Control-Lyapunov and Control-Barrier Functions based Quadratic Program for Spatio-temporal Specifications

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***Abstract*—This paper presents a method for control synthesis under spatio-temporal constraints. First, we consider the problem of reaching a set S in a user-defined or prescribed time T. We define a new class of control Lyapunov functions, called prescribed-time control Lyapunov functions (PT CLF), and present sufficient conditions on the existence of a controller for this problem in terms of PT CLF. Then, we formulate a quadratic program (QP) to compute a control input that satisfies these sufficient conditions. Next, we consider control synthesis under spatio-temporal objectives given as: the closed-loop trajectories remain in a given set S_s at all times; and, remain in a specific set S_i during the time interval $[t_i, t_{i+1})$ for $i = 0, 1, \cdots, N$; and, reach the set S_{i+1} on or before $t = t_{i+1}$. We show that such spatio-temporal specifications can be translated into temporal logic formulas. We present sufficient conditions on the existence of a control input in terms of PT CLF and control barrier functions. Then, we present a QP to compute the control input efficiently, and show its feasibility under the assumptions of existence of a PT CLF. To the best of authors' knowledge, this is the first paper proposing a QP based method for the aforementioned problem of satisfying spatio-temporal specifications for nonlinear control-affine dynamics with input constraints. We also discuss the limitations of the proposed methods and directions of future work to overcome these limitations. We present numerical examples to corroborate our proposed methods.**

I. Introduction

Driving the state of a dynamical system to a given desired set is an important problem, particularly in the fields of robot motion planning and safety-critical control. Various approaches have been developed in past to accomplish this task. Model predictive control (MPC)-based methods [1], [2], rapidly-exploring random tree (RRT) based methods [3], [4], [5], and combinations of them [3] have been studied extensively in the literature. In addition, Lyapunov-based methods, such as vector fields [6], [7] and control Lyapunov functions (CLF) [8], [9], [10] are also popular, in part because these methods are inherently amenable to Lyapunov-based analysis. Control design for systems with input and state constraints is not a trivial task, as these constraints impose limitations on several aspects of the control synthesis. For example, spatial constraints requiring the system trajectories to be in some safe set at all times are common in safety-critical applications. Furthermore, temporal constraints pertaining to convergence within a prescribed time appear in time-critical applications where completion of a task is required within a given time interval. *Spatio-temporal* specifications impose spatial as well as temporal or time constraints on the system trajectories.

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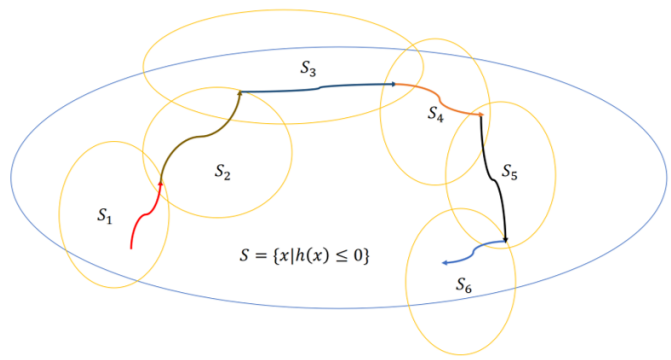

Fig. 1. Motivating problem: The system trajectories need to visit the sets S_i, $i = 1, \ldots, 6$ (orange regions) in a given time sequence, while always remaining in the set S (blue region).

From practical point of view, considering safety constraints, e.g., avoiding collisions in multi-robot systems, avoiding static and dynamic obstacles, and in general, avoiding the unsafe regions in the state space, is crucial. One of the most common methods of incorporating such spatial constraints on the system states is based on control barrier functions (CBF) [11]. Barrier functions are used for the synthesis of safe controllers [11], [12] and barrier certificates are used as a verification tool to guarantee that the closed-loop trajectories remain safe at all times. The authors in [13] present sufficient conditions in terms of existence of a barrier certificate for forward-invariance of a given set, and propose a sum-of-squares formulation to find a Barrier certificate. In order to guarantee safety and convergence, a combination of CLFs and CBFs is used for control design [11], [14], [15]. In the CLF-CBF based controller, convergence is guaranteed due to the CLF and safety is guaranteed due to CBF. [16] utilizes Lyapunov-like barrier functions to guarantee asymptotic tracking of a time-varying output trajectory, while the system output always remains inside a given set. The authors in [11], [14] present conditions using zeroing barrier functions so that the set defined as $\mathcal{C} = \{x \mid h(x) \geq 0\}$, where $h(x)$ is a user-defined smooth function, is forward invariant.

More recently, quadratic program (QP) based approaches have gained popularity for control synthesis; with this approach, the CLF and CBF conditions are formulated as inequalities that are linear in the control input [8], [9], [11], [17]. These methods are suitable for real-time implementation as QPs can be solved very efficiently. The authors in [11] combine the control performance objectives and safety objectives, represented using CLF and CBF, respectively, via a single

QP. Authors in [18] use CBF to encode signal-temporal logic (STL) based specifications and formulate a QP to compute the control input (see [18] for details on STL-based specifications for robot motion planning). The aforementioned work [6], [7], [10]-[17] concerns with designing control laws so that the reachability objectives, such as reaching a desired location or a desired goal set, are achieved as time goes to infinity, i.e., asymptotically.

In contrast to asymptotic stability (AS), which pertains to convergence as time goes to infinity, finite-time stability (FTS) is a concept that guarantees convergence of solutions in finite time. In the seminal work [19], the authors introduce the necessary and sufficient conditions in terms of Lyapunov functions under which continuous, autonomous systems exhibit FTS. The authors in [8] formulate a QP to ensure finite-time convergence of the closed-loop trajectories to a set $S = \{x \mid h(x) \leq 0\}$ with input constraints. Fixed-time stability (FxTS) [20] is a stronger notion than FTS, where the time of convergence does not depend upon the initial conditions. More recently, the authors in [21] used the notion or prescribed-time or user-defined time stability, where the time of convergence can be chosen by the user *a priori*.

