Adversarial Resilience for Sampled-Data Systems using Control Barrier Function Methods

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Abstract—Control barrier functions (CBFs) have recently become a powerful method for rendering a desired safe set forward invariant in single- and multi-agent systems. In the multi-agent case, prior literature has considered scenarios where all agents cooperate to ensure that the corresponding set remains invariant. However, these works do not consider scenarios where a subset of the agents are behaving adversarially with the intent to violate safety bounds. In addition, prior results on multi-agent CBFs assume that control inputs are continuous and do not explicitly consider sampled-data dynamics. This paper presents a method for normally behaving agents in a multi-agent system with heterogeneous control-affine sampled-data dynamics to render a safe set forward invariant in the presence of adversarial agents. The proposed approach considers several aspects of practical control systems including input constraints, clock asynchrony and disturbances, and distributed calculation of control inputs. The efficacy of these results are demonstrated through simulations.

I. INTRODUCTION

Guaranteeing the safety of autonomous systems is a critical challenge in modern control theory. Safety is frequently modeled by defining a safe subset of the state space for a given system, and generating control inputs that render this subset forward invariant. Control barrier function (CBF) methods [1]–[5] that leverage quadratic programming (QP) techniques have risen as a powerful framework for establishing forward invariance of a safe set. Both single-agent [6]–[9] and multi-agent systems [4], [10]–[13] have been considered, where agents have control-affine dynamics.

Prior work on multi-agent CBF methods typically assumes that all agents apply the nominally specified control law. This assumption does not encompass faulty or adversarial behavior of agents within the system. Adversarial agents may apply control laws specifically crafted in an attempt to violate set invariance conditions within given control constraints. Much prior and recent work has considered the accomplishment of control objectives in the presence of faulty or adversarial agents [14]–[20]. However, to the authors’ best knowledge, this is the first work that has explicitly considered the control actions of adversarial agents in a multi-agent CBF setting. Our specific contributions are as follows:

• We present a method under which a set of normally-behaving agents in a system with sampled-data dynamics can collaboratively render a safe set forward invariant despite the actions of adversarial agents. Both synchronous and asynchronous sampling times, as well as distributed calculation of agents’ control inputs.

• We present a linear program that computes normal agents’ best-effort control inputs towards establishing set invariance. Under mild assumptions, this linear program is shown to always be feasible. We present an associated novel condition under which a safe set can be rendered forward invariant by this best-effort controller.

The organization of this paper is as follows: in Section II the notation and problem formulation are given, in Section III the main results are presented, in Section IV simulations are shown demonstrating this paper’s results, and in Section V a brief conclusion is given.

II. NOTATION AND PROBLEM FORMULATION

The nonnegative and strictly positive integers are denoted \( \mathbb{Z}_{\geq 0} \) and \( \mathbb{Z}_{>0} \), respectively. We use the notation \( h \in C_{1}\text{loc} \) to denote a continuously differentiable function \( h \) whose gradient \( \nabla h \) is locally Lipschitz continuous. Let \( x_i \in \mathbb{R}^{n_i} \), \( n_i \in \mathbb{Z}_{\geq 1} \) for \( i = 1, \ldots, N \) be a set of vectors, and let \( \bar{n} = \sum_{i=1}^{N} n_i \). We let \( \bar{x} = [x_1^T, \ldots, x_N^T]^T \) denote the vector concatenating all \( x_i \) vectors. The partial Lie derivative
sets of a function \( f(\bar{x}) \) with respect to (w.r.t.) \( x_i \) is denoted \( L_{f_i} h^{x_i}(\bar{x}) = \frac{\partial h^{x_i}(\bar{x})}{\partial x_i} f(\bar{x}) \). The \( n \)-ary Cartesian product of sets \( S_1, \ldots, S_N \) is denoted \( \bigtimes_{i=1}^{N} S_i = S_1 \times \ldots \times S_N \). The Minkowski sum of sets \( S_1, S_2 \) is denoted \( S_1 \oplus S_2 \). The open and closed norm balls of radius \( \epsilon > 0 \) centered at \( \bar{x} \in \mathbb{R}^n \) are respectively denoted \( B(\bar{x}, \epsilon) \), \( \overline{B}(\bar{x}, \epsilon) \).

