Robust Semi-Cooperative Multi-Agent Coordination in the Presence of Stochastic Disturbances

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***Abstract*— This paper presents a robust distributed coordination protocol that achieves generation of collision-free trajectories for multiple unicycle agents in the presence of stochastic uncertainties. We build upon our earlier work on semi-cooperative coordination and we redesign the coordination controllers so that the agents counteract a class of state (wind) disturbances and measurement noise. Safety and convergence is proved analytically, while simulation results demonstrate the efficacy of the proposed solution.**

I. Introduction

Coordination in multi-agent systems has attracted much attention over the last decade with a plethora of theoretical and practical problems that this paper can not cite in their entirety; for recent overviews the reader is referred to [1]–[4].

A fundamental problem of interest in the area of distributed coordination and control is the decentralized multi-agent motion planning, which mainly focuses on generating collision-free trajectories for multiple agents (e.g., unmanned vehicles, robots) so that they reach preassigned goal locations under limited sensing, communication, and interaction capabilities. Numerous elegant methodologies on planning the motion for a single agent (robot) have appeared in recent years, with the most popular being (i) sampling-based methods, including probabilistic roadmaps [5], and rapidly-exploring random trees [6], [7], (ii) Lyapunov-based methods, including either the definition of closed-form feedback motion plans via potential functions or vector fields, or computation of Lyapunov-based feedback motion plans via sum-of-squares programming [8], [9], and (iii) graph search and decision-theoretic methods, see also [10], [11] for a detailed presentation. Although each method has its own merits and caveats, arguably Lyapunov-based methods (often termed reactive) are particularly popular for multi-agent motion planning problems, as they offer scalability with the number of agents, and the merits of Lyapunov-based control design and analysis.

In addition, robustness against modeling and/or measurement uncertainties is of primary importance for real-world systems and applications. Hence the problems of modeling, quantifying, and treating uncertainty are of particular interest when it comes to multi-agent coordination. A straightforward way to model uncertainty in multi-agent systems is by considering them as *bounded disturbances*, see [12]–[16]. Another way of modeling uncertainty is by *Gaussian random processes*. In [17]–[19], communication noises are described by a standard Brownian motion, and the mean square consensus in the multi-agent system is achieved by proposing a stochastic approximation-type gain vector, which attenuates the effect of noises. In [20], it is assumed that the measurements for each agent are disturbed by white noises. Robust consensus can be achieved by applying stochastic Lyapunov analysis. Based on this idea, in [21], the average-consensus problem of first-order multi-agent systems is considered, and a necessary and sufficient condition is proposed for robust consensus by using probability limit theory. The aforementioned methods are efficient in solving the coordination problems with stochastic uncertainties in measurement or system dynamics; however, safety (i.e., the generation of collision-free trajectories) is not considered.

In contrast to the aforementioned results, in this paper we consider the problem of generating collision-free trajectories for multiple agents in the presence of uncertainty. We propose a robust, Lyapunov-based coordination protocol that achieves collision-free motion for multiple agents in a distributed fashion, in the presence of state and measurement uncertainties. The method builds upon our earlier work in [22], in which the nominal (uncertainty-free) case was considered. More specifically, we redesign the semi-cooperative coordination protocol in [22] so that it accommodates the case of state and measurement uncertainties. Our approach yields a method on the safe and robust motion planning of multiple agents that is based on analytic vector fields, hence offers scalability with the number of agents along with provable guarantees. In summary, the contributions of this paper are: (i) a robust, semi-cooperative coordination protocol that accommodates for a class of stochastic disturbances in the agents' dynamics and measurements, and (ii) the derivation of analytical bounds on the navigation (estimation) and final state errors of the agents in terms of the considered uncertainties.

II. Modeling and Problem Statement

Let us consider N identical agents $i \in \{1, \dots, N\}$, which are assigned to move to goal locations of position coordinates $\boldsymbol{r}_{gi} = \begin{bmatrix} x_{gi} & y_{gi} \end{bmatrix}^T$ relative to some global frame $\mathcal{G}$, while avoiding collisions. The motion of each agent i is modeled under unicycle kinematics with additive disturbances that stand for state and output uncertainty, for instance due to

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wind effects and sensor imperfections, respectively, as:

$$\dot{\boldsymbol{q}}_i = \boldsymbol{f}(\boldsymbol{q}_i, \boldsymbol{u}_i) + \boldsymbol{\Gamma}\boldsymbol{w} \Rightarrow \begin{bmatrix}\dot{x}_i\\ \dot{y}_i\\ \dot{\theta}_i\end{bmatrix} = \begin{bmatrix}u_i c\theta_i\\ u_i s\theta_i\\ \omega_i\end{bmatrix} + \begin{bmatrix}w_x\\ w_y\\ 0\end{bmatrix}, \quad \text{(1a)}$$

$$\boldsymbol{y}_i = \boldsymbol{h}_i(\boldsymbol{q}_i) + \boldsymbol{v}_i, \quad \text{(1b)}$$