In most of the aforementioned work, only one safety and one convergence objectives are considered. In this paper, we consider a multi-task problem of designing a control input. The considered objectives are of the following form: (i) the system trajectories should stay in a given set S_s at all times, (ii) the system trajectories should stay in a set S_i in the time-interval $[t_i, t_{i+1})$ for $i = 0, 1, 2, \cdots, N$, where $\{t_0, t_1, \cdots, t_N,\}$ is a user-defined time sequence, (iii) the trajectories should reach the set S_{i+1} before time instant $t = t_{i+1}$, and (iv) the control input should satisfy control constraints at all times. We show that such spatio-temporal specifications can be translated into a STL formula. In [18], the authors consider the problem of generating controller to satisfy STL specifications under the assumption that the system dynamics are equivalent to a single-integrator dynamics. To the best of authors' knowledge, this is the first paper proposing a QP based method for the aforementioned problem of satisfying spatio-temporal specifications without making any assumptions on the system dynamics. We first study the problem of reaching a given set in a user-defined time T for a general class of control-affine systems with input constraints. We extend the proposed formulation to guarantee that once the trajectories reach the desired set S, they stay in the set S for all future times. We define the new notion of prescribed-time CLF (PT CLF), and use it to solve the problem of reaching a given goal set within a given prescribed time T. Then, we present sufficient conditions in terms PT CLF and CBF to guarantee that the given spatio-temporal specifications are met. Finally, we present a QP-based optimization problem that can compute a control input for the same, and show its feasibility under some mild conditions. In contrast to earlier work [8], [9], [11], [18], our proposed framework is able to accommodate spatio-temporal, i.e., both state and time, constraints in the presence of control input constraints. Furthermore, in contrast to the results in [8], [22], where under the traditional notion of FTS, as defined in [19], the convergence time depends upon the initial conditions, the closed-loop system trajectories resulting from our controller reach the given set in a prescribed time that can be chosen arbitrarily and independently of the initial conditions.

The rest of the paper is organized as follows: In Section II, we present the notations used in the paper and background material on various notions of finite-time stability. In Section III, we study the problem of reaching a set S in a prescribed time and staying there for all future times. In Section IV, we consider the general multi-task problem for spatio-temporal specifications and formulate a QP to find the control input. We show three numerical examples in Section V to corroborate our theoretical results. We discuss the limitations of the proposed methods and propose a direction to relax the assumptions used in deriving the main results and summarize our thoughts on future work in Section VI. Finally, we present the conclusions in Section VII.

II. Mathematical Preliminaries

A. Notation

$\mathbb{R}$ denotes the set of reals and $\mathbb{R}_+$ denotes the set of non-negative reals. The boundary of a closed set S is denoted by ∂S and its interior by $\text{int}(S) \triangleq S \setminus \partial S$. The Lie derivative of a function $h : \mathbb{R}^n \to \mathbb{R}$ along a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is denoted as $L_f h \triangleq \frac{\partial h}{\partial x} f$. We use $\|x\|_p$ to denote the p-norm of the vector $x \in \mathbb{R}^n$ and simply use $\|\cdot\|$ to denote the Euclidean norm.

B. Preliminaries

Consider the system:

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous with $f(0) = 0$. As defined in [19], the origin is said to be an FTS equilibrium of (1) if it is Lyapunov stable and *finite-time convergent*, i.e., for all $x(0) \in \mathcal{N} \setminus \{0\}$, where $\mathcal{N}$ is some open neighborhood of the origin, $\lim_{t \to T} x(t) = 0$, where $T = T(x(0)) < \infty$, depends upon the initial condition $x(0)$. The authors in [20] presented the following result for FxTS, where the time of convergence does not depend upon the initial condition.

Theorem 1 ([20]). *Suppose there exists a positive definite function V for system (1) such that*

$$\dot{V}(x) \leq -(aV(x)^p + bV(x)^q)^k, \tag{2}$$

with $a, b, p, q, k > 0$, $pk < 1$ and $qk > 1$. Then, the origin of (1) is FxTS with continuous settling time function

$$T \leq \frac{1}{a^k(1 - pk)} + \frac{1}{b^k(qk - 1)}. \tag{3}$$

If the settling-time T can be chosen a priori by the user, then the origin is called as user-defined or prescribed-time stable [21].

III. Prescribed-time Set Reachability

In this section, we consider the problem of reaching a set $S = \{x \mid h(x) \leq 0\}$ in a user-defined or prescribed time T, where $h : \mathbb{R}^n \to \mathbb{R}$ is a user-defined function. Consider the system

$$\dot{x}(t) = f(x(t)) + g(x(t))u, \; x(t_0) = x_0, \tag{4}$$

where $x \in \mathbb{R}^n$ is the state-vector, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are system vector fields, and $u \in \mathbb{R}^m$ is the control input. The problem statement can be formally written as:

Problem 1. *Design a control input $u(t) \in \mathcal{U} = \{v \mid A_u v \leq b_u\}$, so that the closed-loop trajectories of (4) reach the set $S = \{x \mid h(x) \leq 0\}$ in a prescribed time T, where $h(x)$ is a user-defined continuously differentiable function, and $A_u \in \mathbb{R}^{l\times m}, b_u \in \mathbb{R}^l$ are user-defined matrices.*

Input constraints of the form $u(t) \in \mathcal{U} = \{v \mid A_u v \leq b_u\}$ are very commonly considered in the literature [11]. Now, we present sufficient conditions for existence of a control input u that solves Problem 1. First, we define a new class of CLF with prescribed-time convergence guarantees:

Definition 1. ***PT CLF-**S: A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called PT CLF-S for (4) with parameters a_1, a_2, b_1, b_2, if it is positive definite with respect to the set S, i.e., $V(x) > 0$ for all $x \notin S$, $V(x) = 0$ for all $x \in \partial S$, and the following holds:*

$$\inf_{u\in\mathcal{U}} \{L_f V + L_g V u\} \leq -a_1 V^{b_1} - a_2 V^{b_2}, \tag{5}$$

for all $x \notin int(S)$, where $a_1, a_2 > 0$, $b_1 > 1$ and $0 < b_2 < 1$. satisfy

$$\frac{1}{a_1(b_1 - 1)} + \frac{1}{a_2(1 - b_2)} \leq T, \tag{6}$$

where $T > 0$ is the prescribed time.

Definition 1 provides a CLF that guarantees convergence of the solutions to the origin within prescribed time T. Note that the traditional notions of CLF [8] and exponential CLF [23] are special cases of Definition 1, with $a_1 = a_2 = 0$, and $a_2 = 0$, $b_1 = 1$, respectively. Based on this definition, we can readily state the following result.

