New Results on Finite-Time Stability: Geometric Conditions and Finite-Time Controllers

Kunal Garg Dimitra Panagou

***Abstract*— This paper presents novel controllers that yield finite-time stability for linear systems. We first present a necessary and sufficient condition for the origin of a scalar system to be finite-time stable. Then we present novel finite-time controllers based on vector fields and barrier functions to demonstrate the utility of this geometric condition. We also consider the general class of linear controllable systems, and present a continuous feedback control law to stabilize the system in finite time. Finally, we present simulation results for each of these cases, showing the efficacy of the designed control laws.**

I. Introduction

Finite Time Stability (FTS) has been a well-studied concept, motivated in part from a practical viewpoint due to properties such as achieving convergence in finite time, as well as exhibiting robustness with respect to (w.r.t.) disturbances [1]. Classical optimal control theory provides several examples of systems that exhibit convergence to the equilibrium in finite time [2]. A well-known example is the double integrator with bang-bang time-optimal feedback control [3]; these approaches typically involve solutions that render discontinuous system dynamics. The approach in [4] considers finite-time stabilization using time-varying feedback controllers. The authors in [5] focus on continuous autonomous systems and present Lyapunov-like necessary and sufficient conditions for a system to exhibit FTS, while in [6] they provide geometric conditions for homogeneous systems to exhibit FTS. [7] extended the notion of finite-time stability from autonomous to time-varying dynamical systems, see also [8]. The authors in [9] provided necessary and sufficient geometric conditions for the finite-time stability of a scalar system, and used the structure of phase portraits for second order systems to develop a class of finite-time systems. In [10], the authors presented a method to construct a finite-time consensus protocol. [11] presents necessary and sufficient conditions for FTS of linear, time-varying systems, as well as an output feedback controller that yields finite-time stability. [12] addresses the problem of FTS for small-time controllable systems. FTS has regained much attention in the recent few years as well; [13]–[15] present FTS results for neural-network systems, output feedback tracking and control of multi-agent systems, respectively. In [16], the authors consider the problem of finite-time consensus and provide a method to bound the position and velocity errors to a small residual set in finite-time. In [17], the authors analyze the finite-time consensus problem for strongly connected graphs of heterogeneous systems. Other recent work includes [18], [19], in which finite-time stability is studied in hybrid systems framework.

In [9], the authors presented a sufficient geometric condition in terms of the integral of the multiplicative inverse of the system dynamics, evaluated between any initial point p and the origin. In this paper, we present a sufficient condition in terms of the derivative of the system dynamics evaluated at the origin, which is much easier to check than the former one. We also present a necessary and sufficient condition in terms of bounds on the system dynamics and utilize it to design finite-time controllers for different classes of systems. In addition, we consider a general class of linear controllable systems, whereas the aforementioned work considered a very special class of linear or nonlinear systems.

In [20], the authors considered the problem of finite-time stabilization of double integrator systems. In this paper, we prove that under the effect of our controller, the closed-loop trajectories of any linear controllable system would converge to the equilibrium point in finite-time. As case studies, we consider a nonholonomic system guided by a vector field, a single integrator system guided by a barrier-function based controller, and controllable LTI system stabilized at an arbitrary equilibrium point, and design finite-time controllers for each one of them. For the sake of brevity, we skip the proofs of various Lemmas in this paper, which can be found in [21].

The paper is organized as follows: Section II presents an overview of the theory of FTS. In Section III we present new geometric conditions to establish FTS for scalar systems. Section IV presents a finite-time Barrier function based controller for obstacle avoidance and convergence to the goal location. In Section V, we present novel control laws for a class of linear controllable systems for FTS. Section VI evaluates the performance of the proposed finite-time controllers via simulation results. Our conclusions and thoughts on future work are summarized in Section VII.

II. Overview of Finite Time Stability

Let us consider the system:

$$\dot{y} = f(y(t)), \tag{1}$$

where $y \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $f(0) = 0$. In [5], the authors define finite-time stability as follows: The origin is said to be a finite-time-stable equilibrium of (1) if there exists an

The authors are with the Department of Aerospace Engineering, University of Michigan, Ann Arbor, MI, USA; {kgarg, dpanagou}@umich.edu.

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open neighborhood $\mathcal{N} \subset D$ of the origin and a function $T : \mathcal{N} \setminus \{0\} \to (0, \infty)$, called the settling-time function, such that the following statements hold:

1) *Finite-time convergence*: For every $x \in \mathcal{N} \setminus \{0\}$, ψ^x is defined on $[0, T(x)), \psi^x(t) \in \mathcal{N} \setminus \{0\}$ for all $t \in [0, T(x))$, and $\lim_{t \to T(x)} \psi^x(t) = 0$. Here, $\psi^x : [0, T(x)) \to D$ is the unique right maximal solution of system (1).
2) *Lyapunov stability*: For every open neighborhood U_ϵ of 0, there exists an open subset U_δ of $\mathcal{N}$ containing 0 such that, for every $x \in U_\delta \setminus \{0\}$, $\psi^x(t) \in U_\epsilon$, for all $t \in [0, T(x))$.