The boundary and interior of a set \( S \subset \mathbb{R}^n \) are denoted \( \partial S \) and \( \text{int}(S) \), respectively.

### A. Problem Formulation

Consider a group of \( N \in \mathbb{Z}_{>0} \) agents with the set of agents denoted \( \mathcal{V} \) and each agent indexed \( \{1, \ldots, N\} \). Each agent \( i \in \mathcal{V} \) has the state \( x_i \in \mathbb{R}^{n_i} \), \( u_i \in \mathbb{Z}_{>0} \) and input \( u_i \in \mathbb{R}^{m_i}, m_i \in \mathbb{Z}_{>0} \). The system and input vectors \( \bar{x}, \bar{u} \), respectively, denote the vectors that concatenate all agents’ states and inputs, respectively, as \( \bar{x} = [x_1^T, \ldots, x_N^T]^T, \bar{u} = [u_1^T, \ldots, u_N^T]^T \), \( \bar{u} \in \mathbb{R}^n, \bar{n} = \sum_{i=1}^{N} u_i, \bar{m} = \sum_{i=1}^{N} m_i \).

Agents receive estimates of the state in a sampled-data fashion; i.e. each agent \( i \in \mathcal{V} \) has an estimate of \( \bar{h}(\bar{x}(t)) \) only at times \( T_i = \{t_{i1}, t_{i2}, \ldots\} \) where \( t_{ij} \) represents agent \( i \)'s \( j \)th sampling time, with \( t_{ij} < t_{ij+1} \) \( \forall k \in \mathbb{Z}_{>0} \).

In addition, at each \( t_{ij} \in T_i \) the agent \( i \) applies a zero-order-hold (ZOH) control input \( u(t_{ij}) \) which is constant on the time interval \( t \in [t_{ij}, t_{ij+1}) \).

For brevity, we denote \( x_{ik} = x_i(t_{ij}) \) and \( u_{ik} = u(t_{ij}) \).

The sampled-data dynamics of each agent \( i \in \mathcal{V} \) under its ZOH controller on each interval \( t \in [t_{ij}, t_{ij+1}) \) is as follows:

\[
\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t_{ij}) + \phi_i(t).
\] (1)

The functions \( f_i, g_i \) may differ among agents, but are all locally Lipschitz on their respective domains \( \mathbb{R}^{n_i} \). Note that under these definitions for any \( i \in \mathcal{V} \) there exists a matrix \( C_i \in \mathbb{R}^{n_i \times n_i} \) such that \( x_i = C_i \bar{x} \). We abuse notation by sometimes writing an expression \( f(x_i) \) as \( f(\bar{x}) \).

The functions \( \phi_i(t) \) are locally Lipschitz in \( t \) and model disturbances to the system (1).

**Assumption 1.** For all \( i \in \mathcal{V} \), the disturbances \( \phi_i(t) \) satisfy \( \|\phi_i(t)\| \leq \phi_i^{\text{max}} \in \mathbb{R}_{>0}, \forall t \geq 0 \).

Since each control input \( u_i(t) \) is piecewise constant, existence and uniqueness of solutions to (1) are guaranteed by Carathéodory’s theorem [22, Sec. 2.2].

Each agent \( i \in \mathcal{V} \) has control input constraints that are represented by a nonempty, convex, compact polytope, i.e. \( u_i \in \mathcal{U}_i(x_i) = \{u \in \mathbb{R}^{m_i} : A_i(x_i)u \leq b_i(x_i)\} \), where the functions \( A_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{m_i \times n_i}, b_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}^{m_i} \) are locally Lipschitz on their respective domains.

Representation of control input constraints as polytopes is common in prior literature [5], [6], [23]. Similar to prior work, it is assumed there exists a nominal control law \( \bar{u}_{\text{nom}}(\cdot) \) that the system computes in order to accomplish some objective [1]. Examples of such a \( \bar{u}_{\text{nom}}(\cdot) \) might include a feedback control law to track a time-varying trajectory or to converge to a goal set. The nominal control law is designed without any safety consideration, and therefore it is desired to minimally modify \( \bar{u}_{\text{nom}} \) in order to render a safe set \( S \subset \mathbb{R}^n \) forward invariant under the dynamics (1). The set \( S \) is defined as the sublevel sets of a function \( h : \mathbb{R}^n \rightarrow \mathbb{R}, h \in C_{1,1}^{\text{loc}} \) as follows:

\[
S = \{\bar{x} \in \mathbb{R}^n : h(\bar{x}) \leq 0\},
\]
\[
\partial S = \{\bar{x} \in \mathbb{R}^n : h(\bar{x}) = 0\}, \text{ and } \int(S) = \{\bar{x} \in \mathbb{R}^n : h(\bar{x}) < 0\}.
\] (2)