Theorem 2. *If there exist constants $\alpha_1, \alpha_2 > 0$, $\gamma_1 > 1$ and $0 < \gamma_2 < 1$, satisfying*

$$\frac{1}{\alpha_1(\gamma_1 - 1)} + \frac{1}{\alpha_2(1 - \gamma_2)} \leq T, \tag{7}$$

such that h is PT CLF-S with parameters $\alpha_1, \alpha_2, \gamma_1, \gamma_2$, then there exists $u(t) \in \mathcal{U}$, such that the closed-loop trajectories of (4) reach the set S within prescribed time T for all initial conditions $x(0) \notin S$.

Proof: Choose the candidate Lyapunov function $V(x) = h(x)$ for $x \notin S$. From the definition of set S, we know that $x \notin S$ implies $h(x) > 0$, which implies $V(x)$ is positive definite with respect to the set S. Now, since (5) holds for all $x \notin S$ for some $u \in \mathcal{U}$, we have that

$$\dot{V} = \dot{h} \leq -\alpha_1 h^{\gamma_1} - \alpha_2 h^{\gamma_2}.$$

Hence, using Theorem 1, we obtain that for all $t \geq T_0$, $V(x(t)) = h(x(t)) = 0$ where $T_0 \leq \frac{1}{\alpha_1(\gamma_1-1)} + \frac{1}{\alpha_2(1-\gamma_2)} \overset{(7)}{\leq} T$. This implies that the closed-loop trajectories reach the set S within prescribed time T. ■

Theorem 2 deals with reaching the set S before time $t = T$. Next, we present a result that guarantees that the closed-loop trajectories reach the set S within a prescribed time T and stay there for all future times using (12).

Corollary 1. *If there exist constants $\alpha_1, \alpha_2 > 0$, $\gamma_1 > 1$ and $0 < \gamma_2 < 1$ satisfying (7), such that the following holds*

$$\begin{aligned}\inf_{u\in\mathcal{U}} \{L_f h(x) + L_g h(x) u\} \leq &- \alpha_1 \max\{0, h(x)\}^{\gamma_1} \\ &- \alpha_2 \max\{0, h(x)\}^{\gamma_2},\end{aligned} \tag{8}$$

for all x, then the closed-loop trajectories of (4) reach the set S within prescribed time $T < \infty$ for all initial conditions $x(0) \in \mathbb{R}^n$, and stay there for all future times.

Proof: Note that once the trajectories of (4) reach the set S, we have $h(x) = 0$. From (8), we obtain that for $h(x) = 0$, $\dot{h}(x) \leq 0$. Hence, $h(x)$ is non-increasing on the boundary of the set S, and hence, the set S is forward invariant under the control input u satisfying (8). So, the closed-loop trajectories stay in the set S once they reach the set S. ■

As pointed out in [8], QPs can be solved very efficiently and can be used for real-time implementation. So, we present a QP-based formulation to compute the control input that satisfies the conditions of Corollary 1.

Theorem 3. *Let the solution to the following QP*

$$\min_{v,\alpha_1,\alpha_2} \frac{1}{2}\|v\|^2 \tag{9a}$$

$$\begin{aligned}\text{s.t. } L_f h(x) + L_g h(x) v \leq &-\alpha_1 \max\{0, h(x)\}^{\gamma_1} \\ &- \alpha_2 \max\{0, h(x)\}^{\gamma_2}\end{aligned} \tag{9b}$$

$$\frac{2}{T} \leq \alpha_1(\gamma_1 - 1), \tag{9c}$$

$$\frac{2}{T} \leq \alpha_2(1 - \gamma_2), \tag{9d}$$

$$A_u v \leq b_u, \tag{9e}$$

where $\gamma_1 > 1$ and $0 < \gamma_2 < 1$, is denoted as $[\bar{\alpha}_1 \; \bar{\alpha}_2 \; \bar{v}(t)]$. Then, the control input defined as $u(t) = \bar{v}(t)$ satisfies (8), and $\alpha_1 = \bar{\alpha}_1, \alpha_2 = \bar{\alpha}_2$ satisfy (7).

Proof: First, note that the optimization variables in (9) are α_1, α_2 and v. The objective of the optimization problem (9) is quadratic in v and the constraints are linear in the optimization variables. Hence, (9) is a QP. Now, first constraints of (9) is equivalent to (8). Constraints (9c)-(9d) make sure that α_1, α_2 are positive and the time constraint (7) is satisfied:

$$\frac{1}{\alpha_1(\gamma_1 - 1)} + \frac{1}{\alpha_2(1 - \gamma_2)} \overset{(9c)-(9d)}{\leq} T.$$

The last constraint in (9) implies that $u \in \mathcal{U}$. Hence, the solution to (9) satisfies (7) and (8). ■

IV. Control Synthesis for STL specifications

A. Problem formulation

In this section, we consider a general problem of designing control input for (4) such that the closed-loop trajectories satisfy spatio-temporal specifications defined as follows. Let $h_i(x)$ be the function defining the set $S_i = \{x \mid h_i(x) \leq 0\}$ for $i \in \Sigma = \{0, 1, 2, \cdots, N\}$ such that $S_i \bigcap S_{i+1} \neq \emptyset$ for all $0 \leq i \leq N-1$. Let $S_s = \{x \mid h(x) \leq 0\}$ be such that $S_s \bigcap S_0 \neq \emptyset$. Let $[t_0, t_1), [t_1, t_2), \cdots, [t_N, t_{N+1})$ be the set of intervals such that $t_{i+1} - t_i \geq \bar{T}$ for some $0 < \bar{T} < \infty$, for all $0 \leq i \leq N-1$. Assume that the functions $h(x), h_i(x)$ are continuously differentiable. We consider the following problem.