The origin is said to be a globally finite-time-stable equilibrium if it is a finite-time-stable equilibrium with $D = \mathcal{N} = \mathbb{R}^n$. The authors in [5] also presented Lyapunov like conditions for finite-time stability of system (1):

Theorem 1: [5] Suppose there exists a continuous function V: $\mathcal{D} \to \mathbb{R}$ such that the following hold:
(i) V is positive definite
(ii) There exist real numbers $c > 0$ and $\alpha \in (0, 1)$ and an open neighborhood $\mathcal{V} \subseteq \mathcal{D}$ of the origin such that

$$\dot{V}(y) + c(V(y))^\alpha \leq 0, \; y \in \mathcal{V} \setminus \{0\}. \tag{2}$$

Then origin is finite-time stable equilibrium of (1).

III. New condition for Finite-time stability

A. Geometric Conditions for FTS

The authors in [9] stated a geometric condition on the system dynamics for the equilibrium to be finite-time stable:

1) $yf(y) < 0$ for $y \in \mathcal{N} \setminus \{0\}$, and $yf(y) = 0$ when $y = 0$,
2) $\int_p^0 \frac{dy}{f(y)} < \infty$ for all $p \in \mathbb{R}$.

However, these conditions are not useful in practice as, in general, it is difficult to evaluate the integral $\int_p^0 \frac{dy}{f(y)}$ for an arbitrary vector field $f(y)$. In this paper we present conditions which are easier to check. Note that these results follow immediately from [9].

First, we present a necessary and sufficient condition for the origin of a scalar system to be finite-time stable:

Theorem 2: Consider the system:

$$\dot{x} = h(x), \quad x \in D \subset \mathbb{R}, \tag{3}$$

such that $h(0) = 0$, and $xh(x) < 0$, $\forall x \neq 0$, i.e., the origin is a stable equilibrium. Then the origin is finite-time stable equilibrium for system (3) if and only if: $\exists\, D \subset \mathbb{R}$ containing the origin, $k > 0$ and $0 < \alpha < 1$, such that $\forall x \in D$,

$$\operatorname{sign}(x)h(x) \leq -k|x|^\alpha. \tag{4}$$

Proof: First we prove the sufficiency: Choose the candidate Lyapunov function $V(x) = \frac{1}{2}x^2$. Taking its time derivative along the trajectories of (3), we obtain:

$$\dot{V}(x) = xh(x) = |x| \operatorname{sign}(x)h(x).$$

Since $\operatorname{sign}(x)h(x) \leq -k|x|^\alpha$, $k > 0$, $0 < \alpha < 1$, we have: $\dot{V} \leq |x|(-k|x|^\alpha)$. Choosing $\beta = \frac{1+\alpha}{2}$ and $c = k2^\beta$, we get $\dot{V} \leq -cV(x)^\beta$ where $0 < \beta < 1$ and $c > 0$. Hence, from Theorem 1, we get that the origin is finite-time stable.

Now we prove the necessity: If origin of the system (3) is finite-time stable, then there exist a positive definite function $V(x)$ satisfying conditions of Theorem 4.3 in [5]. Since $V(x)$ is a positive definite function, we can bound it from below and above as:

$$k_1 x^2 \leq V(x) \leq k_2 x^2,$$

where $k_1, k_2 > 0$. In some neighborhood $D_0 \in \mathbb{R}$, we can bound the derivative as:

$$k_3|x| \leq \operatorname{sign}(x)\frac{\partial V(x)}{\partial x} \leq k_4|x|,$$

for some $k_3, k_4 > 0$. From Theorem 4.3 in [5], we have that $\dot{V}(x) + c(V(x))^\beta \leq 0$ as the origin is finite-time stable. Thus:

$$\begin{aligned} \dot{V}(x) = \frac{\partial V(x)}{\partial x}h(x) \leq -c(V(x))^\beta \Rightarrow \\ -\Big|\frac{\partial V(x)}{\partial x}\Big||h(x)| \leq -c(V(x))^\beta \Rightarrow \\ \Big|\frac{\partial V(x)}{\partial x}\Big||h(x)| \geq c(V(x))^\beta \Rightarrow \\ k_4\,|x|\,|h(x)| \geq c(V(x))^\beta \end{aligned}$$

Now, for any $\gamma > \beta$, there exists a domain $D_\gamma \subset \mathbb{R}$ containing the origin, such that $V(x)^\beta \geq V(x)^\gamma$. Choose the smallest $\gamma = \gamma_1$ such that $\gamma_1 \in (\frac{1}{2}, 1)$. With this choice of γ, we get:

$$k_4\,|x|\,|h(x)| \geq c(V(x))^\beta \geq cV(x)^{\gamma_1} \geq c(k_1x^2)^{\gamma_1}.$$

Now choose $k = \frac{ck_1^{\gamma_1}}{k_4} > 0$, and $0 < \alpha = 2\gamma_1 - 1 < 1$. It follows that: $|h(x)| \geq k|x|^\alpha$. Now, multiply both sides by $\operatorname{sign}(x)\operatorname{sign}(h(x)) = -1$, to get:

$$\operatorname{sign}(x)h(x) \leq -k|x|^\alpha.$$

Choosing $D = D_0 \cap D_{\gamma_1}$ completes the proof. ■

Next we provide a necessary geometric condition in terms of the derivative of the vector field $h(x)$:

Theorem 3: If the origin of the system (3) is finite-time stable, then:

$$\left.\frac{\partial h(x)}{\partial x}\right|_{x=0} = -\infty.$$

Proof: This can be verified by taking the left and right derivative of the function at 0. First, take the right derivative:

$$\lim_{l \to 0^+} \frac{h(l) - h(0)}{l} = \lim_{l \to 0^+} \frac{h(l)}{l} \overset{(2)}{\leq} \lim_{l \to 0^+} \frac{-kl^\alpha}{l} = -\infty.$$

Now, for the left-derivative:

$$\lim_{l \to 0^-} \frac{h(0) - h(l)}{l} = \lim_{l \to 0^-} \frac{-h(l)}{l}.$$

We note that for $l < 0$, from (4), $-h(l) \leq -k|l|^\alpha$ or $h(l) \geq k|l|^\alpha$. Since $l < 0$, we get

$$\frac{h(l)}{l} \leq \frac{k|l|^\alpha}{l} \implies \lim_{l \to 0^-} \frac{h(l)}{l} \leq \lim_{l \to 0^-} \frac{k|l|^\alpha}{l} = -\infty.$$

This completes the proof. ■

Note that this is not a sufficient condition: take $x(t) = x_0 e^{-t^2}$ as a counter example (see [21]). These conditions

can be used to verify finite-time stability of scalar systems. Now we present some examples to demonstrate how these conditions can be utilized to design finite-time controllers.

B. Example: Trajectory Tracking

Consider a vehicle modeled under unicycle kinematics:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}u\cos\theta\\ u\sin\theta\\ \omega\end{bmatrix}, \tag{5}$$

where $\boldsymbol{q} = \begin{bmatrix}\boldsymbol{r}^T & \theta\end{bmatrix}^T \in X \subset \mathbb{R}^3$ is the state vector of the vehicle, comprising the position vector $\boldsymbol{r} = \begin{bmatrix}x & y\end{bmatrix}^T$ and the orientation θ w.r.t. the global frame $\mathcal{G}$, $\boldsymbol{u} = \begin{bmatrix}u & \omega\end{bmatrix}^T \in \mathcal{U} \subset \mathbb{R}^2$ is the control input vector comprising the linear velocity u and the angular velocity ω of the vehicle. The control objective is to track a C^1 reference trajectory $\boldsymbol{r}_g(t)$.

First, we design a reference vector field as:

$$\mathbf{F}_p = -k\boldsymbol{r}_e(t)\|\boldsymbol{r}_e(t)\|^{\alpha-1} + \dot{\boldsymbol{r}}_g(t), \tag{6}$$

where $k > 0$, $0 < \alpha < 1$ and $\boldsymbol{r}_e(t) = \boldsymbol{r}(t) - \boldsymbol{r}_g(t)$. The proposed control law is given by:

$$u = \|\mathbf{F}_p\|, \tag{7}$$

$$\omega = -k_\omega \operatorname{sign}(\theta - \varphi_p)\,|\theta - \varphi_p|^\alpha + \dot{\varphi}_p, \tag{8}$$

where $\varphi_p \triangleq \arctan\left(\frac{\mathrm{F}_{py}}{\mathrm{F}_{px}}\right)$ is the orientation of the vector field $\mathbf{F}_p$.

Before stating the main theorem, we present an intermediate result that is used also later in the paper:

Lemma 1: The origin of

$$\dot{\boldsymbol{x}} = -k\boldsymbol{x}\|\boldsymbol{x}\|^{\alpha-1}, \quad k > 0, \quad 0 < \alpha < 1, \tag{9}$$

is a finite-time stable equilibrium.

Proof: See [21]. ■

Theorem 4: Under the control law (7)-(8), the system (5) trajectories track the trajectory $\boldsymbol{r}_g(t)$ in finite time.

Proof: Consider the position error: $\boldsymbol{r}_e(t) = \boldsymbol{r}(t) - \boldsymbol{r}_g(t)$. The time derivative along the system trajectories reads: $\dot{\boldsymbol{r}}_e(t) = \dot{\boldsymbol{r}}(t) - \dot{\boldsymbol{r}}_g(t) = \begin{bmatrix}u\cos\theta\\ u\sin\theta\end{bmatrix} - \dot{\boldsymbol{r}}_g(t)$. From Lemma 1, $\boldsymbol{r}_e(t)$ goes to origin in finite-time if $\dot{\boldsymbol{r}}_e(t) = -k\boldsymbol{r}_e\|\boldsymbol{r}_e\|^{\alpha-1}$, with $k > 0$ and $0 < \alpha < 1$. Therefore, we set:

$$\begin{bmatrix}u_d\cos\theta_d\\ u_d\sin\theta_d\end{bmatrix} = -k\boldsymbol{r}_e\|\boldsymbol{r}_e\|^{\alpha-1} + \dot{\boldsymbol{r}}_g(t), \tag{10}$$

