of the origin such that

$$\dot{V}(y) + c(V(y))^\alpha \leq 0, \quad y \in \mathcal{V} \setminus \{0\}, \quad (2)$$

where $\dot{V}(y) \triangleq (D^+(V \circ \psi^x))(0)$ is the upper-right Dini derivative of $V(y)$. Then the origin is a FTS for (1).
3 FTS of Switched Systems

Consider a switched system:

\[ \dot{x}(t) = f_{\sigma(t,x)}(x(t)), \quad x(t_0) = x_0, \tag{3} \]

where \( x \in \mathbb{R}^n \) is the system state, \( \sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \Sigma \) is a piecewise constant, right-continuous switching signal that can depend both upon state and time, and \( \Sigma = \{1, 2, \ldots, N\}, N < \infty \) and \( f_{\sigma(\cdot, \cdot)} : \mathbb{R}^n \to \mathbb{R}^n \) is the system vector field describing the active subsystem (called thereafter mode) under \( \sigma(\cdot, \cdot) \). We denote by \([t_{j_k}, t_{j_k+1})\) the time interval when mode \( j \in \Sigma \) is active for the \( k \)-th time with \( t_{j_k} \in \mathbb{R}_+ \) being the time when the mode \( j \) becomes switched-on and \( t_{j_k+1} \) the time when the mode \( j \) becomes switched-off. Without loss of generality, we assume that the switching signal \( \sigma \) is minimal, i.e., for any \( j \in \Sigma \), \( t_{j_i+1} \neq t_{j_{i+1}} \) for all \( i \geq 1 \). Inspired from [44], we define the following:

Definition 2 (Generalized Lyapunov Function): A continuous, positive definite function \( V_j : \mathbb{R}^n \to \mathbb{R}_+ \) is called a generalized Lyapunov function if there exists a continuous function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying \( \phi(0) = 0 \), such that \( V_j(x(t)) \leq \phi(V_j(x(t_j))) \) for all \( t \in [t_{j_k}, t_{j_k+1}) \).

Definition 3 (Class-GK Function): A function \( \alpha : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a class-GK function if it is increasing, i.e., for all \( x > y \geq 0 \), \( \alpha(x) > \alpha(y) \), and right continuous at the origin with \( \alpha(0) = 0 \).

Note that the class-GK functions have similar composition properties as those of class-K functions, i.e., for \( \alpha_1, \alpha_2 \in \mathcal{GK} \) and \( \alpha \in \mathcal{K} \), we have:

\begin{itemize}
  \item \( \alpha_1 \circ \alpha_2 \in \mathcal{GK} \) and \( \alpha_1 + \alpha_2 \in \mathcal{GK} \);
  \item \( \alpha \circ \alpha_1 \in \mathcal{GK} \), \( \alpha_1 \circ \alpha \in \mathcal{GK} \) and \( \alpha_1 + \alpha \in \mathcal{GK} \).
\end{itemize}

We make the following assumptions:

Assumption 1 The origin is the only equilibrium for the subsystem \( \dot{x} = f_j(x) \) for all \( j \in \Sigma \).

Assumption 2 There is a non-zero dwell-time between any two switches, i.e., \( t_{j_{i+1}} - t_{j_i} \geq t_d \) for all \( i \in \mathbb{N} \) and for all \( j \in \Sigma \), where \( t_d > 0 \) is some positive constant.

In effect, we assume that there are only finite number of switches in any given finite interval of time, i.e., there is no Zeno behavior. Note that we do not need to know the value of the dwell-time \( t_d \). It is sufficient that this value is non-zero.

Assumption 3 ([44]) For any given switching signal \( \sigma(\cdot, \cdot) \), the solution \( x(t) : \mathbb{R}_+ \to \mathbb{R}^n \) of (3) exists, is unique, and is continuous.

3.1 Sufficient conditions for FTS

We first define the notion of FTS for switched systems. The standard notion of stability under arbitrary switching, as employed in [23,7,44,11,22], is restrictive in the following sense. The conditions therein require every single mode of the system (3) to be LS, AS or FTS for the origin of the system (3) to be LS, AS or FTS, respectively. We overcome this limitation by defining the corresponding notions of stability for switched systems (inspired in part, from [33, Theorem 1]) as following:

Definition 4 Let \( \Pi \subset \text{PWC}(\mathbb{R}_+, \Sigma) \) denote the set of all possible switching signals, where \( \text{PWC} \) is the set of all piecewise constant functions mapping from \( \mathbb{R}_+ \) to \( \Sigma \). The origin of the switched system (3) is called LS, AS or FTS, if there exists an open neighborhood \( D \subset \mathbb{R}^n \) such that for all \( y \triangleq x(0) \in D \), there exists a subset of switching signals \( \Pi(y) \subset \Pi \) such that the origin of the system (3) is LS, AS or FTS, respectively, for all \( \sigma \in \Pi(y) \).

Following Theorem 1, one can readily establish FTS of system (3) in terms of existence of a common Lyapunov function:

Corollary 1 If there exists a continuously differentiable generalized Lyapunov function \( V \) such that along the trajectories of (3), the following holds:

\[ \dot{V}(x) = \frac{\partial V}{\partial x} f_1(x) \leq -cV(x)^\beta, \quad \forall x \in D \setminus \{0\}, \forall i \in \Sigma, \tag{4} \]

where \( c > 0 \), \( 0 < \beta < 1 \) and \( D \) is an open neighborhood of the origin, then the origin of (3) is FTS.

Corollary 2 The origin of (3) is FTS if there exist constants \( 0 < \beta < 1 \) and \( k > 0 \) such that \( x^T f_1(x) \leq -k \|x\|^\beta+1 \) for all \( i \in \Sigma \).

Proof. Choose the candidate generalized Lyapunov function as \( V(x) = \frac{1}{2} \|x\|^2 \). Taking its time derivative along the trajectories of (3), we obtain

\[ \dot{V}(x) = x^T f_1(x) \leq -k \|x\|^\beta+1 = -cV(x)^\beta, \]

where \( c = 2^{\frac{1+\beta}{\beta}} k > 0 \) and \( \alpha = \frac{1+\beta}{2} < 1 \). Hence, from Corollary 1, we obtain that the origin of (3) is FTS. \( \blacksquare \)

We need the following result before we proceed with the main theorem.