where $\boldsymbol{q}_i = \begin{bmatrix}\boldsymbol{r}_i^T & \theta_i\end{bmatrix}^T$ is the state vector of agent i, comprising the position vector $\boldsymbol{r}_i = \begin{bmatrix}x_i & y_i\end{bmatrix}^T$ and the orientation θ_i of the agent with respect to (w.r.t.) the global frame $\mathcal{G}$, $\boldsymbol{u}_i = \begin{bmatrix}u_i & \omega_i\end{bmatrix}^T$ is the control input vector comprising the linear velocity u_i and the angular velocity ω_i of agent i, $\boldsymbol{f}(\cdot,\cdot) : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^3$ is the vector valued function of the agent dynamics, and $c(\cdot) \triangleq \cos(\cdot)$, $s(\cdot) \triangleq \sin(\cdot)$ and $\boldsymbol{\Gamma} = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$. The random process $\boldsymbol{w} = \begin{bmatrix}w_x & w_y\end{bmatrix}^T$ is assumed to be Gaussian, white, of known mean $\bar{\boldsymbol{w}} = \begin{bmatrix}\bar{w}_x & \bar{w}_y\end{bmatrix}^T$ and known covariance $\boldsymbol{P}_w \in \mathbb{R}^{2\times 2}$. Furthermore, $\boldsymbol{y}_i \in \mathbb{R}^m$ is the output vector comprising the available measurements, $\boldsymbol{h}_i(\cdot) : \mathbb{R}^3 \to \mathbb{R}^m$ is the output function, and $\boldsymbol{v}_i \in \mathbb{R}^m$ is the measurement noise modeled as a Gaussian, white process of zero mean $\bar{\boldsymbol{v}}_i = \boldsymbol{0}$ and known covariance $\boldsymbol{P}_{v_i} \in \mathbb{R}^{m\times m}$. We assume that the output function is the identity map so that the measurement model reduces to $\boldsymbol{y}_i = \boldsymbol{q}_i + \boldsymbol{v}_i$, and that the measurements are uncorrelated, so that the covariance matrix of $\boldsymbol{v}_i$ reads $\boldsymbol{P}_{v_i} = \mathrm{diag}(\sigma_{v_{i,1}}, \sigma_{v_{i,2}}, \sigma_{v_{i,3}})$.

Each agent i is modeled as a circular disk of radius ϱ_i, and has a communication/sensing region $\mathcal{C}_i$ of radius R_c centered at $\boldsymbol{r}_i = \begin{bmatrix}x_i & y_i\end{bmatrix}^T$, denoted as $\mathcal{C}_i : \{\boldsymbol{r} \in \mathbb{R}^2 \mid \|\boldsymbol{r}_i - \boldsymbol{r}\| \leq R_c\}$. We denote $\mathcal{N}_i$ the set of neighboring agents $k \in \mathcal{C}_i$ of agent i. Each agent i can measure the position $\boldsymbol{r}_k$, orientation θ_k and the linear velocity u_k of any agent $k \in \mathcal{C}_i$.

In our earlier work [22] we considered the nominal case of (1), i.e., the case for $\boldsymbol{w} = \boldsymbol{0}$, $\boldsymbol{v} = \boldsymbol{0}$. Details about the control law are omitted for the sake of brevity and can be found in [23]. Under this protocol we were able to establish collision-free and almost globally convergent motion of the agents towards to their goal locations. In this paper, we seek to design a robust coordination protocol so that each agent i can safely accommodate the effects of state and measurement uncertainties $\boldsymbol{w}(t)$, $\boldsymbol{v}_i(t)$, $t \in [0, \infty)$, respectively. Since we are only concerned about radial convergence of the agents to their respective goal locations, we re-define the radially attractive vector field $\mathbf{F}^r_{gi}$ for $\boldsymbol{r}_i \neq \boldsymbol{r}_{gi}$ as:

$$\mathrm{F}^r_{gix} = \frac{-(x_i - x_{gi})}{(x_i - x_{gi})^2 + (y_i - y_{gi})^2}, \quad \text{(2a)}$$

$$\mathrm{F}^r_{giy} = \frac{-(y_i - y_{gi})}{(x_i - x_{gi})^2 + (y_i - y_{gi})^2}. \quad \text{(2b)}$$

With this globally attractive field, the new reference field is given by (see [22] for details):

$$\mathbf{F}_i = \prod_{j\in\mathcal{N}_i}(1 - \sigma_{ij})\mathbf{F}^r_{gi} + \sum_{j\in\mathcal{N}_i}\sigma_{ij}\mathbf{F}^i_{oj}. \quad \text{(3)}$$

III. Robust Coordination: Design and Analysis

A. Control design under known state-disturbances

We first consider the case where each agent i is subject to known state disturbances, without any measurement uncertainty, i.e., we consider the agent dynamics:

$$\dot{\boldsymbol{q}}_i = \boldsymbol{f}(\boldsymbol{q}_i, \boldsymbol{u}^p_i) + \boldsymbol{\Gamma}\bar{\boldsymbol{w}}, \quad \text{(4a)}$$

$$\boldsymbol{y}_i = \boldsymbol{h}_i(\boldsymbol{q}_i), \quad \text{(4b)}$$

where $\bar{\boldsymbol{w}}$ is the known mean of the state disturbance and $\boldsymbol{u}^p_i = \begin{bmatrix}u^p_i & \omega^p_i\end{bmatrix}^T$ is the control input. Let $\boldsymbol{q} = [\boldsymbol{q}_1^T, \ldots, \boldsymbol{q}_N^T]^T$. We propose the following coordination protocol yielding the feedback control law $\boldsymbol{u}^p_i(\boldsymbol{q}, \bar{\boldsymbol{w}}, d_m)$ for the perturbed dynamics (4) of agent i:

Coordination of linear velocities: The linear velocity of each agent i is governed by the control law:

$$u^p_i = \begin{cases} -\frac{1}{\mu}\log\Big(\sum\limits_{k\in\mathcal{N}_i | J_k<0} e^{-\mu u_{i|k}}\Big), & d_m \leq d_{ik} \leq d_\epsilon,\\ u_{ic}, & d_\epsilon \leq d_{ik}; \end{cases}, \quad \text{(5)}$$

where:

- the linear velocity of agent i when free of conflicts is:
$$u_{ic} = \left\| u_i \frac{\mathbf{F}_i}{\|\mathbf{F}_i\|} - \bar{\boldsymbol{w}} \right\|, \quad \text{(6)}$$
$$u_i = k_{ui}\tanh\left(\|\boldsymbol{r}_i - \boldsymbol{r}_{gi}\|\right), \quad \text{(7)}$$
where $k_{ui} > 0$ and $\mathbf{F}_i$ is given by the nominal vector field (3),
- for $\boldsymbol{a} = \begin{bmatrix}a_1, \ldots, a_n\end{bmatrix}^T$, the function $g(\boldsymbol{a}) = -\frac{1}{\mu}\log\Big(\sum\limits_{i=1}^{N} e^{-\mu a_i}\Big)$ is a smooth approximation of the minimum function $\min\{a_1, \ldots, a_n\}$ as $\mu \to \infty$,
- $u_{i|k}$ is the safe velocity of agent i w.r.t. a neighbor agent $k \in \mathcal{N}_i$, given as:
$$u_{i|k} = u_{ic}\,\frac{d_{ik} - d_m}{d_\epsilon - d_m} + \varepsilon_i\, u_{is|k}\,\frac{d_\epsilon - d_{ik}}{d_\epsilon - d_m}, \quad \text{(8)}$$
with the terms in (8) defined as:
$$u_{is|k} = u^p_k\,\frac{{\boldsymbol{r}_{ki}}^T\boldsymbol{\eta}_k}{{\boldsymbol{r}_{ki}}^T\boldsymbol{\eta}_i},\quad \boldsymbol{\eta}_i = \begin{bmatrix}\cos\theta_i\\ \sin\theta_i\end{bmatrix},\quad J_k = {\boldsymbol{r}_{ki}}^T\boldsymbol{\eta}_i,$$
$$\boldsymbol{r}_{ki} = \boldsymbol{r}_i - \boldsymbol{r}_k, \quad \text{and} \quad 0 < \varepsilon_i < 1.$$

Coordination of angular velocities: The angular velocity of each agent i is governed by the control law:

$$\omega^p_i = -k_{\omega i}(\theta_i - \varphi^p_i) + \dot{\varphi}^p_i, \quad \text{(9)}$$

where $k_{\omega i} > 0$, $\varphi^p_i \triangleq \arctan\left(\frac{\mathrm{F}^p_{niy}}{\mathrm{F}^p_{nix}}\right)$ is the orientation of the normalized vector field $\mathbf{F}^p_{ni} = \frac{\mathbf{F}^p_i}{\|\mathbf{F}^p_i\|}$ for the perturbed system (4) at a point (x, y), with the vector field $\mathbf{F}^p_i$ for the perturbed system (4) given out of:

$$\mathbf{F}^p_i = u_i\frac{\mathbf{F}_i}{\|\mathbf{F}_i\|} - \bar{\boldsymbol{w}}, \quad \text{(10)}$$

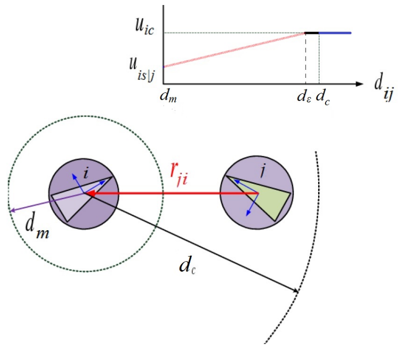

Fig. 1. If $J_j \triangleq \boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_i < 0$, i.e., if agent i moves towards agent j, then agent i adjusts its linear velocity according to the velocity profile shown here, given analytically by (8).

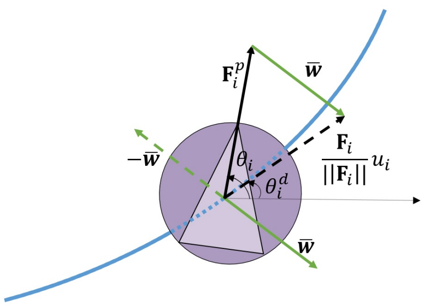

Fig. 2. Construction of new vector field in the presence of wind. Blue line is the desired trajectory, green arrow, dotted black arrow and solid black arrow show the direction of the mean wind speed, nominal vector field $\mathbf{F}_i$ and constructed vector field $\mathbf{F}_i^p$ respectively.

where $\mathbf{F}_i$ is the nominal vector field given out of (3), and u_i is given out of (7). It can be readily seen that steady-state solution of $\dot{\theta}_i = \omega_i^p = -k_\omega(\theta_i - \varphi_i^p) + \dot{\varphi}_i^p$ is $\theta_i = \varphi_i^p$.

The following theorem proves that safety for the multi-agent can be guaranteed under the control law given by (5) and (9):

Theorem 1: If each agent $i \in \{1, \dots, N\}$ subject to the system dynamics (4) follows the control law given by (5) and (9), then $\forall i, j \in \{1, \dots, N\}$, $i \neq j$, it holds that:

$$\|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)\| \geq d_m, \ \forall t \geq 0, \tag{11}$$

and each agent converges to its goal location almost globally, i.e.,

$$\|\boldsymbol{r}_{i\infty} - \boldsymbol{r}_{gi}\| \triangleq \lim_{t\to\infty} \|\boldsymbol{r}_i(t) - \boldsymbol{r}_{gi}\| = 0, \tag{12}$$

except for a set of initial conditions of measure zero.