where u_d and θ_d denote the desired linear speed and orientation, respectively. Let $\angle(\cdot)$ denote signed angle. From (6) and (10), we have that $\varphi_p = \angle\mathbf{F}_p = \theta_d$, and $u = u_d = \|\mathbf{F}_p\|$. Hence, if the system tracks the vector field $\mathbf{F}_p$ in finite-time, it will track the desired trajectory $\boldsymbol{r}_g(t)$ in finite-time. Define $\theta_e = \theta - \theta_d$, and choose the candidate Lyapunov function $V(\theta_e) = \frac{1}{2}\theta_e^2$. Taking its time derivative along (8), we get:

$$\begin{aligned}\dot{V}(\theta_e) &= \theta_e\dot{\theta}_e = \theta_e(\dot{\theta} - \dot{\theta}_d) = \theta_e(\omega - \dot{\theta}_d)\\ &\overset{(8)}{=} \theta_e(-k_\omega \operatorname{sign}(\theta - \varphi_p)\,|\theta - \varphi_p|^\alpha + \dot{\varphi}_p - \dot{\theta}_d).\end{aligned}$$

Since $\theta_d = \varphi_p$, we get:

$$\begin{aligned}\dot{V}(\theta_e) &= \theta_e(-k_\omega \operatorname{sign}(\theta - \varphi_p)\,|\theta - \varphi_p|^\alpha)\\ &= -k_\omega\theta_e \operatorname{sign}(\theta_e)|\theta_e|^\alpha = -k_\omega|\theta_e|^{1+\alpha}\\ &= -k_\omega(2V(\theta_e))^{\frac{1+\alpha}{2}} \le -cV(\theta_e)^\beta,\end{aligned}$$

where $c = k_\omega 2^{\frac{1+\alpha}{2}}$ and $\beta = \frac{1+\alpha}{2} < 1$. Hence, $\theta_e(t)$ converges to zero in finite time. This along with the fact that the magnitude of linear speed given out of (7) is equal to the desired linear speed u_d implies that $\dot{\boldsymbol{r}}(t) = -k\boldsymbol{r}_e(t)\|\boldsymbol{r}_e(t)\|^{\alpha-1}$. Hence, $\boldsymbol{r}_e(t) \to \mathbf{0}$ in finite time and which implies that the system trajectory $\boldsymbol{r}(t)$ converges to $\boldsymbol{r}_g(t)$ in finite-time. ■

It is noteworthy that for the dynamics of the orientation error θ_e, from (8), one has:

$$\begin{aligned}\dot{\theta}_e = -k_\omega \operatorname{sign}(\theta_e)|\theta_e|^\alpha = h(\theta_e) \Rightarrow\\ \operatorname{sign}(\theta_e)h(\theta_e) = -k_\omega|\theta_e|^\alpha\end{aligned}$$

Also, $\frac{\partial h(\theta_e)}{\partial\theta_e} = -k_\omega|\theta_e|^{\alpha-1}$ which implies $\left.\frac{\partial h(\theta_e)}{\partial\theta_e}\right|_{\theta_e=0} = -\infty$ since $\alpha < 1$. Hence, both the conditions presented in Theorem 2 and 3 are getting satisfied.

IV. Finite Time Barrier Function-Based Control

Consider a vehicle modeled as a single integrator as:

$$\dot{\boldsymbol{x}} = \boldsymbol{u}, \tag{11}$$

where $\boldsymbol{x}, \boldsymbol{u} \in \mathbb{R}^n$. The problem of reaching to a specified goal location in finite time can be formulated as follows:

$$\exists t^* < \infty, \text{ such that } \forall t \ge t^*, \ \|\boldsymbol{x}(t) - \boldsymbol{\tau}\| = 0,$$

where $\boldsymbol{\tau} \in \mathbb{R}^n$ is the desired goal location, while the problem of obstacle avoidance can be written as:

$$\|\boldsymbol{x}(t) - \boldsymbol{o}\| \ge d_c, \ \forall t \ge t_0,$$

where $\boldsymbol{o} \in \mathbb{R}^n$ represents the location of the obstacle, and t_0 is the starting time. We model the obstacle as a $n-$sphere of radius ρ_o. Choosing $d_c = d_m + \rho_o$ ensures that vehicle maintains the required minimum distance d_m from the obstacle. We assume that $\|\boldsymbol{o} - \boldsymbol{\tau}\| > 2d_c$, so that the desired location is sufficiently far away from the obstacle. We also assume that the vehicle starts sufficiently far away from the obstacle so that $\|\boldsymbol{x}(t_0) - \boldsymbol{o}\| > d_c$. We seek a continuous feedback control law $\boldsymbol{u} = \gamma(\boldsymbol{x})$ such that the vehicle's trajectories out of (11) reach the goal location while maintaining safe distance from the obstacle. More specifically, we seek a Barrier function-based controller for this problem. First we define the Barrier function as follows:

$$B(\boldsymbol{x}) = \frac{\|\boldsymbol{x} - \boldsymbol{\tau}\|^2}{\|\boldsymbol{x} - \boldsymbol{o}\| - d_c + \frac{1}{\epsilon}}, \tag{12}$$

where $\epsilon \gg 1$ is a very large number. We choose the controller of the form:

$$\boldsymbol{u} = -k_1\nabla B\|\nabla B(\boldsymbol{x})\|^{\alpha-1}, \tag{13}$$

where k_1 and $0 < \alpha < 1$. With this controller, we have the following result:

Theorem 5: Under the control law (13), the point $\boldsymbol{x} = \boldsymbol{\tau}$ is FTS equilibrium for the closed-loop system (11), and the closed-loop system trajectories will remain safe w.r.t. the obstacle.