Lemma 1 If a generalized Lyapunov function \( V_F \) satisfies

\[ \sum_{k=1}^{p} |V_F(x(t_{F_{k+1}})) - V_F(x(t_{F_k+1}))| \leq \alpha(\|x_0\|), \tag{5} \]

where \( \alpha \) is a positive function, then

\[ \Pi \subset \text{PWC}(\mathbb{R}_+, \Sigma) \]
for all $p \in \mathbb{N}$ where $\alpha \in GK$, then there exists $\bar{\alpha} \in GK$ such that the following holds for all $0 \leq \beta < 1$ and $p \in \mathbb{N}$:

$$
\sum_{k=1}^{p} \left( V_F(x(t_{F_k+1}))^{1-\beta} - V_F(x(t_{F_k+1}))^{1-\beta} \right) \leq \bar{\alpha}(\|x_0\|).
$$

(6)

**Proof.** See Appendix A. □

Now we present the conditions for FTS of (3) in terms of multiple generalized Lyapunov functions. Let $\{i^0, i^1, \ldots, i^p, \ldots\}$ be the sequence of modes active on the intervals $[t_0, t_1), [t_1, t_2), \ldots, [t_p, t_{p+1}), \ldots$, respectively for $i^p \in \Sigma, p \in \mathbb{Z}_+$.

**Theorem 2** The origin of (3) is FTS if there exist generalized Lyapunov functions $V_i(x)$ for each $i \in \Sigma$, and the following hold:

(i) There exists $\alpha_1 \in GK$, such that

$$
\sum_{k=0}^{p} \left( V_{i_k}(x(t_{k+1})) - V_{i_k}(x(t_k)) \right) \leq \alpha_1(\|x_0\|),
$$

(7) holds for all $p \in \mathbb{Z}_+$;

(ii) There exists $\alpha_2 \in GK$, such that

$$
\sum_{k=0}^{p} \left( V_{i_k}(x(t_{k+1})) - V_{i_k}(x(t_k)) \right) \leq \alpha_2(\|x_0\|),
$$

(8) holds for all $p \in \mathbb{Z}_+$;

(iii) There exist sequences $c > 0$ and $0 < \beta < 1$ such that the corresponding generalized Lyapunov function $V_F$ satisfies

$$
\dot{V}_F \leq -c V_F^\beta,
$$

(9) for all $t \in [t_{F_k}, t_{F_{k+1}})$, $i \in \Sigma$;

(iv) There exists $\alpha_3 \in GK$ such that the following inequality holds for all $p \in \mathbb{N}$

$$
\sum_{k=1}^{p} |V_F(x(t_{F_k+1})) - V_F(x(t_{F_k+1}))| \leq \alpha_3(\|x_0\|),
$$

(10) holds;

(v) The mode $F$ is active for an accumulated duration $T_F$ defined as

$$
T_F = \gamma(\|x_0\|) \triangleq \frac{\bar{\alpha}(\|x_0\|)}{c(1-\beta)} + \gamma_1(\|x_0\|),
$$

(11) where $\bar{\alpha}, \gamma_1 \in GK$.

Moreover, if all the conditions hold globally and the functions $V_i$ are radially unbounded for all $i \in \Sigma$, then the origin of (3) is GFTS.

Fig. 1. Conditions (i) and (ii) of Theorem 2: change in the values of the generalized Lyapunov functions. The increment shown by blue double-arrow pertains to the condition (i) while the red double-arrow pertains to the condition (ii).

**Remark 1** Condition (i) means that at switching instants, the cumulative value of the differences between the consecutive Lyapunov functions is bounded by a class-$\bar{\alpha}$ function; condition (ii) means that the cumulative increment in the value of the individual Lyapunov functions when the respective modes are active, is bounded by a class-$\alpha_2$ function (see Figure 1); condition (iii) means that there exists an FTS mode $F \in \Sigma$; and condition (iv) means that the accumulated increment in the value of the Lyapunov function $V_F$ during the "switched-off" periods is bounded (see Figure 2). Now we present the proof of Theorem 2.

**Proof.** Let $x_0 \in D$, where $D$ is some open neighborhood of the origin. For all $p \in \mathbb{Z}_+$, we have that

$$
V_{i^p}(x(t_p)) = V_{i^0}(x(t_0)) + \sum_{k=1}^{p} \left( V_{i_k}(x(t_k)) - V_{i_{k-1}}(x(t_k)) \right) \\
+ \sum_{k=0}^{p-1} \left( V_{i_k}(x(t_{k+1})) - V_{i_k}(x(t_k)) \right) \leq \alpha_0(\|x_0\|) + \alpha_1(\|x_0\|) + \alpha_2(\|x_0\|),
$$

where $\alpha_0(r) = \max_{i \in \Sigma, \|x\| \leq r} V_i(x)$. Since $V_i$ are positive def-
inite, $\alpha_0$ is positive definite. Hence, there exists $\alpha \in \mathcal{GK}$ such that $\alpha_0(||x_0||) + \alpha_1(||x_0||) + \alpha_2(||x_0||) \leq \alpha(||x_0||)$ for all $x_0 \in D$. Consequently:

$$V_F(x(t_F)) \leq \alpha(||x_0||). \quad (12)$$

Let $\{V_{F_1}, V_{F_2}, \cdots\}$ and $\{V_{F_1+1}, V_{F_2+1}, \cdots\}$ be the sequences of the values of the generalized Lyapunov function $V_F$ at the beginning and the end of the intervals $T_i = [t_{F_i}, t_{F_i+1})$ for $i \in \mathbb{N}$, respectively. From (9), in each such interval, we have $V_F \leq -cV_F^\beta$. Using the Comparison lemma [17, Section 5.2], for all $i \in \mathbb{N}$, we have:

$$\frac{V_{F_i+1}^{1-\beta} - V_{F_i}^{1-\beta}}{1-\beta} \leq -c(t_{F_i+1} - t_{F_i}).$$

Denote $|T_i| = t_{F_i+1} - t_{F_i}$ as the length of the interval $T_i$, so that we obtain

$$|T_i| \leq \frac{V_{F_i}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)}. \quad (13)$$

From Assumption 2, $|T_i| \geq t_d$ and hence, $T_F = \sum_{i=1}^{M} |T_i| \geq Mt_d$, where $M$ is the total number of time instants when the mode $F$ becomes active, and $T_F$ denotes the accumulated time for which the mode $F$ is active. This further reads that $M \leq \frac{T_F}{t_d}$, which implies that $M < \infty$ if $T_F < \infty$. Using (13), we have:

$$T_F = \sum_{i=1}^{M} |T_i| \leq \sum_{i=1}^{M} \frac{V_{F_i}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)}$$

$$= \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_{i+1}}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)}.$$