**Assumption 2.** The set \( S \) is compact.

**Assumption 3.** For all \( i \in \mathcal{V} \) and \( \forall \bar{x} \in S \), the interior of \( U_i(\bar{x}) \) is nonempty and \( U_i(\bar{x}) \) is uniformly compact near \( \bar{x} \).

**Remark 1.** Note that the conditions for Assumption 3 are trivially satisfied when \( A_i, b_i \) are constant and the interior set \( \{u \in \mathbb{R}^{m_i} : A_iu < b_i\} \) is nonempty. For a specific example satisfying Assumption 3 when \( U_i(\cdot) \) is not constant, see equation (28) in Section IV of this paper.

For multi-agent systems that apply continuous controllers \( u_i(t) \) to the dynamics (1), forward invariance can be collaboratively guaranteed by satisfying the sufficient condition \( h(\bar{x}(t)) \leq -\alpha(h(\bar{x}(t))) \) based on Nagumo’s theorem [24], where \( \alpha(\cdot) \) is an extended class-\( K_{\infty} \) function and locally Lipschitz on \( \mathbb{R} \).

The dependence of \( \bar{x}(t) \) on \( t \) will be omitted for brevity. For the multi-agent system (1), expanding the term \( h(\bar{x}) \) yields

\[
\sum_{i \in \mathcal{V}} (L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x})u_i + L_{\phi_i} h^{x_i}(\bar{x})) \leq -\alpha(h(\bar{x})),
\] (3)

where the partial Lie derivative notation \( L_{f_i} h^{x_i}(\bar{x}) \) is defined at the beginning of Section II. When all agents behave normally, methods exist for agents to locally solve for appropriate local control inputs that together satisfy the condition in (3) (e.g. [2]).

In contrast to prior work, this paper considers systems containing agents which are faulty or adversarial. More specifically, this paper considers a subset of agents \( \mathcal{A} \subset \mathcal{V} \) that apply the following control input for all sampling times \( t_{ik} \), \( k \in \mathbb{Z}_{>0}, j \in \mathcal{A} \):

\[
u_j^{\text{max}}(x_{ik}) = \arg \max_{u \in \mathcal{U}_j} [L_{f_j} h^{x_j}(\bar{x}) + L_{g_j} h^{x_j}(\bar{x})u_j + L_{\phi_j} h^{x_j}(\bar{x})].
\] (4)

The agents in \( \mathcal{A} \) are called adversarial. Agents that are not adversarial are called normal. The set of normal agents is denoted \( \mathcal{N} = \mathcal{V} \setminus \mathcal{A} \). Dividing the LHS of (3) into normal and adversarial parts yields the following sufficient condition for set invariance in the presence of adversaries:

\[
\sum_{j \in \mathcal{A}} (L_{f_j} h^{x_j}(\bar{x}) + L_{g_j} h^{x_j}(\bar{x})u_j^{\text{max}} + L_{\phi_j} h^{x_j}(\bar{x})) + \sum_{i \in \mathcal{N}} (L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x})u_i + L_{\phi_i} h^{x_i}(\bar{x})) \leq -\alpha(h(\bar{x})).
\] (5)

Again, the equation (5) being satisfied for all \( t \geq 0 \) is equivalent to \( h(\bar{x}(t)) \leq \alpha(h(\bar{x}(t))) \) being satisfied for all \( t \geq 0 \) which implies forward invariance of the set \( S \).

The form of (5) reflects sampled-data adversarial agents seeking to violate the set invariance condition in (3) by maximizing their individual contributions to the left hand side (LHS)
Problem 1. Determine control inputs for the normal agents $i \in \mathcal{V}$ which render the set $S$ forward invariant under the perturbed sampled-data dynamics (1) in the presence of a set of worst-case adversarial agents $A$.