Proof: To prove (11) it is required that

$$\frac{d}{dt} d_{ij}\big|_{d_{ij}=d_m} \geq 0,$$

so that the inter-agent distance does not decrease further once the agents i, j are at the minimum allowed separation d_m. The derivative of inter-agent distance is given in (17):

$$\frac{d}{dt}\ d_{ij} = \frac{u_i^p\ \boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_i - u_j^p\ \boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_j}{d_{ij}}. \tag{18}$$

The worst case neighbor for agent i is defined as the agent $j \in \{\mathcal{N}_i \mid J_j \triangleq \boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_i < 0\}$ towards whom the rate of change of inter-agent distance d_{ij} given by (18), due to the motion of agent i, is maximum. More specifically, the decision making process works as follows: The term $J_j < 0$ describes the set of neighbor agents j of agent i towards whom agent i is moving [24]. Thus agent i computes safe velocities $u_{i|j}$ w.r.t. each neighbor $j \in \{\mathcal{N}_i \mid J_j < 0\}$, and picks the minimum $u_{i|j}$ of the safe velocities so that the first term in (18) is as less negative as possible. Now, the value of the safe velocity $u_{i|j}$ when $d_{ij} = d_m$ is by construction equal to $\varepsilon_i u_{is|k} = \varepsilon_i u_j \frac{\boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_j}{\boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_i}$. Plugging this value into (18) reads:

$$\frac{d}{dt}\ d_{ij} = \frac{(\varepsilon_i - 1)u_j\ \boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_j}{d_{ij}} \geq 0. \tag{19}$$

To see why this condition is true, recall that $\varepsilon_i - 1 < 0$, $u_j \geq 0$, and $\boldsymbol{r}_{ji}{}^T\boldsymbol{\eta}_j \leq 0$: this is because agent j is either following a vector field $\mathbf{F}_j$ that points away from agent i, or happens to move away from agent i in the first place. This implies that the inter-agent distance d_{ij} can not become less than d_m (see [22] for details).

To verify this argument, it is sufficient to show that in the presence of known disturbances $\bar{\boldsymbol{w}}$, the motion of any agent $i, j \in \{1, 2, \dots, N\}$ under the control law (5) and (9) is along their nominal vector fields given by (3), as follows: Let θ_i^d be the nominal direction that agent i is supposed to follow under the nominal vector field $\mathbf{F}_i$, i.e., $\theta_i^d \triangleq \arctan\left(\frac{\mathrm{F}_{ix}}{\mathrm{F}_{iy}}\right)$. From the steady-state solution of $\dot{\theta}_i = \omega_i$ under the control law (9), we have $\theta_i = \varphi_i^p$, where θ_i is the orientation of the agent i. Let $\angle(\cdot)$ be signed angle, defined to be positive in the clockwise direction and negative in the counter-clockwise direction. Now, from (4) we have: $\angle(\dot{\boldsymbol{r}}_i - \bar{\boldsymbol{w}}) = \theta_i$. Also, by construction of the new vector field (10), $\angle\left(u_i\frac{\mathbf{F}_i}{||\mathbf{F}_i||} - \bar{\boldsymbol{w}}\right) = \varphi_i^p \overset{(9)}{=} \theta_i$ out of the steady-state solution of (9), see also Figure 2. Thus we have that $\angle(\dot{\boldsymbol{r}}_i - \bar{\boldsymbol{w}}) = \angle\left(u_i\frac{\mathbf{F}_i}{||\mathbf{F}_i||} - \bar{\boldsymbol{w}}\right)$, which makes $\angle\dot{\boldsymbol{r}}_i = \angle\mathbf{F}_i = \theta_i^d$. Hence, the motion of agent i is along the desired nominal vector field $\mathbf{F}_i$. Analysis of convergence of the agents to their goal locations is given in [22]. Note that the analysis given in [22] is carried out for dipole type attractive vector field, which will still hold for the radially attractive vector field. ■

B. Extension of the controller for stochastic disturbances

In order to estimate the states of the system in the presence of stochastic disturbances, we design a system observer based on the Extended Kalman Filter:

$$\dot{\hat{\boldsymbol{q}}}_i = \boldsymbol{f}(\hat{\boldsymbol{q}}_i, \boldsymbol{u}_i) + \boldsymbol{\Gamma}\bar{\boldsymbol{w}} + \boldsymbol{K}_i(\boldsymbol{y}_i - \hat{\boldsymbol{y}}_i), \tag{20a}$$
$$\dot{\boldsymbol{P}}_i = \boldsymbol{A}_i\boldsymbol{P}_i + \boldsymbol{P}_i\boldsymbol{A}_i^T - \boldsymbol{K}_i\boldsymbol{H}_i\boldsymbol{P}_i + \boldsymbol{\Gamma}\boldsymbol{P}_w\boldsymbol{\Gamma}^T, \tag{20b}$$
$$\hat{\boldsymbol{y}}_i = \boldsymbol{h}(\hat{\boldsymbol{q}}_i) = \hat{\boldsymbol{q}}_i, \tag{20c}$$
$$\boldsymbol{K}_i = -\boldsymbol{P}_i\boldsymbol{H}_i^T\boldsymbol{P}_{v_i}^{-1}, \tag{20d}$$
$$\boldsymbol{u}_i = \boldsymbol{u}_i(\hat{\boldsymbol{q}}, \bar{\boldsymbol{w}}, d_m', \epsilon_J), \tag{20e}$$

$$\dot{d}_{ij} = \frac{(x_i - x_j)(\dot{x}_i - \dot{x}_j) + (y_i - y_j)(\dot{y}_i - \dot{y}_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} \overset{(4)}{=} \frac{\Big((x_i - x_j)(u_i^p c\theta_i - u_j^p c\theta_j) + (y_i - y_j)(u_i^p s\theta_i - u_j^p s\theta_j)\Big)}{d_{ij}}. \quad (17)$$