Using (12), we obtain that

$$\frac{V_{F_i}^{1-\beta}}{c(1-\beta)} \leq \alpha(||x_0||)^{1-\beta}.\quad (13)$$

Define $\gamma_1(||x_0||) \triangleq \frac{\alpha(||x_0||)^{1-\beta}}{c(1-\beta)}$ so that $\gamma_1 \in \mathcal{GK}$. From condition (iv) and Lemma 1, there exists $\bar{\alpha} \in \mathcal{GK}$ such that:

$$\sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_{i+1}}^{1-\beta}}{c(1-\beta)} \leq \bar{\alpha}(||x_0||) \frac{1}{c(1-\beta)}. \quad (14)$$

Define $\gamma \triangleq \gamma_1 + \frac{\bar{\alpha}(||x_0||)^{1-\beta}}{c(1-\beta)}$ so that we obtain:

$$T_F + \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_{i+1}}^{1-\beta}}{c(1-\beta)} \leq \gamma(||x_0||).$$

Clearly, $\gamma \in \mathcal{GK}$. Now, if $T_F = \gamma(||x_0||)$, we obtain

$$T_F + \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \gamma(||x_0||) = T_F,$$

which implies $\frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq 0$. But $V_F \geq 0$, which implies $V_{F_{M+1}} = 0$. Hence, if mode $F$ is active for the accumulated time $T_F$, the value of the function $V_F$ converges to 0 as $t \to t_{F_{M+1}}$.

Now we show that the time of convergence is finite, i.e., $t_{F_{M+1}} < \infty$. From the above analysis, we have that if $\sum_{i=1}^{M} |T_i| = T_F$, then there exists an interval $[t_{F_i}, t_{F_{M+1}})$ such that $V_F(x(t_{F_{M+1}})) = 0$. Since $V_F(x(t_F)) \leq \alpha(||x_0||)$, from (13), we obtain that

$$t_{F_{M+1}} - t_{F_M} \leq \frac{V_{F_{M+1}}^{1-\beta} - V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \frac{V_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \frac{\bar{\alpha}(||x_0||)^{1-\beta}}{c(1-\beta)} < \infty$$

for all $x_0 \in D$. Now, there are two cases possible. If $t_{F_M} < \infty$, then, we obtain that $t_{F_{M+1}} \leq t_{F_M} + \frac{\bar{\alpha}(||x_0||)^{1-\beta}}{c(1-\beta)} < \infty$ for all $x_0 \in D$. If $t_{F_M} = \infty$, we obtain that the time of activation for mode $F$ reads $T_F = \sum_{i=1}^{M-1} |T_i| < \gamma(||x_0||)$, which contradicts condition (v). Hence, for condition (v) to hold, it is required that $t_{F_M} < \infty$ and hence, we obtain that $t_{F_{M+1}} < \infty$.

If all the conditions (i)-(v) hold globally and the functions $V_i$ are radially unbounded, we obtain that $\alpha_0$ is also radially unbounded. Using the fact that $\bar{\alpha}(||x_0||) \frac{1}{c(1-\beta)} < \infty$ for all $||x_0|| < \infty$, we obtain $t_{F_{M+1}} \leq t_{F_M} + \frac{\bar{\alpha}(||x_0||)^{1-\beta}}{c(1-\beta)} < \infty$ for all $x_0$, which implies GFTS.

We give some remarks on Theorem 2 and its proof.

**Remark 2** In general, the inequality (14) can be difficult to obtain directly. Consider the case when we only know that the mode $F$ is homogeneous, with negative degree of homogeneity. From [6, Theorem 7.2], we know that in this case, condition (iii) of Theorem 2 holds for some $\beta$, but its exact value might not be known. In this case, it is not possible to bound the left-hand side (LHS) of (14). Lemma 1 allows one to bound this LHS with a class-$\mathcal{GK}$ function without explicitly knowing the value of $\beta$.

**Remark 3** Note that if Assumption 2 does not hold, one can construct a counter-example where even though $\sum |T_i| = T_F$, the time of convergence $t = t_{F_{M+1}} \to \infty$. Consider the case when $|T_i| = T_F^i$. It is clear that for
With Assumption 2, we obtain that the system would need to execute a Zeno behavior for FTS. With Assumption 2, we obtain that $k \leq \frac{T_f}{\tau_d} < \infty$, which rules out this possibility.

We only used the minimum dwell-time condition as stated in the Assumption 2 for the FTS mode. We relax this assumption as following.

**Assumption 4** There is no Zeno behavior executed by the trajectories of (3) and there is a non-zero dwell-time for the FTS mode $F \in \Sigma$, i.e., $t_{F_{i+1}} - t_{F_i} \geq \tau_d$ for all $i \geq 1$, where $\tau_d > 0$ is a positive constant.

**Corollary 3** Under Assumption 4 with conditions (i)-(v) of Theorem 2, the origin of (3) is FTS. If all the conditions hold globally and the functions $V_i$ are radially unbounded, the origin of (3) is GFS.

So far we have considered the cases when only one of the modes of the system (3) is FTS. Next, we present a result for the case when all modes of the system are FTS.

**Theorem 3** The origin of (3) is FTS if $\dot{x} = f_i(x)$ is FTS for all $i \in \Sigma$ and there exist generalized Lyapunov functions $V_i$ for each $i \in \Sigma$ such that conditions (i), (iii) and (iv) of Theorem 2 are satisfied.

**Proof.** Let $T_i$ denote the total time duration for which mode $i \in \Sigma$ is active. As the generalized Lyapunov function $V_i$ satisfies condition (iv) of Theorem 2, it satisfies the condition (ii) as well with $\alpha_2 = 0$. Let $T_f(||x_0||) = \max_i T_i$. In any time interval $[0, T_f(\|x_0\|)]$, at least one mode $i$ satisfies $T_i \geq \frac{T}{N}$. Hence, for $T \geq T_M = N T_f(||x_0||)$, we obtain that there exists at least one mode $i$ satisfying $T_i \geq \frac{T}{N} \geq \frac{T}{N} = T_f(||x_0||) \geq \alpha_i(\|x_0\|)$. This implies that condition (v) and hence, all the conditions of Theorem 2 are satisfied. Therefore, we obtain that the system is FTS with settling time $t \leq T_M$.