Remark 2. Since faulty or adversarial agents’ states are generally modeled as being uncontrollable under the nominal system control law, the function $h(\bar{x})$ can be defined to consider only the safety of normal agents.

Remark 3. This paper assumes the identities of the adversarial agents are known to the normal agents. Methods for identifying misbehavior are beyond the scope of this paper.

III. MAIN RESULTS

A. Preliminaries

The results of this subsection will be needed for our later analysis. The minimum and maximum value functions $\gamma_i^{\min}(\cdot), \gamma_i^{\max}(\cdot)$ for $i \in \mathcal{V}$ are defined as follows:

$$\begin{align*}
\gamma_i^{\min}(\bar{x}) &= \min_{u_i \in \mathcal{U}_i} \left[ L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i \right], \\
\gamma_i^{\max}(\bar{x}) &= \max_{u_i \in \mathcal{U}_i} \left[ L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i \right].
\end{align*} \tag{6}$$

Each $\gamma_i^{\min}(\cdot)$ and $\gamma_i^{\max}(\cdot)$ can be calculated by solving a parametric linear program

$$\begin{align*}
\min_{u_i \in \mathbb{R}^{m_i}} \quad & c(\bar{x})^T u_i \\
\text{s.t.} \quad & A_i(\bar{x}) u_i \leq b_i(\bar{x}), \tag{7}
\end{align*}$$

where the vector $c(\bar{x})^T = L_{g_i} h^{x_i}(\bar{x})$ when calculating $\gamma_i^{\min}$ and $c(\bar{x})^T = -L_{g_i} h^{x_i}(\bar{x})$ when calculating $\gamma_i^{\max}$. Note that (7) is feasible for all $\bar{x} \in S$ under Assumption 3. For an adversarial agent $i \in A$, the function $\gamma_i^{\max}(\cdot)$ represents the bound on the worst-case contribution of $i$ to the sum on the LHS of (5). Similarly, the function $\gamma_i^{\min}(\cdot)$ for a normal agent $i \in \mathcal{N}$ represents the bound on agent $i$’s best control effort towards minimizing the LHS of (5). The following result presents a sufficient condition under which $\gamma_i^{\min}(\cdot)$ and $\gamma_i^{\max}(\cdot)$ are locally Lipschitz on the set $S$.

Lemma 1. If the interior of $\mathcal{U}_i(\bar{x})$ is nonempty for all $\bar{x} \in S$ and $\mathcal{U}_i(\bar{x})$ is uniformly compact near $\bar{x}$ for all $\bar{x} \in S$, then the functions $\gamma_i^{\min}(\cdot)$ and $\gamma_i^{\max}(\cdot)$ defined by (6) are locally Lipschitz on $S$.

Proof. The proofs for $\gamma_i^{\min}(\cdot)$ and $\gamma_i^{\max}(\cdot)$ are identical except for trivially changing the sign of the objective function; therefore only the proof for $\gamma_i^{\min}(\cdot)$ is given. Define the set of optimal points

$$P_i(\bar{x}) = \left\{ u_i^*: u_i^* = \arg \min_{u_i \in \mathcal{U}_i} \left[ L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i \right] \right\}. $$

The result in [25, Thm. 5.1] states that if $\mathcal{U}_i(\bar{x})$ is nonempty and uniformly compact near $\bar{x} \in \mathbb{R}^d$ and if the Mangasarian-Fromovitz (M-F) conditions hold at each $u_i^* \in P_i(\bar{x})$, then $\gamma_i^{\min}(\cdot)$ is locally Lipschitz near $\bar{x}$ (see [25] for the definition of the M-F conditions). The first two conditions hold by assumption, and so we next prove that the M-F conditions hold at each $u_i^* \in P(\bar{x})$. Let $A_{i,j}(\cdot)$ denote the $j$th row of $A_i(\cdot)$ and $b_{i,j}(\cdot)$ denote the $j$th entry of $b_i(\cdot)$.