where $\boldsymbol{A}_i = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_i}\Big|_{\hat{\boldsymbol{q}}_i}$ is the state matrix of the linearized dynamics evaluated at $\hat{\boldsymbol{q}}_i$, $\boldsymbol{H}_i = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{q}_i}\Big|_{\hat{\boldsymbol{q}}_i}$ is the linearized output matrix evaluated at $\hat{\boldsymbol{q}}_i$, $\boldsymbol{P}_i$ is the covariance matrix of the estimation error $\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i - \hat{\boldsymbol{q}}_i$, and $\boldsymbol{K}_i$ is the Kalman gain. Note that the control law (20e) has the same form as (5) but it uses the estimated states $\hat{\boldsymbol{q}}$ for feedback, the mean wind speed $\bar{\boldsymbol{w}}$ for feed-forward command, and involves d'_m as the new safety parameter. The last parameter ϵ_J is introduced to re-define the worst-case neighbor, as per (21).

Stability of EKF-based estimator: In [25], authors have shown that the EKF is stable if system is uniformly observable and $\exists\, \bar{c}, \underline{p}, \underline{q}, \delta, k > 0$ such that:

- $\|\boldsymbol{H}(t)\| \leq \bar{c}$
- $\underline{p} \leq \boldsymbol{P}_v \leq \delta \boldsymbol{I}$
- $\underline{q} \leq \boldsymbol{P}_w \leq \delta \boldsymbol{I}$
- $\|H.O.T\| \leq k\|\tilde{\boldsymbol{q}}_i\|^2$

where $\tilde{\boldsymbol{q}}_i = \boldsymbol{q}_i - \hat{\boldsymbol{q}}_i$ and $H.O.T$ stands for higher-order terms of the linearization . Furthermore, if these conditions hold, then the estimation error is exponentially mean-square bounded. Using this result, we now show that EKF based estimator will remain bounded for system (1) whose control law is given by (20e). In our system, $\boldsymbol{H}$ is just an identity mapping, which allows us to choose $\bar{c} = 1$. Since $\boldsymbol{H}$ is identity, we have that the system is always observable. By the assumption on the type of disturbance we consider in (1), the covariance matrices $\boldsymbol{P}_{v_i}$ and $\boldsymbol{P}_w$ are non-zero constant matrices and hence are bounded. We assume that the initial estimation error is bounded. Now, we show that our closed-loop system also satisfies the condition on boundedness of $H.O.T$. From (5) and (9) we have that the control law (20e) is a bounded, continuously differentiable function of $\boldsymbol{q}$, since the vector field $\mathbf{F}_i^p$, its partial derivatives w.r.t. x_i and y_i, and the linear control input u_i^p, are continuously differentiable and bounded. Hence the closed loop function $\boldsymbol{f}(\boldsymbol{q}_i)$ is a continuously differentiable, bounded function of the states of the agent i and all its derivatives $\frac{d^k \boldsymbol{f}}{d\boldsymbol{q}_i^k}$ are bounded in $\mathbb{R}^n$. Using the expression for Lagrange Remainder for Taylor series expansion [26], we have that $\|H.O.T\| \leq \frac{1}{2}\|\frac{d^2 \boldsymbol{f}}{d\boldsymbol{q}_i^2}\|\|\tilde{\boldsymbol{q}}_i\|^2$ and $\|\frac{d^2 \boldsymbol{f}}{d\boldsymbol{q}_i^2}\| \leq L$. Hence $\|H.O.T\| \leq k\|\tilde{\boldsymbol{q}}_i\|^2$ with $k = L/2$.

Thus, the EKF-based estimator (20) is stable, therefore we can bound the norms of the estimation errors in individual states and position of agent i by small, positive numbers. Define the maximum of the errors in the estimation of the states for any agent i as $\epsilon_x \triangleq \max_i \sqrt{\boldsymbol{P}_{i11}}$, $\epsilon_y \triangleq \max_i \sqrt{\boldsymbol{P}_{i22}}$ and $\epsilon_\theta \triangleq \max_i \sqrt{\boldsymbol{P}_{i33}}$ where $\boldsymbol{P}_{ill}, l \in \{1, \ldots, n\}$, are the diagonal terms of error covariance matrix $\boldsymbol{P}_i$. Also, we define the maximum error in the estimation of the position as $\epsilon_d \triangleq \sqrt{\epsilon_x^2 + \epsilon_y^2}$. Thus we have that $\|x_i - \hat{x}_i\| \leq \epsilon_x$, $\|y_i - \hat{y}_i\| \leq \epsilon_y$, $\|\theta_i - \hat{\theta}_i\| \leq \epsilon_\theta$ and $\|\boldsymbol{r}_i - \hat{\boldsymbol{r}}_i\| \leq \epsilon_d$. Referring to Definition 4.1 of [25], we can choose a function $\Lambda(\boldsymbol{q}) = -1$, so that $\|\Lambda\| = 1$. Then, using Theorem 7 in [27], we have that $\|\boldsymbol{P}_i(t)\| \leq \|\boldsymbol{P}_i(0)\| + 1$. We choose $\boldsymbol{P}_i(0) = \boldsymbol{P}_{v_i}$, so that $\max \|\boldsymbol{P}(t)\| = \max_i \|\boldsymbol{P}_{v_i}\| + 1$. With this bound on the covariance matrix, we can express $\epsilon_x = \epsilon_y = \epsilon_\theta = \sqrt{\max_i \|\boldsymbol{P}_{v_i}\| + 1}$.