First we present some useful Lemmas:

Lemma 2: In the domain $\bar{D}_o = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{o}\| > d_c\}$, the Barrier function $B(\boldsymbol{x})$ is bounded as $B(\boldsymbol{x}) \leq \epsilon \|\boldsymbol{x} - \boldsymbol{\tau}\|^2$.

Proof: See [21]. ■

Lemma 3: The gradient $\nabla B(\boldsymbol{x})$ of the barrier function $B(\boldsymbol{x})$ is non-zero everywhere, except at $\boldsymbol{x} = \boldsymbol{\tau}$, and at

$$\boldsymbol{x} = \boldsymbol{\tau} + 2\frac{\|\boldsymbol{o} - \boldsymbol{\tau}\| + d_c - \frac{1}{\epsilon}}{\|\boldsymbol{o} - \boldsymbol{\tau}\|}(\boldsymbol{o} - \boldsymbol{\tau}). \tag{14}$$

Proof: See [21]. ■

Lemma 4: In any domain $D \subset \mathbb{R}^n$ containing point $\boldsymbol{\tau}$ and excluding the region $\bar{D} = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{p}\| < r \quad \boldsymbol{p} = \boldsymbol{\tau} + \theta(\boldsymbol{o} - \boldsymbol{\tau}) \ , \ \theta \geq 1\}$, where r is an arbitrary small positive number, $\exists c > 0$ such that $\forall \boldsymbol{x} \in D$

$$\|\nabla B(\boldsymbol{x})\| \geq c\|\boldsymbol{x} - \boldsymbol{\tau}\|. \tag{15}$$

Proof: See [21]. ■

Now we are ready to prove Theorem 5:

Proof: From Lemma 3, we have that $\nabla B(\boldsymbol{x}) = \boldsymbol{0}$ at the equilibrium point $\boldsymbol{x} = \boldsymbol{\tau}$, and at the point $\boldsymbol{x} = \boldsymbol{\tau} + \mu(\boldsymbol{o} - \boldsymbol{\tau})$ where μ takes values as per Lemma 3. Consider the domain around the goal location D_o as per Lemma 2. Define $\mathcal{D} = D_o \setminus \bar{D}$ where $\bar{D}$ is defined as per Lemma 4. Since $\bar{D}$ is a closed domain and D_o is open, domain $\mathcal{D}$ is an open domain around the equilibrium $\boldsymbol{\tau}$. Choose the candidate Lyapunov function

$$V(\boldsymbol{x}) = B(\boldsymbol{x}).$$

For simplifying the notation, define $\boldsymbol{x}_e = \boldsymbol{x} - \boldsymbol{\tau}$ and drop the argument $\boldsymbol{x}$ for the functions B and ∇B. Taking the time derivative of $V(\boldsymbol{x})$ along the trajectories of (11), we have:

$$\begin{aligned}\dot{V}(\boldsymbol{x}) =& (\nabla B)^T(-k_1 \nabla B \|\nabla B\|^{\alpha-1}) \\ =& -k_1\|\nabla B\|^{1+\alpha}\end{aligned}$$

From Lemma 4, we have that $\|\nabla B\| \geq c\|\boldsymbol{x}_e\|$, hence

$$\dot{V} \leq -k_1 c^{1+\alpha}\|\boldsymbol{x}\|^{1+\alpha}$$

Using the result from Lemma 2, we have that:

$$\begin{aligned}V(\boldsymbol{x}) = B(\boldsymbol{x}) \leq \epsilon\|\boldsymbol{x}_e\|^2 \Rightarrow -\|\boldsymbol{x}_e\|^2 \leq -\frac{1}{\epsilon}V(\boldsymbol{x}) \\ \implies \dot{V}(\boldsymbol{x}) \leq -k_1 c^{1+\alpha}|\boldsymbol{x}_e\|^{1+\alpha} \leq -KV(\boldsymbol{x})^\beta\end{aligned}$$

for any $\boldsymbol{x} \in \mathcal{D}$, where $K = (\frac{1}{\epsilon})^\beta k_1 c^{1+\alpha}$ and $\beta = \frac{1+\alpha}{2} < 1$. Thus, we have that the equilibrium $\boldsymbol{x} = \boldsymbol{\tau}$ is finite-time stable. Safety is trivial from the construction of Barrier function: since $\dot{V}(\boldsymbol{x})$ is bounded, $B(\boldsymbol{x})$ would remain bounded and hence, the denominator of $B(\boldsymbol{x})$ would have non-zero positive value. Choose ϵ greater than 1 over this minimum non-zero value would ensure $\|\boldsymbol{x} - \boldsymbol{o}\| - d_c > 0$. ■

V. Finite Time Stability of LTI Systems

Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \tag{16}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in U \subset \mathbb{R}^n$, $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$. The objective is to stabilize the origin of (16) in finite time. Mathematically, we seek a continuous feedback law so that conditions of Theorem 1 are satisfied.