Problem 2. *Assume $x(t_0) \in S_0 \bigcap S_s$. Design a control input $u(t) \in \mathcal{U} = \{u \mid A_u u \leq b_u\}$, so that the closed-loop trajectories satisfy the following for all $i \in \Sigma$:*

$$x(t) \in S_s \ \forall\, t \geq t_0, \tag{10a}$$

$$x(t) \in S_i \ \forall\, t \in [t_i, t_{i+1}). \tag{10b}$$

Note that (10b) inherently requires that $x(t_{i+1}) \in S_{i+1}$ for $1 \leq i \leq N-1$, i.e., the trajectories should reach the set S_{i+1} on or before $t = t_{i+1}$, while staying in the set S_i for all times $t \in [t_i, t_{i+1})$. Problem 2 can be readily translated into temporal logic formulas for the form of specifications that are encountered, for instance, in mission planning problems. The STL specifications, given by formula ϕ include the following semantics (see [18] for more details):

- $(x,t) \models \phi \iff h(x(t)) \leq 0$;
- $(x,t) \models \neg\phi \iff h(x(t)) > 0$;
- $(x,t) \models \phi_1 \wedge \phi_2 \iff (x,t) \models \phi_1 \wedge (x,t) \models \phi_2$;
- $(x,t) \models G_{[a,b]}\phi \iff h(x(t)) \leq 0, \forall t \in [a,b]$;
- $(x,t) \models F_{[a,b]}\phi \iff \exists t \in [a,b]$ such that $h(x(t)) \leq 0$,

where $\phi =$ true if $h(x) \leq 0$ and $\phi =$ false if $h(x) > 0$. So, Problem 2 can be written in the STL semantics as follows.

Problem 3. *Design control input $u \in \mathcal{U}$ so that the closed-loop trajectories satisfy*

$$\begin{aligned}(x,t) \models & G_{[t_0,t_N]}\phi_s \wedge G_{[t_0,t_1]}\phi_0 \wedge F_{[t_0,t_1]}\phi_1 \wedge G_{[t_1,t_2]}\phi_1 \\ & \wedge F_{[t_1,t_2]}\phi_2 \wedge \cdots \wedge G_{[t_{N-1},t_N]}\phi_{N-1} \wedge F_{[t_{N-1},t_N]}\phi_N,\end{aligned} \tag{11}$$

where ϕ (respectively, ϕ_i) $=$ true if $h(x)$ (respectively, $h_i(x)$) ≤ 0, and false otherwise.

Remark 1. *If the STL-based specifications satisfy certain assumptions, then these specifications can be posed as an instance of Problem 2. For illustration, consider Example 2 from [18]. The STL specification $\phi = \phi_1 \wedge \phi_2$, where $\phi_1 = F_{[5,15]}(\|x - [10\ 0]^T\| \leq 5)$ and $\phi_2 = G_{[5,15]}(\|x - [10\ 5]^T\| \leq 10)$, means that the closed loop trajectories should reach the set $S_1 = \{x \mid \|x - [10\ 5]^T\| \leq 10\}$ on or before $t = 5$ sec, remain in the set S_1 for $t \in [5, 15]$ and reach the set $S_2 = \{x \mid \|x - [10\ 0]^T\| \leq 5\}$ on or before $t = 15$. Since $S_1 \bigcap S_2 \neq \emptyset$, we can use the problem set of Problem 2 to address these specifications. In Section V, we present an example on how to address problems that do not satisfy the setup of Problem 2, i.e., if the functions $h(x)$ or $h_i(x)$ are non-smooth or $S_i \bigcap S_{i+1} = \emptyset$, e.g., the case study in [24].*

B. Main results

In this work, we use the conditions of zeroing CBF (ZCBF) to ensure safety or forward invariance of the safe set S_s. The ZCBF is defined by the authors in [11] as following.

Definition 2. *A continuously differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ is called as ZCBF for (4) for set S_s if $B(x) < 0$ for $x \in int(S_s)$, $B(x) = 0$ for $x \in \partial S_s$, and there exists a continuous, increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha(0) = 0$, such that*

$$\inf_{u \in \mathcal{U}} \{L_f B(x) + L_g B(x) u\} \leq \alpha(-B(x)), \tag{12}$$

for all $x \in S_s$.

One special case of (12) is

$$\inf_{u \in \mathcal{U}} \{L_f B(x) + L_g B(x) u\} \leq -\rho B(x), \tag{13}$$

for some $\rho \in \mathbb{R}$. In [11, Remark 6], the authors mention that B is is a ZCBF if (13) holds with $\rho > 0$. We note that this restriction is not needed for guaranteeing safety. We present sufficient conditions in terms of PT CLF-ZCBF like inequalities for existence of control input u that solves Problem (2).

Theorem 4. *If there exist parameters a_{i1}, a_{i2}, $\gamma_{i1} > 1$ and $0 < \gamma_{i2} < 1$ for $i \in \Sigma$ such that*

$$\bar{T} \geq \max_{i \in \Sigma} \left\{ \frac{1}{a_{i1}(\gamma_{i1} - 1)} + \frac{1}{a_{i2}(1 - \gamma_{i2})} \right\}, \tag{14}$$

and a control input $u(t)$ such that the following holds

$$\inf_{u \in \mathcal{U}} \{L_f h(x) + L_g h(x) u\} \leq -\lambda_h h(x), \tag{15a}$$

$$\inf_{u \in \mathcal{U}} \{L_f h_i(x) + L_g h_i(x) u\} \leq -\lambda_i h_i(x), \tag{15b}$$

$$\begin{aligned}\inf_{u \in \mathcal{U}} \{L_f h_{i+1} + L_g h_{i+1} u\} \leq & -a_{i1} \max\{0, h_{i+1}\}^{\gamma_{i1}} \\ & - a_{i2} \max\{0, h_{i+1}\}^{\gamma_{i2}},\end{aligned} \tag{15c}$$

for $t \in [t_i, t_{i+1})$, for each $i \in \Sigma$, then, under the effect of control input u, the closed-loop trajectories satisfy (10).

Proof: Since $x(t_0) \in S_s \bigcap S_0$, we have that $h(x(t_0)) \leq 0$. Note that (15a) is independent of i, i.e., it is needed that (15a) holds for all $t \in [t_0, t_{N+1})$. If the control input satisfies (15a), then the set S_s is forward invariant (Corollary 1). Similarly, using (15b), we conclude that for $t \in [t_0, t_1)$, the set S_0 is forward-invariant. Finally, for $x \notin S_1$, from (15c), we obtain $\dot{h}_1 \leq -a_{01} h_1^{\gamma_{01}} - a_{02} h_1^{\gamma_{02}}$. Using Theorem 2, we obtain that the closed-loop trajectories satisfy $h_1(x(t)) = 0$ for $t \geq t_0 + T_0$, where $T_0 \leq \bar{T}$. Hence, we obtain that $t_0 + T_0 \leq t_0 + \bar{T} \leq t_1$, which implies that the closed-loop trajectories reach the set S_1 on or before $t = t_1$. Also, once trajectories reach the set S_1, we have that $\dot{h}_1 \leq 0$, i.e., the set S_1 is forward

invariant. Hence, the closed-loop trajectories reach the set S_1 on or before $t = t_1$ and stay in the set S_1 till $t = t_1$. So, at $t = t_1$, we have $x(t_1) \in S_s \bigcap S_1$.