C. Safety Analysis

We re-define the worst case neighbor (defined earlier) in terms of the estimated states as:

$$j \in \{\mathcal{N}_i \mid \hat{\boldsymbol{r}}_{ji}^T \hat{\boldsymbol{\eta}}_i \leq -\epsilon_J \triangleq -\frac{2\epsilon_d + s(\epsilon_\theta)(d_m + 2\epsilon_d)}{c(\epsilon_\theta)}\}. \quad (21)$$

To proceed with the analysis, we first prove the following Lemma:

Lemma 1: If $\hat{\boldsymbol{r}}_{ji}^T \hat{\boldsymbol{\eta}}_i \leq -\epsilon_J$, where ϵ_J is given as in (21), then $\boldsymbol{r}_{ji}^T \boldsymbol{\eta}_i \leq 0$.

Proof: Since we are concerned about safety of the system, let's assume that inter-agent distance $d_{ij} = d_m$. We use the fact that $\|x_i - \hat{x}_i\| \leq \epsilon_x$, $\|y_i - \hat{y}_i\| \leq \epsilon_y$, $\|\theta_i - \hat{\theta}_i\| \leq \epsilon_\theta$ and $\|\boldsymbol{r}_i - \hat{\boldsymbol{r}}_i\| \leq \epsilon_d$. Therefore we have that

$$d_m - 2\epsilon_d \leq \|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j\| \leq d_m + 2\epsilon_d.$$

Term $J_j = \boldsymbol{r}_{ji}{}^T \boldsymbol{\eta}_i = \boldsymbol{r}_{ji}{}^T \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} = (x_i - x_j)c\theta_i + (y_i - y_j)s\theta_i$ which further reads as in (22). Since ϵ_θ and ϵ_d are small positive numbers, we have $\cos(\epsilon_\theta) > 0$ and $\epsilon_J > 0$. Hence, if $((\hat{x}_i - \hat{x}_j)c\hat{\theta}_i + (\hat{y}_i - \hat{y}_j)s\hat{\theta}_i) \leq -\epsilon_J$, then we have that $\boldsymbol{r}_{ji}^T \boldsymbol{\eta}_i \leq 0$. ∎

Now we use this result to show the safety of the system. In order to maintain the safety of the agents in the presence of disturbances, the safety parameter d'_m in the control law (20e) is given by:

$$d'_m = d_m + 2\epsilon_d. \quad (23)$$

Theorem 2: If each agent i under the system dynamics (1) follows the control law given by (20e), and if d'_m is given by (23), then, $\forall t \geq 0, \forall\, i, j \in \{1, \ldots, N\},\ j \neq i$:

$$\|\boldsymbol{r}_i(t) - \boldsymbol{r}_j(t)\| \geq d_m. \quad (24)$$

Proof: Using triangle inequality,

$$\|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j\| \leq \|\hat{\boldsymbol{r}}_i - \boldsymbol{r}_i\| + \|\hat{\boldsymbol{r}}_j - \boldsymbol{r}_j\| + \|\boldsymbol{r}_i - \boldsymbol{r}_j\| \Rightarrow$$
$$\|\hat{\boldsymbol{r}}_i - \hat{\boldsymbol{r}}_j\| \leq 2\epsilon_d + \|\boldsymbol{r}_i - \boldsymbol{r}_j\| \Rightarrow d'_m \leq 2\epsilon_d + \|\boldsymbol{r}_i - \boldsymbol{r}_j\|.$$

Choosing d'_m as per (23), we get $\|\boldsymbol{r}_i - \boldsymbol{r}_j\| \geq d_m$. To complete the proof, it is sufficient to show that when $\hat{d}_{ij} = d'_m$, the time derivative of inter-agent distance is positive, i.e. $\dot{d}_{ij} > 0$. In the presence of uncertainties, to guarantee that the system is safe, we modify the definition of the worst case neighbor as $j \in \{\mathcal{N}_i \mid \hat{J}_j \triangleq \hat{\boldsymbol{r}}_{ji}^T \hat{\boldsymbol{\eta}}_i < -\epsilon_J\}$ so that, from

$$
\begin{aligned}
{\boldsymbol{r}_{ji}}^T\boldsymbol{\eta}_j &= \big((x_i-\hat{x}_i)-(x_j-\hat{x}_j)+(\hat{x}_i-\hat{x}_j)\big)c\theta_i+\big((y_i-\hat{y}_i)-(y_j-\hat{y}_j)+(\hat{y}_i-\hat{y}_j)\big)s\theta_i\\
&\le (\hat{x}_i-\hat{x}_j)c\theta_i+(\hat{y}_i-\hat{y}_j)s\theta_i+2\epsilon_x c\theta_i+2\epsilon_y s\theta_i \le (\hat{x}_i-\hat{x}_j)c(\hat{\theta}_i+\epsilon_{\theta_i})+(\hat{y}_i-\hat{y}_j)s(\hat{\theta}_i+\epsilon_{\theta_i})+2\epsilon_d\\
&\le (\hat{x}_i-\hat{x}_j)\big(c\hat{\theta}_i c(\epsilon_\theta)-s\hat{\theta}_i s(\epsilon_\theta)\big)+(\hat{y}_i-\hat{y}_j)\big(s\hat{\theta}_i c(\epsilon_\theta)+c\hat{\theta}_i s(\epsilon_\theta)\big)+2\epsilon_d\\
&= \Big((\hat{x}_i-\hat{x}_j)c\hat{\theta}_i+(\hat{y}_i-\hat{y}_j)s\hat{\theta}_i\Big)c(\epsilon_\theta)-s(\epsilon_\theta)\Big((\hat{x}_i-\hat{x}_j)s\hat{\theta}_i-(\hat{y}_i-\hat{y}_j)c\hat{\theta}_i\Big)+2\epsilon_d\\
&\le \Big((\hat{x}_i-\hat{x}_j)c\hat{\theta}_i+(\hat{y}_i-\hat{y}_j)s\hat{\theta}_i\Big)c(\epsilon_\theta)+s(\epsilon_\theta)(d_m+2\epsilon_d)+2\epsilon_d
\end{aligned}
\tag{22}
$$