A. Multi-Input Case

Theorem 6: Consider the system (16) and assume that $\boldsymbol{B}$ is of full rank. Then, the feedback control law:

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{K}_1\boldsymbol{x} + \boldsymbol{K}_2\boldsymbol{x}_\alpha, \tag{17}$$

where $\boldsymbol{x}_\alpha = \boldsymbol{x}\|\boldsymbol{x}\|^{\alpha-1}$, $0 < \alpha < 1$, $\boldsymbol{K}_2 = -\boldsymbol{B}^{-1}$, and $\boldsymbol{K}_1$ is such that $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1$ is Hurwitz, yields the origin of the closed-loop system (16) a finite-time stable equilibrium.

Proof: Since $\boldsymbol{B}$ is of full rank, $(\boldsymbol{A}, \boldsymbol{B})$ is controllable. Therefore, there exists a gain $\boldsymbol{K}_1$ such that: $Re(\text{eig}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1)) < 0$, i.e., there exists a positive definite matrix $\boldsymbol{P}$ such that $(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1)^T\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{B}\boldsymbol{K}_1) = -\boldsymbol{Q}$, where $\boldsymbol{Q}$ is a positive definite matrix. Choose the candidate Lyapunov function $V(x) = \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}$. Taking the time derivative of $V(x)$ along the closed-loop trajectories of system (16), we get:

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &= \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}) + (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u})^T\boldsymbol{P}\boldsymbol{x} \\ &= \boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}_1\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}_2\boldsymbol{x}_\alpha) \\ &+ (\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}_1\boldsymbol{x} + \boldsymbol{B}\boldsymbol{K}_2\boldsymbol{x}_\alpha)^T\boldsymbol{P}\boldsymbol{x} \\ &= \boldsymbol{x}^T(\boldsymbol{P}\tilde{\boldsymbol{A}} + \tilde{\boldsymbol{A}}^T\boldsymbol{P})\boldsymbol{x} + \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}_2\boldsymbol{x}_\alpha \\ &+ \boldsymbol{x}_\alpha^T\boldsymbol{K}_2^T\boldsymbol{B}^T\boldsymbol{P}\boldsymbol{x},\end{aligned}$$

where $\tilde{\boldsymbol{A}} = \boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}_1$. Choose $\boldsymbol{K}_2 = -\boldsymbol{B}^{-1}$. It follows that $(\boldsymbol{P}\boldsymbol{B}\boldsymbol{K}_2 + \boldsymbol{K}_2^T\boldsymbol{B}^T\boldsymbol{P}) = -2\boldsymbol{P}$. Therefore:

$$\dot{V}(\boldsymbol{x}) = -\boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} - 2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}_\alpha \leq -2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}_\alpha.$$

Now, for any symmetric matrix $\boldsymbol{P}$, $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{y}$ can be bounded as: $\lambda_{min}(\boldsymbol{P})\boldsymbol{x}^T\boldsymbol{y} \leq \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{y} \leq \lambda_{max}(\boldsymbol{P})\boldsymbol{x}^T\boldsymbol{y}$, if $\boldsymbol{x}^T\boldsymbol{y} > 0$. We have that $\boldsymbol{x}^T\boldsymbol{x}_\alpha > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$. Since $\boldsymbol{P}$ is a positive definite matrix, we have that $\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}_\alpha \geq \lambda_{min}(\boldsymbol{P})\boldsymbol{x}^T\boldsymbol{x}_\alpha$. Also, we can bound $V(\boldsymbol{x})$ by:

$$\begin{aligned}\lambda_{min}(\boldsymbol{P})\|\boldsymbol{x}\|^2 \leq \boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} \leq \lambda_{max}(\boldsymbol{P})\|\boldsymbol{x}\|^2 \Rightarrow \\ V(\boldsymbol{x}) \leq \lambda_{max}(\boldsymbol{P})\|\boldsymbol{x}\|^2 \Rightarrow V(\boldsymbol{x})^\beta \leq k\|\boldsymbol{x}\|^{2\beta} = k\|\boldsymbol{x}\|^{1+\alpha},\end{aligned}$$

where $k = (\lambda_{max}(\boldsymbol{P}))^\beta > 0$, and $\beta = \frac{1+\alpha}{2} < 1$. Hence:

$$\begin{aligned}\dot{V}(\boldsymbol{x}) &\leq -2\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x}_\alpha \leq -2\lambda_{min}(\boldsymbol{P})\boldsymbol{x}^T\boldsymbol{x}_\alpha \\ &= -2\lambda_{min}(\boldsymbol{P})\boldsymbol{x}^T\boldsymbol{x}\|\boldsymbol{x}\|^{\alpha-1} = -2\lambda_{min}(\boldsymbol{P})\|\boldsymbol{x}\|^{1+\alpha} \\ &\leq -2\lambda_{min}(\boldsymbol{P})\frac{V(\boldsymbol{x})^\beta}{k} \leq -cV(x)^\beta,\end{aligned}$$

where $c = \frac{2\lambda_{min}(\boldsymbol{P})}{k} > 0$, and $0 < \beta < 1$, since $0 < \alpha < 1$. Therefore, from Theorem 1, we have that the origin of the closed-loop system is finite-time stable. ■

Note that the above is a restrictive case, since we assumed that the matrix $\boldsymbol{B}$ is of full rank. Before we present the most general case, we state the following result:

Lemma 5: Consider the scalar system

$$\dot{x} = ax + bu, \quad b \neq 0. \tag{18}$$

$x(t)$ converges to $x^d(t)$ in finite time under the control law:

$$u = \frac{1}{b}(-ax - k\,\mathrm{sign}(x - x^d)|x - x^d|^\alpha + \dot{x}^d(t)), \tag{19}$$

where $k > 0$ and $0 < \alpha < 1$.