Using the same arguments for each $i = 1, 2, \cdots, N-1$, we obtain that the closed-loop trajectories satisfy (10) under the effect of control input u satisfying (15). ■

Note that inequalities (15a) and (15b) are ZCBF conditions that render the set S and S_i forward-invariant, while (15c) is the PT CLF condition that guarantees fixed-time convergence to set S_{i+1}, as well forward invariance of the set S_{i+1} once trajectories reach the set S_{i+1}.

Remark 2. *In contrast to [18], where the authors assume that $g(x)g(x)^T$ is positive definite, we do not make any assumptions on the system vector fields f and g. In fact, for $m < n$, this condition is not satisfied for (4). Furthermore, [18] does not consider any input constraints.*

Lastly, we formulate a QP based optimization problem in order to find the parameters $a_{i1}, a_{i2}, \lambda_h, \lambda_i$ for each $i \in \Sigma$ and the control input $u(t)$ so that (14) and (15) are satisfied. Consider the optimization problem

$$\min_{v, a_{i1}, a_{i2}, \lambda_h, \lambda_i} \frac{1}{2}\|v\|^2 \tag{16a}$$

$$\text{s.t. } L_f h(x) + L_g h(x) v + \lambda_h h(x) \leq 0, \tag{16b}$$

$$L_f h_i(x) + L_g h_i(x) v + \lambda_i h_i(x) \leq 0, \tag{16c}$$

$$L_f h_{i+1} + L_g h_{i+1} v \leq -a_{i1} \max\{0, h_{i+1}\}^{\gamma_{i1}} - a_{i2} \max\{0, h_{i+1}\}^{\gamma_{i2}}, \tag{16d}$$

$$A_u v \leq b_u, \tag{16e}$$

$$\frac{2}{\bar{T}} \leq a_{i1}(\gamma_{i1} - 1), \tag{16f}$$

$$\frac{2}{\bar{T}} \leq a_{i2}(1 - \gamma_{i2}), \tag{16g}$$

where $\gamma_{i1} > 1$ and $0 < \gamma_{i2} < 1$. Let the solution to (16) is denoted as $[\bar{a}_{i1}\ \bar{a}_{i2}\ \bar{\lambda}_h\ \bar{\lambda}_i\ \bar{v}_i]$ for $t \in [t_i, t_{i+1})$, for $i \in \Sigma$. We can now state the main result of the paper.

Theorem 5. *If the functions h_i are PT CLF-S_i for all $i \in \Sigma$, then, the solution to (16) exists, and the control input defined as*

$$u(t) = \bar{v}_i(t),\ t \in [t_i, t_{i+1}),\ i \in \Sigma \tag{17}$$

satisfies (15), and $a_{i1} = \bar{a}_{i1}, a_{i2} = \bar{a}_{i2}$ satisfy (14).

Proof: First, note that the optimization variables in (16) are $a_{i1}, a_{i2}, \lambda_h, \lambda_i$ and v. The constraints are linear in these variables, while the objective function is quadratic in v. Hence, the optimization problem (16) is a QP. It is easy to show that the problem (16) is feasible if h_{i+1} is a PT CLF-S_{i+1} with respect to a_{i1}, a_{i2} satisfying (14). To see why this is true, note that there exists v satisfying (16d)-(16e), since h_{i+1} is a PT CLF-S_{i+1}. With this v, one can choose λ_h, λ_i satisfying (16b)-(16c), respectively, and a_{i1}, a_{i2} satisfying (16f)-(16g), respectively. Hence, there exists a solution to the QP (16). Note that the initial four constraint are equivalent to the three inequalities in (15). Next, (16f)-(16g) imply that $\frac{1}{a_{i1}(\gamma_{i1}-1)} + \frac{1}{a_{i2}(1-\gamma_{i2})} \leq \bar{T}$, so, (14) is also satisfied. Hence, with the last two constraints in (9), all the conditions of Theorem 4 are satisfied. Hence, the input defined as (17) satisfies (15). ■

The constraints of the QP (16) change at time instant t_i for $1 \leq 1 \leq N$. Note that we assume that the functions $h(x), h_i(x)$ are continuously differentiable to be able to use (15) or (16). In Section VI, we discuss how to overcome this limitation.

V. Simulations

We present three numerical examples to demonstrate the efficacy of the proposed methods. In the first scenario, we consider the example of reaching a set S_1 in a prescribed time T, and stay there for all future times, while also remaining in a S_2 at all times. Mathematically, the closed-loop trajectories are required to satisfy $x(t) \in S_1,\ \forall t \geq T, \quad x(t) \in S_2,\ \forall t \geq 0$, with $x(0) \in S_2$. The system dynamics are considered as

$$\dot{x}_1 = -x_2 + x_1^2 + x_1 u$$
$$\dot{x}_2 = x_1 + x_2 \tanh x_2 + x_2 u,$$

where the state-vector is $x = [x_1\ x_2]^T \in \mathbb{R}^2$ and the control input is $u \in \mathbb{R}$. Note that the open-loop trajectories for these dynamics diverge to infinity, i.e., the origin is unstable for the open-loop system. We choose $S_1 = \{x \mid \|x\| \leq 1\}$ and $S_2 = \{x \mid \frac{x_1^2}{9^2} + \frac{x_2^2}{0.9^2} \leq 1\}$ and $T = 10$ sec. Figure 2 shows the closed-loop trajectories for four different initial conditions. The trajectories reach the set S_1 in prescribed time and stay there at all the future times, while remaining in the set S_2 at all times.

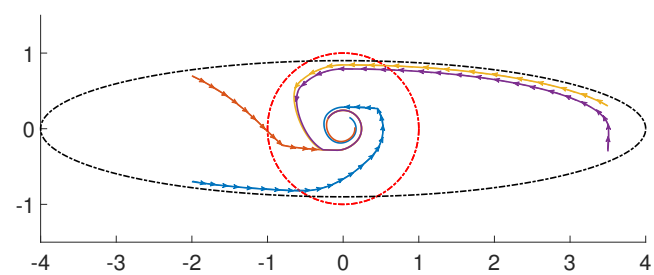

Fig. 2. Scenario 1: Closed-loop trajectories.