### 3.2 Finite-time Stabilizing Switching Law

In this subsection, we present a method of designing a switching law, based upon Theorem 2, so that the origin of the switched system is FTS. The approach is inspired by [44] where a method of designing an asymptotically stabilizing switching law is presented. Suppose there exist continuous functions $\mu_{ij} : \mathbb{R}^n \to \mathbb{R}$ satisfying:

$$
\begin{align*}
\mu_{ij}(0) &= 0, \\
\mu_{ij}(x) &= 0 \quad \forall x, \\
\mu_{ij} + \mu_{jk} &\leq \min\{0, \mu_{ik}\},
\end{align*}
$$

for all $i, j, k \in \Sigma$. Define the following sets:

$$
\begin{align*}
\Omega_i &= \{ x \mid V_i(x) - V_j(x) + \mu_{ij}(x) \leq 0, j \in \Sigma \}, \\
\Omega_{ij} &= \{ x \mid V_i(x) - V_j(x) + \mu_{ij}(x) = 0, i \neq j \},
\end{align*}
$$

where $V_i$ is a generalized Lyapunov function for each $i \in \Sigma$. Also suppose that we have continuous functions $\nu : \mathbb{R}^n \to \mathbb{R}$ for each $i \in \Sigma$ satisfying:

$$
\begin{align*}
\nu_i(0) &= 0, \\
\sum_{k=1}^p \nu_k(x_k) &\leq \alpha_1(||x(0)||),
\end{align*}
$$

for all $p \geq 1$, $x_k \in \mathbb{R}^n$, where $\alpha_1 \in \mathcal{G}K$. Let $F \in \Sigma$ be a finite-time stable mode in the system (3), $V_F$ satisfy condition (iii) of Theorem 2 and $\nu_F$ satisfy

$$
\begin{align*}
\nu_F(x) &\geq 0, \forall x, \\
\sum_{k=1}^p \nu_F(x_k) &\leq \alpha_2(||x(0)||),
\end{align*}
$$

for all $p \in \mathbb{N}$, $x_k \in \mathbb{R}^n$, where $\alpha_2 \in \mathcal{G}K$. Define the sets $\Omega_i$ for $i \in \Sigma$ as:

$$
\tilde{\Omega}_i(y) = \{ x \mid V_i(x) - V_j(y) \leq \nu_i(y) \}.
$$

For mode $F$, define the set $\tilde{\Omega}_F$ as:

$$
\tilde{\Omega}_F(y) = \{ x \mid |V_F(x) - V_F(y)| \leq \nu_F(y) \}.
$$

![Fig. 3. Sets $\tilde{\Omega}_i$ and $\tilde{\Omega}_F$: Blue, red and black circles denote the level sets $V_i = 10, 20$ and $30$, respectively. The annular $A_1 = \{ x \mid 10 \leq V_1(x) \leq 20 \}$ is shaded with orange stripes, $A_2 = \{ x \mid 20 \leq V_1(x) \leq 30 \}$ is shaded with black stripes and $A_0 = \{ x \mid V_1(x) \leq 10 \}$ is the circle shaded by blue stripes. With $\nu_1(y) = 10$ and $V_1(y) = 20$ for some $y$, the set $\tilde{\Omega}_1(y) = A_0 \cup A_1 \cup A_2$ and if $i = F$, $\tilde{\Omega}_F(y) = A_1 \cup A_2$.](image)
Now we are ready to define the switching law. Let \( \sigma(t_0) = i \) and \( i, j \in \Sigma \) be any arbitrary modes. For all times \( t \geq t_0 \), define the switching law as:

\[
\sigma(t) = \begin{cases} 
  i, & \sigma(t^{-}) = i, \ x(t) \in \text{int}(\Omega_i) ; \\
  j, & \sigma(t^{-}) = F, \ x(t) \in \Omega_{Fj}, \ \Delta_t \geq t_d ; \\
  j, & \sigma(t^{-}) = i, \ x(t) \notin \Omega_i(F) \bigcap \Omega_i(x_{on}) ; \\
  F, & \sigma(t^{-}) = i, \ x(t) \in \Omega_i(F)(x_{off}) \bigcap \Omega_i(x_{on}) ; 
\end{cases} 
\]

(21)

where (see Figure 4)

- \( \Delta_t = t - t_k \) is the time duration from the last switching instant \( t_k \);
- \( t_d > 0 \) is some positive dwell-time;
- \( x_{on} = x(t_k) \) is the value of \( x \) at last switching instant \( t_k \);
- \( x_{off} = x(t_{F+1}) \) is the value of \( x \) when the mode \( F \) switched-off last time (at \( t_{F+1} \)) before time \( t \);

Note that the condition for switching from mode \( F \) to mode \( j \) includes a dwell-time of \( t_d \), so that Assumption 4 is satisfied. We now state the following result:

**Theorem 4** Let the switching signal for (3) is given by (21). Let \( V_i \) be generalized Lyapunov functions for \( i = 1, 2, \cdots, N \), and \( \mu_{ij} \) and \( \nu_i \) satisfy (15) and (17)-(18), respectively. Assume that the following hold:

(i) There exists a finite-time stable mode \( F \in \Sigma \) satisfying condition (iii) and (v) of Theorem 2;

(ii) The functions \( \mu_{ij} \) are continuously differentiable and satisfy

\[
\frac{\partial \mu_{ij}}{\partial x} f_i \leq 0, \ i, j = 1, 2, \cdots, N. 
\]

(22)

(iii) No sliding mode occurs at any switching surface.

Then, the origin of (3) is FTS.

**Proof.** We show that all the conditions of Theorem 2 and Assumption 4 are satisfied to establish FTS of the origin for (3) using Corollary 3, when the switching law is defined as per (21). As per the analysis in [44, Theorem 3.18], we obtain that the condition (i) of Theorem 2 is satisfied if the switching signal is chosen as per (21), the sets \( \Omega_i \) and \( \Omega_{ij} \) are defined as (16), and the functions \( \mu_{ij} \) satisfy (15) and (22). Let \( \{\bar{p}, \bar{i}, \cdots, \bar{p}, \bar{i}, \cdots\} \) be the sequence of modes active on the intervals \([l_0, t_1), [t_1, t_2), \cdots, [t_p, t_{p+1}), \cdots\) respectively for \( \bar{p} \in \Sigma, \bar{i} \in \mathbb{Z}_+ \). Then, from (19) and (21), we obtain that at each time of switching \( t_{k+1} \), the following holds:

\[
V_i(x(t_{k+1})) - V_i(x(t_k)) \leq \nu_i(x(t_k)). 
\]