Proof: See [21]. ■

Now we present the general case of linear controllable systems, and show that a linear controllable system can reach its equilibrium in finite time.

B. Linear Controllable System

We consider the controllable canonical form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u, \tag{20}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $u \in \mathbb{R}$ and the system matrices are of the form:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_n \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\boldsymbol{x}^d = \begin{bmatrix} x_1^d & 0 & \cdots & 0 \end{bmatrix}^T$ be the desired state. We propose a continuous state-feedback control law u, so that $\exists T < \infty$, such that $\forall t \geq T$, $\boldsymbol{x}(t) = \boldsymbol{x}^d$.

Theorem 7: The system (20) reaches the desired state $\boldsymbol{x}^d$ in finite time under the control law:

$$u = \dot{x}_n^d - \sum_{i=1}^{n} a_i x_i - k_n\,\mathrm{sign}(x_n - x_n^d)|x_n - x_n^d|^\alpha, \tag{21}$$

where $k_i > 0$, x_i^d is given out of (23), and $\frac{n-1}{n} < \alpha < 1$. Furthermore, the controller u remains bounded.

Proof: The system (20) can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \\ \sum_{i=1}^n a_i x_i + u \end{bmatrix}. \tag{22}$$

For $x_1 \to x_1^d$ in finite-time, the desired time-rate of x_1, i.e., the desired x_2 (denoted as x_2^d) should be:

$$x_2^d = -k_1\,\mathrm{sign}(x_1 - x_1^d)|x_1 - x_1^d|^\alpha + \dot{x}_1^d,$$

where $k_1 > 0$ and $0 < \alpha < 1$ (see Lemma 5). As we assume x_1^d to be constant, we have $\dot{x}_1^d = 0$. Similarly, in general form, one can write:

$$x_{i+1}^d = -k_i\,\mathrm{sign}(x_i - x_i^d)|x_i - x_i^d|^\alpha + \dot{x}_i^d, \tag{23}$$

where $1 \leq i \leq n-1$, and $k_i > 0$.

Choose the candidate Lyapunov function $V(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{1}{2}(x_i - x_i^d)^2$. Taking the time derivative along the trajectories of the closed-loop system under (21), we have:

$$\begin{aligned} \dot{V}(\boldsymbol{x}) &= \sum_{i=1}^{n-1}(x_i - x_i^d)(x_{i+1} - \dot{x}_i^d) + (x_n - x_n^d)(\dot{x}_n - \dot{x}_n^d) \\ &= \sum_{i=1}^{n-1}(x_i - x_i^d)(-k_i\,\mathrm{sign}(x_i - x_i^d)|x_i - x_i^d|^\alpha) \\ &+ (x_n - x_n^d)(-k_n\,\mathrm{sign}(x_n - x_n^d)|x_n - x_n^d|^\alpha) \\ &= \sum_{i=1}^{n} -k_i|x_i - x_i^d|^{\alpha+1}. \end{aligned}$$

Define $\bar{k} = \min_i k_i$, so that we can write:

$$\dot{V}(\boldsymbol{x}) \leq -\bar{k}\sum_{i=1}^{n}|x_i - x_i^d|^{\alpha+1} = -\bar{k}\|\boldsymbol{x}_e\|_{1+\alpha}^{1+\alpha},$$

where $\boldsymbol{x}_e \triangleq \begin{bmatrix} x_1 - x_1^d & x_2 - x_2^d & \dots & x_n - x_n^d \end{bmatrix}^T$, and $\|\boldsymbol{x}_e\|_{1+\alpha}^{1+\alpha}$ is the $(1+\alpha)-$norm of vector $\boldsymbol{x}_e$, raised to power of $(1+\alpha)$. Using the norm inequality for equivalent norms, we have that $\|\boldsymbol{x}_e\|_2 \leq \|\boldsymbol{x}_e\|_{1+\alpha}$ since $1+\alpha < 2$, hence:

$$\begin{aligned} \dot{V}(\boldsymbol{x}) \leq -\bar{k}\|\boldsymbol{x}_e\|_2^{1+\alpha} = -\bar{k}(\|\boldsymbol{x}_e\|_2^2)^{\frac{1+\alpha}{2}} = -\bar{k}(2V(\boldsymbol{x}))^{\frac{1+\alpha}{2}} \\ \Rightarrow \dot{V}(\boldsymbol{x}) \leq -cV(\boldsymbol{x})^\beta, \end{aligned}$$