In second scenario, we take Example 2 from [18] and use our proposed method to satisfy the STL specifications $\phi = \phi_1 \wedge \phi_2$, where $\phi_1 = F_{[5,15]}(\|x - [10\ 0]^T\| \leq 5)$ and $\phi_2 = G_{[5,15]}(\|x - [10\ 5]^T\| \leq 10)$, with $S_1 = \{x \mid \|x - [10\ 5]^T\| \leq 10\}$ and $S_2 = \{x \mid \|x - [10\ 0]^T\| \leq 5\}$. The robot dynamics are modeled as $\dot{x} = u$ where $x, u \in \mathbb{R}^2$. We use $\|u\| \leq 10$ as the control input constraints. In order to translate the input constraint in the form of (16e), we define $A_u = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$ and $b_u = \begin{bmatrix} 7 & 7 & 7 & 7 \end{bmatrix}^T$, so that $u_x, u_y \in [-7,\ 7]$. Figure 3 shows the closed-loop trajectories for various initial conditions outside the set S_1. It can be seen that the trajectories reach

the set S_1 and stay in S_1 at all future times, and then reach set S_2. Figure 4 shows the norm of the control input $u(t)$ with time. As can be seen from the figure, the control input jumps at $t = 5$ sec, when the system trajectories reach the set S_1.

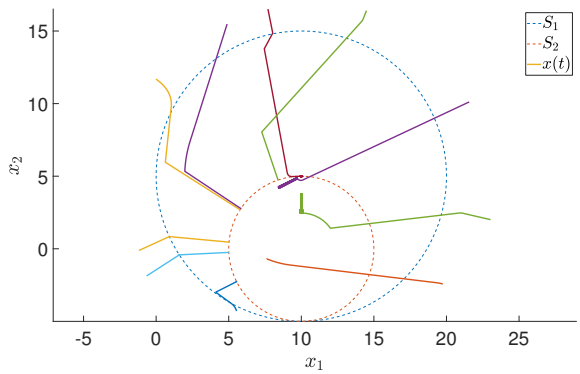

Fig. 3. Scenario 2: Closed-loop trajectories for various initial conditions.

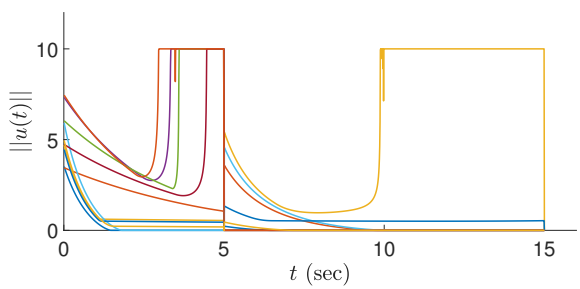

Fig. 4. Scenario 2: Control input for various initial conditions.

In the third scenario, we present a method of constructing sets S_i for applications such as robot motion planning, where the conditions of Theorem 4 are not met. The closed-loop trajectories, starting from $x(0) \in C_1$, are required to satisfy the following spatio-temporal specifications$(x_1, t) \models G_{[0,T_4]}\phi_s \wedge F_{[0,T_1]}\phi_2 \wedge F_{[T_1,T_2]}\phi_3 \wedge F_{[T_2,T_3]}\phi_4 \wedge F_{[T_3,T_4]}\phi_1$, which is explained in details below (see Figure 5):

- $x(t) \in S_s = \{x \mid \|x\|_1 \leq 2 \bigcap \|x\|_2 \geq 1\}$ for all $t \geq 0$, i.e., the closed-loop trajectories should stay inside the solid-blue square and outside the red-dotted circle at all times;
- Before a given $0 < T_1 < \infty$, $x(T_1) \in C_2 = \{x \mid \|x - [1.5\ 1.5]^T\|_1 \leq 0.5\}$;
- Before a given $T_1 < T_2 < \infty$, $x(T_2) \in C_3 = \{x \mid \|x - [1.5\ \ -1.5]^T\|_1 \leq 0.5\}$;
- Before a given $T_2 < T_3 < \infty$, $x(T_3) \in C_4 = \{x \mid \|x - [-1.5\ \ -1.5]^T\|_1 \leq 0.5\}$;
- Before a given $T_3 < T_4 < \infty$, $x(T_4) \in C_1 = \{x \mid \|x - [-1.5\ 1.5]^T\|_1 \leq 0.5\}$.

This problem is an extended version of the case study considered in [24]. Note that the sets C_i are not overlapping with each other, and the corresponding functions $h_i(x)$ are not continuously differentiable. Now, in order to be able to use QP-based formulation (16), we need to find the sets $\bar{S}_i$ such that $\bar{S}_i \bigcap \bar{S}_{i+1} \neq \emptyset$.

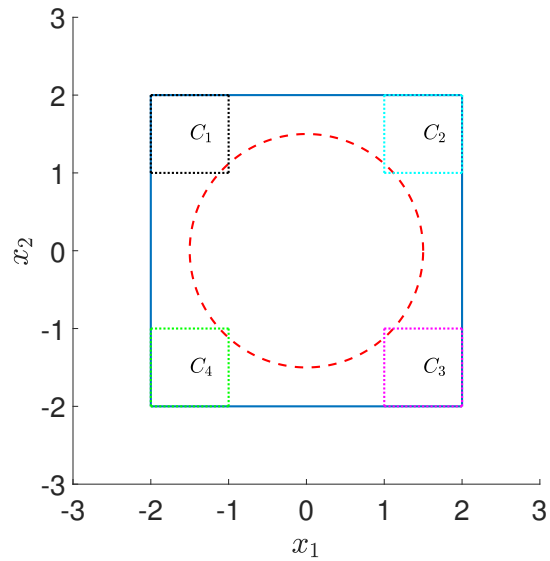

Fig. 5. Scenario 3: Problem setting.