Hence, using (17), we obtain that

\[
\sum_{k=0}^{p} (V_i(x(t_{k+1})) - V_i(x(t_k))) \leq \sum_{k=0}^{p} \nu_i(x(t_k)) \leq \hat{\alpha}_1(\|x(0)\|), 
\]

which implies that condition (ii) of Theorem 2 is satisfied. Also, at the time of switching \( t = t_{F+1} \) from any other mode \( i \) to mode \( F \), as per (20) and (21), we have:

\[
|V_F(x(t_{F+1})) - V_F(x(t_{F+1}))| \leq \nu_F(x(t_{F+1})), 
\]

where \( t_{F+1} \) is the last time instant when the mode \( F \) switched-off before \( t \). Using (18), we obtain that

\[
\sum_{l=1}^{p} (V_F(x(t_{l+1})) - V_F(x(t_{l+1}))) \leq \sum_{l=1}^{p} \nu_F(x(t_{l+1})) \leq \hat{\alpha}_2(\|x(0)\|), 
\]

which implies that condition (iv) of Theorem 2 is satisfied. From (i), we obtain that conditions (iii) and (v) of Theorem 2 hold as well. Thus, all the conditions of the Theorem 2 and Assumption 4 are satisfied. Hence, from Corollary 3, we obtain that the origin of (3) with switching law defined as per (21) is FTS. \( \square \)

**Remark 4** Note that an arbitrary switching signal \( \sigma(t) \) may not satisfy the conditions of Theorem 2, particularly condition (v), where the mode \( F \) is required to be active for \( T_F(x_0) \) time duration. For any given initial condition \( x_0 \), the switching signal can be defined as per (21) to render the origin of (3) FTS. Definition 4 allows us to choose the switching signal \( \sigma(t) \) as per (21) so that the switched system (3) satisfies the conditions of Theorem 2.
3.3 FTS Observer for Switched Systems

In the subsection, we consider a switched linear system with a single observable mode, and design an observer to estimate the system state in a finite time. Consider the system:

\[
\begin{align*}
\dot{x} &= A y x, \\
y &= C y x,
\end{align*}
\]

where \(x \in \mathbb{R}^n, y \in \mathbb{R}\) are the system states and output of the system, respectively, \(A_i \in \mathbb{R}^{n \times n}\) are the state matrices and \(C_i \in \mathbb{R}^{1 \times n}\) are the output matrices. The switching signal \(\sigma\) is a piecewise constant continuous function. We make the following assumption:

**Assumption 5** There exists a mode \(\sigma_o \in \Sigma\) such that \((A_{\sigma_o}, C_{\sigma_o})\) is observable and the matrices \(A_{\sigma_o}, C_{\sigma_o}\) are given as

\[
A_{\sigma_o} = \begin{bmatrix}
    a_1 & 0 & \cdots & 0 \\
a_2 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
a_{n-1} & 0 & \cdots & 1 \\
a_n & 0 & \cdots & 0
\end{bmatrix}, \quad C_{\sigma_o} = \begin{bmatrix}
    1 & 0 & \cdots & 0
\end{bmatrix}.
\]

The objective is to design an observer for (23) so that the complete state vector \(x\) can be reconstructed in finite time. The form of the observer is:

\[
\dot{x} = A_{\sigma_o} \dot{x} + g_o (C_{\sigma_o} x - C_{\sigma_o} \dot{x}).
\]

Following [34, Theorem 10], define the function \(g : \mathbb{R} \to \mathbb{R}^n\) as:

\[
g(y) = \begin{bmatrix}
    a_1 y + k_1 \text{sign}(y)|y|^{\alpha_1} \\
a_2 y + k_2 \text{sign}(y)|y|^{\alpha_2} \\
    \vdots \\
a_n y + k_n \text{sign}(y)|y|^{\alpha_n}
\end{bmatrix},
\]

where \(k_i\) are such that the matrix \(A\) defined as

\[
A = \begin{bmatrix}
    -k_1 & 1 & 0 & \cdots & 0 \\
    -k_2 & 0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    -k_{n-1} & 0 & 0 & \cdots & 1 \\
    -k_n & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

is Hurwitz, and the exponents \(\alpha_i\) are chosen as \(\alpha_i = \frac{\alpha - i}{n} < \alpha < 1\). Define the function \(g_o\) as:

\[
g_o(y) = \begin{cases}
    g(y), & \sigma(t) = \sigma_o; \\
    0, & \sigma(t) \neq \sigma_o;
\end{cases}
\]

Let the observation error be \(e = x - \dot{x}\), with \(e_i = x_i - \dot{x}_i\) for \(i = 1, 2, \cdots, N\). Its time derivative reads:

\[
\dot{e} = A_{\sigma_o} e - g_o(C_{\sigma_o} e).
\]

We now state the following result:

**Theorem 5** Let the switching signal \(\sigma(t)\) for (23) be given by (21) with \(F = \sigma_o\). Assume that there exist functions \(\mu_{ij}, \nu_i\) as defined in (15) and (17)-(18), respectively, and that the conditions (i)-(iii) of Theorem 4 are satisfied. Then, the origin of (29) is an FTS equilibrium.

**Proof.** It is easy to verify that the origin is the only equilibrium of (29). From the analysis in Theorem 4, we know that the conditions (i) and (ii) of Theorem 2 are satisfied. The observation-error dynamics for mode \(\sigma_o\) reads:

\[
\dot{e} = \begin{bmatrix}
    e_2 - k_1 \text{sign}(e_1)|e_1|^{\alpha_1} \\
    e_3 - k_2 \text{sign}(e_1)|e_1|^{\alpha_2} \\
    \vdots \\
    e_n - k_{n-1} \text{sign}(e_1)|e_1|^{\alpha_{n-1}} \\
    -k_n \text{sign}(e_1)|e_1|^{\alpha_n}
\end{bmatrix}.
\]

Now, using [34, Theorem 10], we obtain that the origin is an FTS equilibrium for (30), i.e., for mode \(\sigma_o\) of (29). From [34, Lemma 8], we also know that (30) is homogeneous with degree of homogeneity \(d = \alpha - 1 < 0\). Hence, using [6, Theorem 7.2], we obtain that there exists a Lyapunov function \(V_o\) satisfying \(\dot{V}_o \leq -cV_o^{\beta}\) where \(c > 0\) and \(0 < \beta < 1\). Hence, condition (iii) of Theorem 2 is also satisfied. From the proof of Theorem 4, we obtain that the conditions (iv)-(v) of Theorem 2 and Assumption 4 are also satisfied. Hence, using Corollary 3, we obtain that the origin of (29) is an FTS equilibrium.