Lemma 1 we have that $\boldsymbol{r}_{ji}^T\boldsymbol{\eta}_i \le 0$, which implies $\dot{d}_{ij} \ge 0$ at $\hat{d}_{ij} \triangleq \|\hat{\boldsymbol{r}}_i-\hat{\boldsymbol{r}}_j\| = d'_m$. Therefore, we have that: (i) $\hat{d}_{ij} = d'_m \implies d_{ij} \ge d_m$, and (ii) $\frac{d}{dt}d_{ij}\big|_{\hat{d}_{ij}=d'_m} \ge 0$, i.e. the inter-agent distance does not decrease beyond this point. Hence the multi-agent system always remains safe. ■

D. Analysis of convergence to goal location

Theorem 3: If agent i under the system dynamics (1) follows the control law given by (20e), then:

$$\lim_{t\to\infty}\|\boldsymbol{r}_i(t)-\boldsymbol{r}_{gi}\| = \|\boldsymbol{r}_{i\infty}-\boldsymbol{r}_{gi}\| \le \epsilon_f, \tag{25}$$

where $\epsilon_f = \epsilon_d+\epsilon$, and ϵ is a small positive constant.

Proof: Using triangle inequality,

$$\|\boldsymbol{r}_{i\infty}-\boldsymbol{r}_{gi}\| \le \|\boldsymbol{r}_{i\infty}-\hat{\boldsymbol{r}}_{i\infty}\|+\|\hat{\boldsymbol{r}}_{i\infty}-\boldsymbol{r}_{gi}\|.$$

System (20a) can be seen as a perturbed form of (4) with $\boldsymbol{y}_i-\hat{\boldsymbol{y}}_i$ being the constantly acting, bounded perturbation. From Theorem 1, we have that $\boldsymbol{r}_{gi}$ is an (almost globally) asymptotically stable equilibrium of (4). Furthermore, (assuming no interactions among agents in the vicinity of the goal locations) one has that $\boldsymbol{r}_{gi}$ is a stable equilibrium of (20a). To verify this argument, use the candidate Lyapunov function $V = \|\hat{\boldsymbol{r}}_i-\boldsymbol{r}_{gi}\|^2$. Define $\boldsymbol{r}_e \triangleq \hat{\boldsymbol{r}}_i-\boldsymbol{r}_{gi}$.

Now, taking the time derivative of candidate Lyapunov function along the system (20a), we get

$$\dot{V} = \boldsymbol{r}_e^T\dot{\hat{\boldsymbol{r}}}_i = \boldsymbol{r}_e^T\begin{bmatrix}u_i^p c\varphi_i^p+\bar{w}_x\\ u_i^p s\varphi_i^p+\bar{w}_y\end{bmatrix}+\boldsymbol{r}_e^T\boldsymbol{\Gamma}^T\boldsymbol{K}_i(\boldsymbol{y}_i-\hat{\boldsymbol{y}}_i).$$

Note that $\begin{bmatrix}u_i^p c\varphi_i^p+\bar{w}_x\\ u_i^p s\varphi_i^p+\bar{w}_y\end{bmatrix}$ is the $\dot{\boldsymbol{r}}_i^p$ of agent i in the absence of unknown disturbance with magnitude $|\dot{\boldsymbol{r}}_i^p| = u_i$ and is along the attractive vector field $\mathbf{F}_{gi}^r$ which points in the direction $-(\hat{\boldsymbol{r}}_i-\boldsymbol{r}_{gi})$. Hence, we have that $\dot{\boldsymbol{r}}_i^p = -u_i\frac{\boldsymbol{r}_e}{\|\boldsymbol{r}_e\|}$. Finally, since the EKF based estimator is stable, the perturbation term $\boldsymbol{K}_i(\boldsymbol{y}_i-\hat{\boldsymbol{y}}_i)$ can be bounded by $\|\boldsymbol{K}_i\|\epsilon_d \le \delta$. Hence, we have

$$
\begin{aligned}
\dot{V} &= -\boldsymbol{r}_e^T u_i\frac{\boldsymbol{r}_e}{\|\boldsymbol{r}_e\|}+\boldsymbol{r}_e^T\boldsymbol{\Gamma}^T\boldsymbol{K}_i(\boldsymbol{y}_i-\hat{\boldsymbol{y}}_i)\\
&\le -k_{ui}\|\boldsymbol{r}_e\|\tanh(\|\boldsymbol{r}_e\|)+\delta\|\boldsymbol{r}_e\|.
\end{aligned}
$$

Hence, we have $\dot{V} \le 0$ for $\|\boldsymbol{r}_e\| = \|\hat{\boldsymbol{r}}_i-\boldsymbol{r}_{gi}\| \ge \tanh^{-1}\frac{\delta}{k_{ui}}$ where k_{ui} is chosen to be greater than δ. Using the stability of perturbed systems under the effect of constantly acting (non-vanishing) perturbations [28], we have that $\boldsymbol{r}_{gi}$ is a stable equilibrium of (20a), which ensures that $\|\hat{\boldsymbol{r}}_{i\infty}-\boldsymbol{r}_{gi}\| \le \epsilon$ for some small positive number $\epsilon > 0$. Therefore,

$$\|\boldsymbol{r}_{i\infty}-\boldsymbol{r}_{gi}\| \le \|\boldsymbol{r}_{i\infty}-\hat{\boldsymbol{r}}_{i\infty}\|+\epsilon \Rightarrow \|\boldsymbol{r}_{i\infty}-\boldsymbol{r}_{gi}\| \le \epsilon_d+\epsilon.$$