where $\beta = \frac{1+\alpha}{2} < 1$, and $c = \bar{k}2^\beta$. Hence, from Theorem 1, we have that the $\boldsymbol{x} \to \boldsymbol{x}^d$ in finite time. Note that the above inequality holds for any $\boldsymbol{x} \in \mathbb{R}^n$. This, along with the fact that $V(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}_e\|^2$ is radially unbounded, implies that $\boldsymbol{x}^d$ is a global-finite time stable equilibrium of the closed-loop system. Furthermore, with $\alpha > \frac{n-1}{n}$, it can be verified that the controller (21) remain bounded: from (23), $\dot{x}_n^d = -k_{n-1}|x_i - x_d|^{2\alpha-1} + \ddot{x}_{n-1}^d$. Define $(v)^{(k)}$ as the $k-th$ time derivative of v, so that we get

$$(x_{i+1}^d)^{n-i} = -k_i|x_i - x_i^d|^{(n-i+1)\alpha-(n-i)} + (x_i^d)^{n-i+1}.$$

Each $(x_{i+1}^d)^{n-i}$ is bounded if $\alpha > \frac{n-i}{n-i+1}$ and $(x_i^d)^{n-i+2}$ is bounded. $(x_2^d)^{n-1} = -k_1|x_1 - x_1^d|^{n\alpha-(n-1)}$ is bounded if $\alpha > \frac{n-1}{n}$. Hence, with this choice of α, all the derivatives $(x_i^d)^{(n-i+2)}$ and the controller remain bounded. ■

VI. Simulations

A. Simulation results for Section III-B

We consider a sinusoidal trajectory as the desired trajectory, i.e. $\boldsymbol{r}_g(t) = \begin{bmatrix} t & \cos(t) \end{bmatrix}^T$. Figure 1 shows the errors or deviations of coordinates $x(t)$ and $y(t)$ from the desired coordinates $x_g(t)$ and $y_g(t)$ while Figure 2 shows the actual and desired trajectory for the closed-loop system.

B. Simulation results for Section IV

We consider the desired goal location for the system (11) as $\boldsymbol{\tau} = \begin{bmatrix} 10 & 20 \end{bmatrix}^T$ and the obstacle at $\boldsymbol{o} = \begin{bmatrix} 2 & 8 \end{bmatrix}^T$ of radius 1. We use the safe distance $d_c = 2$. Figure 3 shows the path of the vehicle.

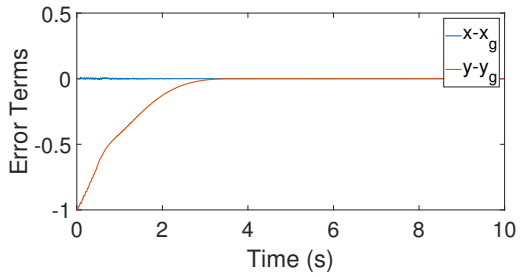

Fig. 1. Deviation from Desired Trajectory with time.

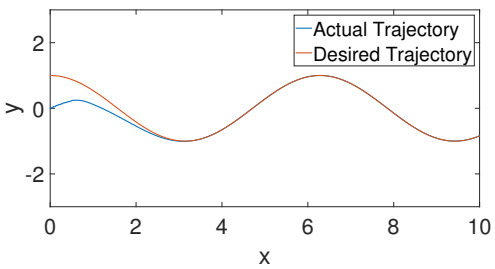

Fig. 2. Actual and Desired Trajectories

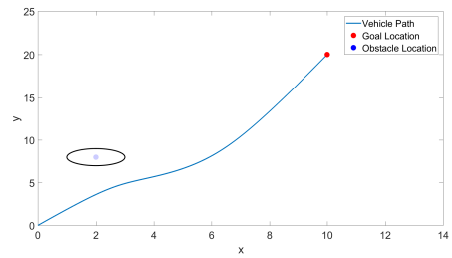

Fig. 3. Vehicle Path in presence of Obstacle.

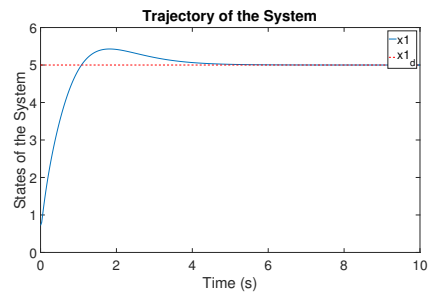

Fig. 4. Time Evolution of the system states with time.

C. Simulation results for Section V-B

We consider 4 states for system (20). Desired location is chosen as $\boldsymbol{x}_d = [5, 0, 0, 0]^T$. Figure 4 shows state $x_1(t)$.

VII. Conclusions and Future Work

We presented new geometric conditions for scalar systems in terms the system dynamics evaluated to establish finite-time stability. We demonstrated the utility of the condition through 2 examples: a vector-field based controller for finite-time convergence and finite-time Barrier function based control law for obstacle avoidance. Finally, we presented a novel continuous finite-time feedback controller for a general class of linear controllable systems. Our current research focuses on Hybrid and Switched systems and we would like to devise conditions equivalent to Branicky's condition for Switching systems to be finite-time stable under arbitrary switching. We would also like to expand our collection of finite-time controllers for a general class of non-linear systems.

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