4 FTS of Hybrid Systems

In this section, we consider the class of hybrid systems \(H = \{F, G, C, D\}\) described as

\[
\begin{align*}
\dot{x}(t) &= f_{\sigma_o}(x(t)) \quad x(t) \in C, \\
x(t^+) &= g_{\sigma_o}(x(t)) \quad x(t) \in D,
\end{align*}
\]

(31)
where \( x \in \mathbb{R}^n \) is the state vector with \( x(t_0) = x_0, f_i(x) \in F \triangleq \{ f_i \} \) for \( i \in \Sigma_f \triangleq \{ 1, 2, \ldots, N_f \} \) is the continuous flow for the system which are allowed on the subset of the state space \( C \subset \mathbb{R}^n \) and \( g_j(x) \in G \triangleq \{ g_j \} \) for \( j \in \Sigma_g \triangleq \{ 1, 2, \ldots, N_g \} \) defines the discrete behavior, which is allowed on the subset \( D \subset \mathbb{R}^n \). Define \( x^+(t) \triangleq x(t^+) \). The switching signals \( \sigma_f \) and \( \sigma_g \) are assumed to be piecewise constant, right-continuous signals, which can depend upon both state and time. Details about the definition and the solution concept of the hybrid system (31) can be found in [12].

Denote by \( T_{i_k} = [t_{i_k}, t_{i_{k+1}}] \) the interval in which the flow \( f_j \) is active for the \( k \)-th time for \( i \in \Sigma_f \) and \( k \in \mathbb{N} \), and \( t_{j_m} \) as the time when discrete jump \( x^+) = g_j(x) \) takes place for the \( m \)-th time for \( j \in \Sigma_g \) and \( m \in \mathbb{N} \). Define \( J_i = \{ t_{j_m} | t_{j_m} \in T_{i_k}, j \in \Sigma_g, m \in \mathbb{N} \} \) as the set of all time instances when a discrete jump takes place along the continuous flow \( f_i \). Assume that the solution \( x(t) \) to (31) exists and is maximal. Furthermore, we also assume that \( f_i(0) = g_j(0) = 0 \) for all \( f_i \in F \) and \( g_j \in G \), i.e., the origin is an equilibrium point for all the continuous flows and discrete jumps. In this work, we are only concerned with the case when there is a unique continuous flow and discrete jumps. In this work, we define \( \Sigma_g \) as the time when discrete jump \( x(t) \) is active for the \( k \)-th time for \( i \in \Sigma_f \) and \( k \in \mathbb{N} \).

For each interval \( T_{i_k} \), define the largest connected subinterval \( T_{i_k} \subset T_{i_k} \), such that there is no discrete jump in the system state during \( T_{i_k} \). Note that if \( \bigcup_{i \in F} \bigcup_{i \in F} J_i = \emptyset \), we recover the switched system (3). Hence, in rest of the section we assume that \( \bigcup_{i \in F} J_i \neq \emptyset \). The conditions for FTS of the origin of (31) in terms of a common Lyapunov function are already given in [20]. In this work, we direct our focus on conditions in terms of multiple generalized Lyapunov functions.

Let \( T_{F_k} = [t_{F_k}, t_{F_{k+1}}] \) with \( t_{F_{k+1}} - t_{F_k} \geq t_d \), and \( \{ V_{F_1}, V_{F_2}, \ldots, V_{F_k} \} \) and \( \{ V_{F_1+1}, V_{F_2+1}, \ldots, V_{F_{k+1}} \} \) be the sequence of the values of the generalized Lyapunov function \( V_F \) at the beginning and end of the intervals \( T_{i_k} \) respectively. Let \( i^0, i^1, \ldots, i^l, \ldots \in \Sigma_f \) be the sequence of modes active on the intervals \([t_0, t_1), [t_1, t_2), \ldots, [t_i, t_{i+1}), \ldots \), respectively. We now present our main result for FTS of hybrid systems.

**Theorem 6** The origin of (31) is FTS if there exist generalized Lyapunov functions \( V_i(x) \) for each \( i \in \Sigma_f \) and the following hold:

(i) There exists \( \alpha_1 \in \mathcal{GK} \), such that

\[
\sum_{k=0}^P \left( V_{i_{k+1}}(x(t_{k+1})) - V_{i_k}(x(t_{k+1})) \right) \leq \alpha_1(||x_0||),
\]

holds for all \( p \in \mathbb{Z}_+ \);

(ii) There exists \( \alpha_2 \in \mathcal{GK} \) such that:

\[
\sum_{k=0}^P \left( V_{i_k}(x(t_{k+1})) - V_{i_k}(x(t_k)) \right) \leq \alpha_2(||x_0||),
\]

holds for all \( p \geq 0 \);

(iii) There exists \( \alpha_3 \in \mathcal{GK} \) such that for all \( i \in \Sigma_f \),

\[
\sum_{t \in J_i} \left( V_i(x^+(t)) - V_i(x(t)) \right) \leq \alpha_3(||x_0||);
\]

(iv) There exists \( F \in \Sigma_f \) such that \( \dot{y} = f_F(y) \) is FTS, and there exist a positive definite generalized Lyapunov function \( V_F \) and constants \( c > 0, 0 < \beta < 1 \) such that \( V_F \leq -cV^2_F \) for all \( t \in [t_{F_k}, t_{F_{k+1}}) \setminus J_F \);

(v) There exists \( \alpha_4 \in \mathcal{GK} \) such that for all \( p \geq 1 \):

\[
\sum_{k=1}^P |V_{F_{k+1}} - V_{F_k}| \leq \alpha_4(||x_0||),
\]

(vi) The accumulated duration of activation for mode \( F \) without any discrete jumps is satisfied \( T_F = \sum_{i \in F} |T_{i_k}| = \gamma(||x_0||) \) where \( \gamma \in \mathcal{GK} \) such that \( T_{i_k} \) is the length of the interval \( T_{i_k} \).

Moreover, if all the conditions hold globally and the functions \( V_i \) are radially unbounded for all \( i \in \Sigma_f \), then the origin of (3) is GFTS.