Choosing $\epsilon_f = \epsilon_d+\epsilon$ completes the proof. ■

IV. Simulations

We consider two scenarios involving N = 20 agents which are assigned to move towards goal locations while avoiding collisions. The goal locations are selected sufficiently far apart so that the agents' communication regions do not overlap when agents lie on their goal locations. The covariance matrix of the state disturbance is taken as $\boldsymbol{P}_w = \mathrm{diag}(0.01, 0.01)$ and measurement noise as $\boldsymbol{P}_{vi} = \mathrm{diag}(0.01, 0.01, 0.01)$. With these uncertainties, we have $\epsilon_d = 1.42\ m$ and $\epsilon_f = 1.52\ m$. The radii of the agents are $\varrho = 0.4\ m$, the minimum separation is set equal to $d_m = 2\varrho = 0.8\ m$, and the communication radius is set equal to $R_c = 2d'_m = 7.28\ m$. In the first case, we assume the mean wind speed to be constant with time, with $\bar{\boldsymbol{w}} = \begin{bmatrix}-0.2 & 0.7\end{bmatrix}^T\ (m/sec)$. In the second case, we assume the mean wind speed to be time varying with $\bar{w}_x(t), \bar{w}_y(t) \in [-1, 1]\ (m/sec)$.

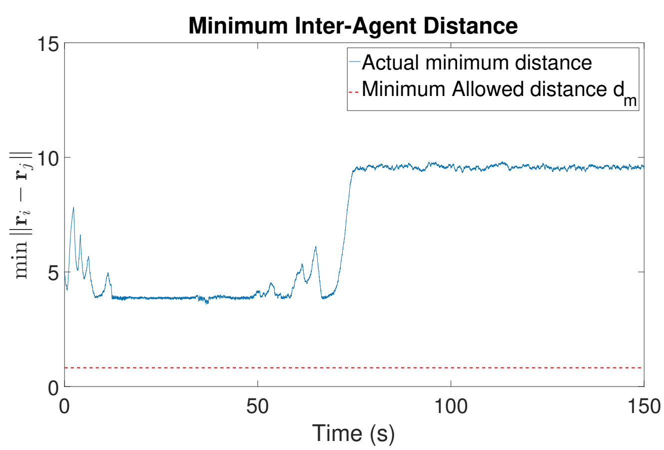

Fig. 3. The smallest pairwise distance at each time instant for constant $\bar{\boldsymbol{w}}$.

V. Conclusions and Future Work

We presented a safe semi-cooperative multi-agent coordination protocol under state and measurement uncertainty. The nominal case of our earlier work is redesigned by feedforward control, vector-field-based feedback control, and nonlinear estimation techniques, so that safety and convergence of the agents up to some bound around the desired destination is guaranteed. In the future, we would like to study the case when the mean state disturbance has a spatial distribution.

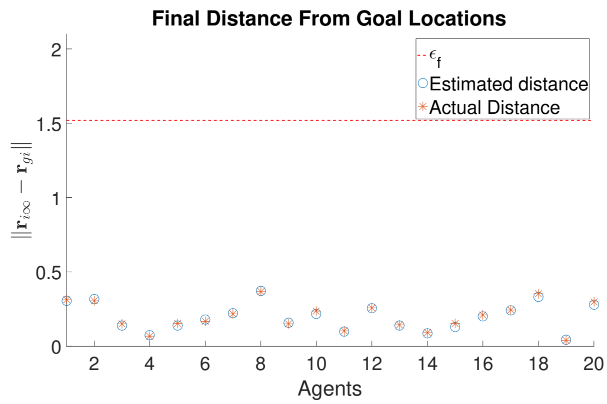

Fig. 4. Final distance from the goal location for constant $\bar{\boldsymbol{w}}$.

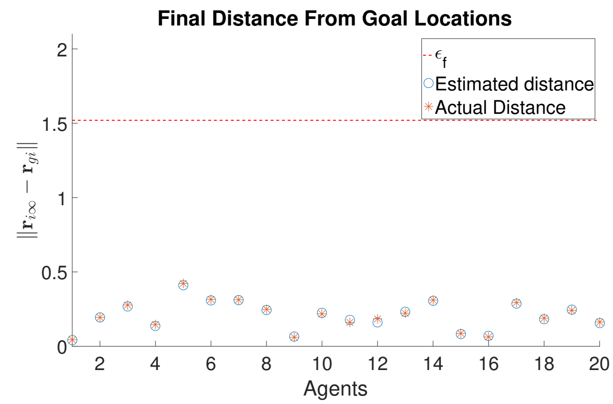

Fig. 6. Final distance from the goal location for time varying $\bar{\boldsymbol{w}}$.

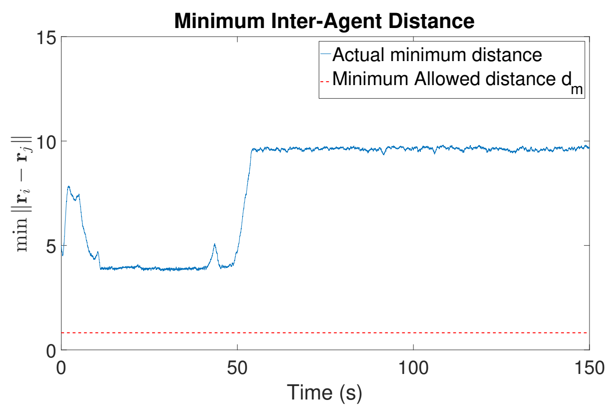

Fig. 5. Smallest pairwise distance for time varying $\bar{\boldsymbol{w}}$.

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