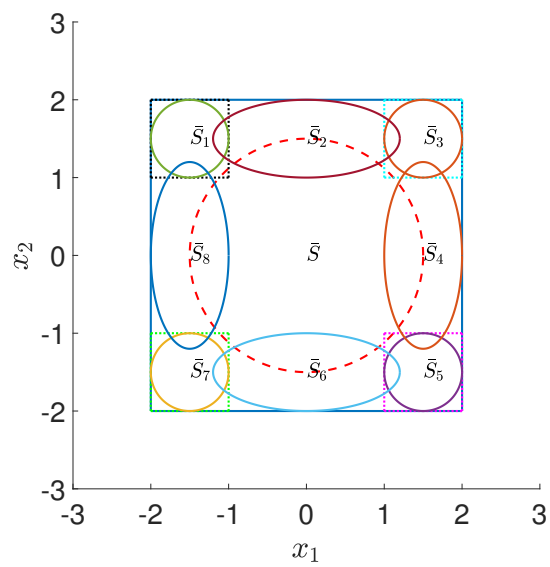

Fig. 6. Scenario 3: Construction of sets $\bar{S}, \bar{S}_1, \cdots, \bar{S}_8$.

The set $\bar{S} = \{x \mid \|x\| \leq 1.5\}$ and sets $\bar{S}_i$ are defined as follows (see Figure 6):

- $\bar{S}_1 = \{x \mid \|(x - [-1.5\ 1.5]^T)\| \leq 1\}$;
- $\bar{S}_2 = \{x \mid \|(x - [0\ 1.5]^T)\|_{P_1} \leq 1\}$;
- $\bar{S}_3 = \{x \mid \|(x - [1.5\ 1.5]^T)\| \leq 1\}$;
- $\bar{S}_4 = \{x \mid \|(x - [1.5\ 0]^T)\|_{P_2} \leq 1\}$;
- $\bar{S}_5 = \{x \mid \|(x - [1.5\ \ -1.5]^T)\| \leq 1\}$;
- $\bar{S}_6 = \{x \mid \|(x - [0\ \ -1.5]^T)\|_{P_1} \leq 1\}$;
- $\bar{S}_7 = \{x \mid \|(x - [-1.5\ \ -1.5]^T)\| \leq 1\}$;
- $\bar{S}_8 = \{x \mid \|(x - [-1.5\ 0]^T)\|_{P_2} \leq 1\}$;

where $\|z\|_{P_1} = \sqrt{\frac{z_1^2}{1.2^2} + \frac{z_2^2}{0.5^2}}$ and $\|z\|_{P_2} = \sqrt{\frac{z_1^2}{0.5^2} + \frac{z_2^2}{1.2^2}}$. The problem can be re-formulated to design a control input $u(t)$ such that for $x(0) \in \bar{S}_1$,

- For a given $0 < t_0 < T_1$, $x(t_0) \in \bar{S}_2 \setminus \bar{S}$;

- For a given $t_0 < t_1 \leq T_1$, $x(t_1) \in \bar{S}_3 \setminus \bar{S}$;
- For a given $T_1 < t_2 < T_2$, $x(t_2) \in \bar{S}_4 \setminus \bar{S}$;
- For a given $t_2 < t_3 \leq T_2$, $x(t_3) \in \bar{S}_5 \setminus \bar{S}$;
- For a given $T_2 < t_4 < T_3$, $x(t_4) \in \bar{S}_6 \setminus \bar{S}$;
- For a given $t_4 < t_5 \leq T_3$, $x(t_5) \in \bar{S}_7 \setminus \bar{S}$;
- For a given $T_3 < t_6 < T_4$, $x(t_6) \in \bar{S}_8 \setminus \bar{S}$;
- For a given $t_6 < t_7 \leq T_4$, $x(t_7) \in \bar{S}_1 \setminus \bar{S}$,

which can be written as an STL formula as in (18).

We can now use the formulation (16) to compute the control input. We use $|u| \leq 10$ as the input constraints and use the same approach as in scenario 2, to translate these constraints in the form of (16e). The time constraints are chosen as $T_i = 2$ for $i \in \{1, 2, 3, 4\}$ and $t_j = 1$ for $j \in \{t_0, t_1, \cdots, t_7\}$. We choose $\mu = 5$, so that $\gamma_1 = 1.2$ and $\gamma_2 = 0.8$. Figures 7-8 illustrate the closed-loop position trajectories of the robot for one initial condition; it is evident that the robot position always remains in the safe set S_s, while visiting the sets C_2, C_3, C_4 and C_1 sequentially.Figure 9 illustrates the control input trajectories and verifies that the control input constraint $\|u_i(t)\| \leq 10$ is satisfied at all times.

VI. Direction for future work

Note that Theorem 2, Corollary 1 and Theorem 4 are restrictive because of the following two reasons. First, it is needed that the functions $h(x)$ and $h_i(x)$ are CLF/CBF for the system (4), otherwise the respective inequalities used in the aforementioned results do not hold. Second, these results also need that the functions $h(x)$ and $h_i(x)$ are continuously differentiable. Although, this is a very common assumption in the literature (see [8], [11] and other similar work), this limits the choice of sets S and S_i that can be considered in the setup of Theorem 2 or Theorem 4. One approach is to use non-smooth analysis (e.g., [25]) to formulate the constraints of (16), so that the sets characterized by non-differentiable $h(x)$ can also be incorporated. Another plausible approach is to look for the CLF and the control input u simultaneously. We propose sufficient conditions to characterize the CLF and the control input for Problem 1.

Proposition 1. *If there exist continuously differentiable function V and constants $a_1, a_2 > 0$, $\gamma_1 > 1$ and $0 < \gamma_2 < 1$ satisfying $\frac{1}{a_1(\gamma_1 - 1)} + \frac{1}{a_2(1-\gamma_2)} \leq T$ such that the following holds for $x \notin S$*

$$h(x) \leq V(x) \leq h(x) + c \tag{19a}$$

$$\inf_{u \in \mathcal{U}} \{L_f V(x) + L_g V(x) u + a_1 h(x)^{\gamma_1} + a_2 h(x)^{\gamma_2}\} \leq 0, \tag{19b}$$

where $c \geq 0$, then the closed-loop trajectories of (4) reach the set S within prescribed time T for all initial conditions.