**Proof.** Conditions (i)-(iii) implies that \( V_i(x(t_p)) \leq \sum_{q=0}^P \alpha_p(||x_0||) \) for all \( p \in \mathbb{Z}_+ \), where \( \alpha_0(r) \triangleq \max_{i \in \Sigma_f ||x_0|| \leq r} V_i(x) \) is a positive definite function. Hence, there exists \( \alpha \in \mathcal{GK} \) such that \( \sum_{q=0}^P \alpha_p(||x_0||) \leq \alpha(||x_0||) \). We also know that during \( T_{F_k} \), there is no discrete jump for all \( k \in \mathbb{N} \). As
per condition (iv), \( \dot{V}_F \leq -cV_F^\beta \) for all \( t \in \bigcup \mathcal{T}_{F_k} \). Using this and the Comparison lemma, we obtain:

\[
|\mathcal{T}_{F_k}| \leq \frac{\dot{V}_F^{1-\beta}}{c(1-\beta)} - \frac{\dot{V}_{F_k}^{1-\beta}}{c(1-\beta)} \\
\Rightarrow \sum_{k=1}^{M} |\mathcal{T}_{F_k}| \leq \sum_{k=1}^{M} \left( \frac{\dot{V}_F^{1-\beta}}{c(1-\beta)} - \frac{\dot{V}_{F_k}^{1-\beta}}{c(1-\beta)} \right).
\]

Similar to the analysis in the proof of Theorem 2, we obtain

\[
\sum_{k=1}^{M} |\mathcal{T}_{F_k}| + \frac{\dot{V}_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq \sum_{k=1}^{M} \frac{\dot{V}_F^{1-\beta} - \dot{V}_{F_k}^{1-\beta}}{c(1-\beta)} \leq \gamma(\|x_0\|),
\]

where \( \gamma = \frac{\alpha(\|x_0\|)^{1-\beta}}{c(1-\beta)} + \gamma_2 \) with \( \gamma_2 \) defined as in the proof of Theorem 2 using (35). Hence, per condition (vi), \( \sum_{k=1}^{M} |\mathcal{T}_{F_k}| = T_F = \gamma(\|x_0\|) \), and hence, we obtain that \( \frac{\dot{V}_{F_{M+1}}^{1-\beta}}{c(1-\beta)} \leq 0 \), which implies that \( \dot{V}_{F_{M+1}} = 0 \). From Assumption 6, \( |\mathcal{T}_{F_k}| \geq t_d \) for all \( k \in \mathbb{N} \), and hence \( Mt_d \leq \sum_{i=1}^{M} |\mathcal{T}_{F_i}| = \gamma(\|x_0\|) \), which implies that

\[
M \leq \frac{\gamma(\|x_0\|)}{t_d} < \infty.\]

Hence, the trajectories of (31) reach the origin with finite number of active intervals of the continuous flow \( f_F \). Similar arguments as in the proof of Theorem 2 can be used to show that the time of convergence is finite and to complete the proof of GFTS. \( \blacksquare \)

**Remark 5** Compared to [20,21], our method is less conservative since we allow generalized Lyapunov functions to increase during the continuous flow (per (33)) as well as at the discrete jump (per (34)). Also, during the continuous flow the generalized Lyapunov functions are allowed to grow when switching from one continuous flow to another (per (32)), whereas the aforementioned work imposes that the common Lyapunov function is always non-increasing.

### 5 Simulations

We present two numerical examples. In the first example, we consider the switched linear system from Section 3.3. The simulation parameters are:

- The matrices \( A_i \) are chosen as
  \[
  A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},
  \]
  \[
  A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1.2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 2 \end{bmatrix};
  \]

- Generalized Lyapunov functions are chosen as \( V_i(x) = x^TP_ix \) where matrices \( P_i \) are chosen as:
  \[
  P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix},
  \]
  \[
  P_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 6 & 1 \\ 1 & 3 \end{bmatrix};
  \]

and \( V_5(x) = \frac{1}{2\alpha} \|x\|^{2\alpha} + \frac{1}{2} \|x_2\|^2 \);

- Functions \( v_i = 10\|x\|^2 \) for all \( i \in \Sigma \) and \( \mu_{ij} \) as
  \[
  \mu_{ij}(x) = \begin{cases} -\|x\|^2, & i \in \{1,2,4\}; \\ 0, & i \in \{3,5\}; \end{cases}
  \]

for all \( j \in \sigma \).

Note that mode 1 is stable, mode 3 is asymptotically stable, and modes 2 and 4 are unstable. Figure 5 shows the switching signal \( \sigma(t) \) with time. Figures 6 and 9 show the norm of the state vector \( x(t) \) with respect to time on log scale for initial error \( e(0) = e_1 = [1.879 \ 1.829]^T \) and \( e(0) = e_2 = [-5.783 \ -385.211]^T \), respectively. We use log plot so that the behavior of the norm \( \|e(t)\| \), when it is very small, can be observed. It is clear from Figures 6 and 9 that the observation error goes to zero in a finite time.

![Fig. 5. Switching signal \( \sigma(t) \) for the considered linear switched system for \( e(0) = e_1 \).](image-url)
system switches between all the 5 modes and spends at least \( t_d = 0.1 \) seconds in mode 5 each time. Figure 7 shows the values of the generalized Lyapunov functions \( V_i(t) \) for \( e(0) = e_2 \). The generalized Lyapunov functions can be seen to increase both at the switching instants as well as during the continuous flows. Particularly, we can see that \( V_i \) are increasing in mode 2 and 4, constant in mode 1 and decreasing in mode 3 and 5.

Figure 7. Generalized Lyapunov functions \( V_i(t) \) for \( t \in [0, 10] \) sec for \( e(0) = e_2 \).

Next, we present a numerical example of a FTS hybrid system with non-linear modes. The number of modes is \( N = 5 \) and the fifth mode is FTS, i.e., \( F = 5 \). The system modes are chosen as:

\[
\begin{align*}
    f_1 &= \begin{bmatrix} 0.01x_1^2 + x_2 \\ -0.01x_1^3 + x_2 \end{bmatrix}, & f_2 &= \begin{bmatrix} 0.01x_1 - x_2 \\ -x_1^2 + 0.01x_2 \end{bmatrix}, \\
    f_3 &= \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2 \end{bmatrix}, & f_4 &= \begin{bmatrix} 0.01x_1^2 + 0.01x_1x_2 \\ -0.01x_1^3 + x_2^2 \end{bmatrix}, \\
    f_5 &= \begin{bmatrix} x_2 - 20\text{sign}(x_1)|x_1|^{10} \\ -10\text{sign}(x_1)|x_1|^{2-2\alpha} \end{bmatrix}.
\end{align*}
\]

We simulated the observation error dynamics (29) for 500 different initial conditions \( e(0) \). Figure 10 shows the accumulated time \( T_F \) the mode \( F \) is active for various initial errors \( \|e(0)\| \). It can be seen that \( T_F(\|e(0)\|) \leq \gamma(\|e(0)\|) \) for all \( e(0) \), where \( \gamma = 40\|e(0)\|^{2-2\alpha} \in \mathcal{G} \). This shows that in practice, the FTS mode needs to be active for a smaller amount of time than required by the estimate in Theorem 2.