Proof: If $h(x)$ is a smooth function, one can choose $c = 0$, so that $V(x) = h(x)$. Note that (19a) implies that $V(x) \leq h(x)$ for $x \notin S$. If there exists a control input u, such that (19b) holds, then from (19a), we obtain that

$$\begin{aligned} \dot{V} + a_1 V(x)^{\gamma_1} + a_2 V(x)^{\gamma_2} &\leq \dot{V} + a_1 h(x)^{\gamma_1} + a_2 h(x)^{\gamma_2} \\ &\leq 0. \end{aligned}$$

Hence, using Theorem 1, we obtain that $V(x(t)) = 0$ for all $t \geq \bar{T}$, where $\bar{T} \leq \frac{1}{a_1(\gamma_1 - 1)} + \frac{1}{a_2(1-\gamma_2)} \leq T$. Now, from (19a), we know that for $V(x) = 0 \implies h(x) \leq 0$, which implies $x \in S$. ■

We illustrate, via a simple example, how Proposition 1 can be used for the case when $h(x)$ is non-smooth. Consider the case when the set S in Problem 1 is defined as $S = \{x \mid \|x\|_1 \leq 1\}$, i.e., using the 1-norm of x, so that S is a square. Using the fact that $S_n = \{x \mid x_1^{2n} + x_2^{2n} - 1\} \to S$, as $n \to \infty$, one can choose $V = x_1^{2n} + x_2^{2n} - 1$, for large positive integer n and look for n, along with $u, a_1, a_2 > 0$, $\gamma_1 > 1$ and $0 < \gamma_2 < 1$ so that conditions of Proposition 1 hold. A similar set of sufficient conditions can be derived for Theorem 4, which would allow a larger class of problems to be solved. It is part of our future investigations to study methods to solve for V and u, simultaneously, in an efficient way. In future, we would also like to study properties of the system dynamics and the functions $h(x), h_i(x)$, so that the resulting closed-loop trajectories are smooth.

VII. Conclusions

In this paper, we considered the problem of trajectory planning under spatio-temporal, and control input constraints. We defined a new class of CLF, called PT CLF, to guarantee that the closed loop trajectories reach a given set within the prescribed time. We formulated a QP to find a control input that guarantees prescribed time convergence. Then, we considered a general problem of control synthesis under multiple spatiotemporal objectives. We first presented sufficient conditions for the existence of a control input in terms of PT CLF and CBF. Then, we presented a QP based formulation to efficiently compute the control input that guarantees safety and prescribed time convergence in the presence of control input constraints.

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$$(x_1,t) \models G_{[0,T_4]}\bar{\phi}_s \wedge G_{[0,t_0]}\bar{\phi}_1 \wedge F_{[0,t_0]}\bar{\phi}_2 \wedge G_{(t_0,T_1]}\bar{\phi}_2 \wedge F_{(t_0,T_1]}\bar{\phi}_3 \wedge G_{(T_1,t_2]}\bar{\phi}_3 \wedge F_{(T_1,t_2]}\bar{\phi}_4 \wedge G_{(t_2,T_2]}\bar{\phi}_4 \wedge F_{(t_2,T_2]}\bar{\phi}_5$$
$$\wedge\, G_{(T_2,t_4]}\bar{\phi}_5 \wedge F_{(T_2,t_4]}\bar{\phi}_6 \wedge G_{(t_4,T_3]}\bar{\phi}_6 \wedge F_{(t_4,T_3]}\bar{\phi}_7 \wedge G_{(T_3,t_6]}\bar{\phi}_7 \wedge F_{(T_3,t_6]}\bar{\phi}_8 \wedge G_{(t_6,T_4]}\bar{\phi}_8 \wedge F_{(t_6,T_4]}\bar{\phi}_1. \quad (18)$$

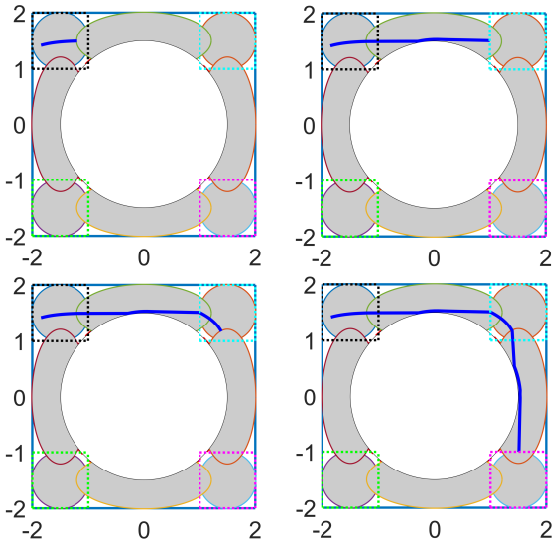

Fig. 7. Scenario 3: Closed-loop trajectory: snapshot at $t = 1, 2, 3$ and 4 sec.

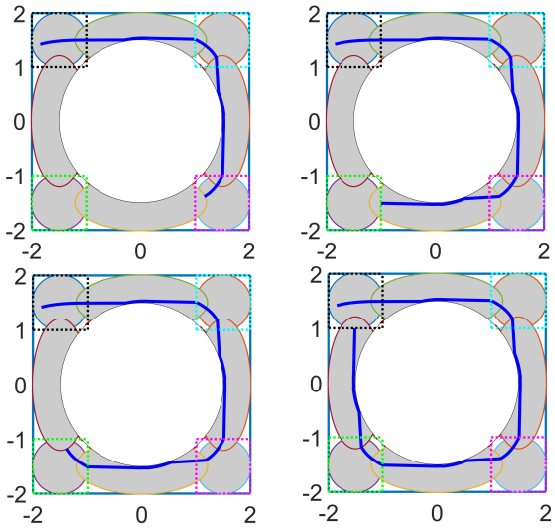

Fig. 8. Scenario 3: Closed-loop trajectory: snapshot at $t = 5, 6, 7$ and 8 sec.

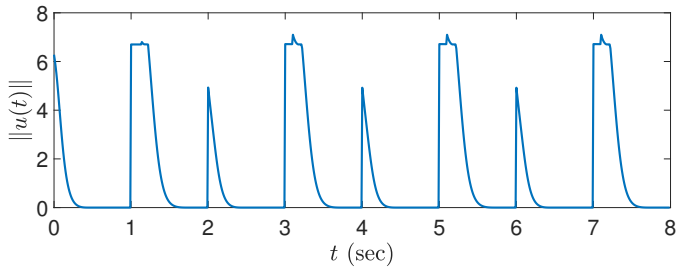

Fig. 9. Scenario 3: Norm of the control input $\|u(t)\|$.

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