Figure 10. Accumulated active time for mode \( F \). The blue line shows the actual time \( T_F \) the mode \( F \) is active while red line is the required active time \( \gamma(\|e(0)\|) \).

Next, we present a numerical example of a FTS hybrid system with non-linear modes. The number of modes is \( N = 5 \) and the fifth mode is FTS, i.e., \( F = 5 \). The system modes are chosen as:

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    f_5 &= \begin{bmatrix} x_2 - 20\text{sign}(x_1)|x_1|^{10} \\ -10\text{sign}(x_1)|x_1|^{2-2\alpha} \end{bmatrix}.
\end{align*}
\]
The discrete jumps are defined by \( g = \begin{bmatrix} -1.1 & 0 \\ 0 & -1.1 \end{bmatrix} \) so that the states \( x_1 \) and \( x_2 \) change their signs and increase in the magnitude at the discrete jumps. In this case, we allowed the switches between the continuous flows after 0.2 sec and discrete jumps after 0.1 sec, so that \( t_d = 0.1 \) sec (see Assumption 6). Figure 11 shows the switching signal \( \sigma_f(t) \). The system switches between all the 5 modes and remains in the FTS mode \( F \) for at least 0.2 sec.

![Fig. 11. Switching signal \( \sigma_f(t) \) for the considered hybrid system.](image)

Figure 12 shows the states \( x_1(t) \) and \( x_2(t) \) for the first 10 seconds of the simulation. The states change sign at the discrete jumps. It can be seen in the figure that the system moves away from the origin while operating in the unstable modes. Figure 13 shows the norm of the state vector \( x(t) \) on log scale. Figure 14 shows the evolution of generalized Lyapunov functions \( V_i \) with respect to time for first 10 seconds of the simulation. The generalized functions for this example are same as the ones for linear FTS observer example. As can be seen from the figure, the generalized Lyapunov functions increase at the time of discrete jumps, the switching instants, as well as during the continuous flows along the unstable modes.

![Fig. 12. System states \( x_1(t) \) and \( x_2(t) \) for \( t \in [0, 10] \) sec. At the discrete jumps \( t_d \), the states change sign.](image)

![Fig. 13. The norm of the state vector \( x(t) \) for the considered hybrid system.](image)

![Fig. 14. Generalized Lyapunov functions \( V_i(t) \) for \( t \in [0, 10] \) sec for the considered hybrid system.](image)

The provided examples validate that the system can achieve FTS even if one or more modes are unstable, if the FTS mode is active for long enough. Also, we observed that the actual time duration \( T_F \) for which the mode \( F \) is active before the system reaches the origin is less than the time required as per Theorem 2.

6 Conclusions and Future Work

In this paper, we studied FTS of switched and hybrid systems. We showed that under some mild conditions on the bounds on the difference of the values of generalized Lyapunov functions, if the FTS mode is active for a minimum required time, then FTS can be achieved. We also presented a method of designing a finite-time stabilizing switching law. We then proposed an FTS observer for a class of linear switched systems when only one of the modes is observable. Finally, we presented conditions in terms of multiple generalized Lyapunov functions for FTS of hybrid systems. Our proposed method allows the individual generalized Lyapunov functions to increase both during the continuous flow as well as at the discrete jumps.

Our ongoing research focuses on incorporating input and state constraints in the hybrid framework for specifications involving both spatial and temporal requirements. More specifically, we are investigating how to
incorporate prescribed-time, rather than merely finite-time, convergence for the system modes, so that the overall framework can be used in the synthesis and analysis of controllers with spatiotemporal specifications.

Appendix

A Proof of Lemma 1

Proof. Let the function $V_F$ satisfy (5) for all $k\mathbb{N}$, where $\alpha \in \mathcal{GK}$. Lemma 3.3, 3.4 of [49] establish the following set of inequalities for $z_i \geq 0$ and $0 < r \leq 1$

$$\left(\sum_{i=1}^{M} z_i\right)^r \leq \sum_{i=1}^{M} z_i^r \leq M^{1-r} \left(\sum_{i=1}^{M} z_i\right)^r. \quad (A.1)$$

Hence, we have that for $a \geq b \geq 0$ and $0 < r \leq 1$, $a^r = (b + (a-b))^r \leq b^r + (a-b)^r$, or equivalently,

$$a^r - b^r \leq (a-b)^r. \quad (A.2)$$

Define $a_i \triangleq V_F(x(t_{F,i+1}))$ and $b_i \triangleq V_F(x(t_{F,i+1}))$ so that whenever $a_i \geq b_i$, or,

$$V_F(x(t_{F,i+1})) \geq V_F(x(t_{F,i+1})),$$

we have that $a_i^r - b_i^r \leq (a_i - b_i)^r$. Denote $i \in I_1 \triangleq \{i_1, i_2, \ldots, i_m\}$ for some $m \leq k$, for which $a_i \geq b_i$ and $i \in I_2 \triangleq \{i_{m+1}, i_{m+2}, \ldots, i_k\}$ for which $a_i \leq b_i$ (or equivalently, $a_i^r - b_i^r \leq 0$). Hence, we have that for any $0 < r \leq 1$,

$$k \sum_{i=1}^{k} (a_i^r - b_i^r) = \sum_{i \in I_1} (a_i^r - b_i^r) + \sum_{i \in I_2} (a_i^r - b_i^r) \leq \sum_{i \in I_1} (a_i^r - b_i^r) \leq \sum_{i \in I_1} (a_i - b_i)^r \leq m^{1-r} \left(\sum_{i \in I_1} (a_i - b_i)\right)^r.$$

Using this, we obtain that

$$\begin{align*}
&\sum_{i=1}^{k} (V_F(x(t_{F,i+1}))^r - V_F(x(t_{F,i+1}))^r) \\
\leq &m^{1-r} \left(\sum_{i \in I_1} (V_F(x(t_{F,i+1})) - V_F(x(t_{F,i+1})))\right)^r \\
\leq &m^{1-r} \left(\sum_{i=1}^{k} [V_F(x(t_{F,i+1})) - V_F(x(t_{F,i+1}))]\right)^r \\
\leq &m^{1-r} \alpha(\|x(0)\|)^r \triangleq \bar{\alpha}(\|x(0)\|)
\end{align*}$$

where $\bar{\alpha}$ is clearly a class $\mathcal{GK}$ function. Hence, we obtain that (6) holds for all $0 < \beta = 1 - r < 1$. \hfill \blacksquare

References