Consider any $\bar{x} \in S$ and $u_i^* \in P_i(\bar{x})$. Denote $J_i(\bar{x}) = \{ j \in \{1, \ldots, q_i \} : A_{i,j}(\bar{x}) u_i^* - b_{i,j}(\bar{x}) = 0 \}$ as the set of constraint indices where equality holds at $u_i^*$. The interior int$(\mathcal{U}_i(\bar{x}))$ being nonempty and convex implies there exists an $r \in \mathbb{R}^{m_i}$ such that $A_{i,j}(\bar{x})(u_i^* + r) - b_{i,j}(\bar{x}) < 0$, which implies $A_{i,j}(\bar{x})r < b_{i,j}(\bar{x}) - A_{i,j}(\bar{x}) u_i^* = 0$. This implies that there exists an $r$ such that $A_i(\bar{x})r < 0$. The point $u_i^*$ is therefore M-F regular. Since this holds for any $u_i^* \in P_i(\bar{x})$ and $\forall \bar{x} \in S$, by [25, Thm. 5.1] it holds that $\gamma_i^{\min}(\cdot)$ is locally Lipschitz on $S$.

We briefly emphasize the difference between the min / max value functions $\gamma_i^{\min}, \gamma_i^{\max}$ in (6) and the min / max point functions defined as

$$\begin{align*}
u_i^{\min}(\bar{x}) &= \arg \min_{u_i \in \mathcal{U}_i} \left[ L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i \right], \tag{8} \\
u_i^{\max}(\bar{x}) &= \arg \max_{u_i \in \mathcal{U}_i} \left[ L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i \right]. \tag{9}
\end{align*}$$

In words, $u_i^{\min}$ and $u_i^{\max}$ represent the control actions such that, respectively, $\gamma_i^{\min}(\bar{x}) = L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i^{\min}$ and $\gamma_i^{\max}(\bar{x}) = L_{f_i} h^{x_i}(\bar{x}) + L_{g_i} h^{x_i}(\bar{x}) u_i^{\max}$. Although the min / max value functions $\gamma_i^{\min}(\cdot), \gamma_i^{\max}(\cdot)$ are locally Lipschitz under the conditions of Lemma 1 and [25], the min / max point functions $u_i^{\min}$ and $u_i^{\max}$ may not be locally Lipschitz in general.

The following Lemma will also be needed for our later analysis, and is based on results in [9], [26, Thm. 3.4]. It establishes an upper bound on the difference between the sampled state $\bar{x}^{\mathcal{K}i}$ and the state $\bar{x}(t)$ on the time interval $t \in [t^k_i, t^{k+1}_i + \Gamma]$, $\Gamma \geq 0$.

Lemma 2. For any $\Gamma > 0$, there exists a $\mu \geq 0, L' > 0$ such that the following holds:

$$\| \bar{x}(t) - \bar{x}^{\mathcal{K}i} \| \leq \frac{\mu}{L'} \left( e^{L' \Gamma} - 1 \right) \forall t \in [t^k_i, t^{k+1}_i + \Gamma].$$

The proof is omitted due to space constraints. For brevity, we define the function $\epsilon : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0} \rightarrow \mathbb{R}$ as

$$\epsilon(\Gamma, \mu, L') = \frac{\mu}{L'} \left( e^{L' \Gamma} - 1 \right). \tag{10}$$

For fixed $\mu, L'$, we abuse notation by writing $\epsilon(\Gamma)$ as a function of $\Gamma$ only. It can be shown that for fixed $\mu, L'$, $\epsilon(\cdot)$ is a class-$\mathcal{K}$ function in $\Gamma$.

B. Synchronous Sampling Times

To facilitate the presentation of the main results, we first consider the case where all agents in the system have synchronous sampling times with a period of $\tilde{T} > 0$, i.e., $T_i = \{ k\Gamma : k \in \mathbb{Z}_{\geq 0} \} \forall i \in \mathcal{N}$. This assumption is later relaxed to consider agents with asynchronous, nonidentical

\footnote{We re-emphasize however that when (9) is applied in a ZOH manner, existence and uniqueness of solutions to (1) is guaranteed by Carathéodory’s theorem [22, Sec. 2.2].}
sampling times. The Cartesian product of the admissible controls for all normal agents is denoted \( \mathcal{U}_N = \times_{i \in N} \mathcal{U}_i \). Under Assumption 3, each \( \mathcal{U}_i(\bar{x}) \) being uniformly compact near all \( \bar{x} \in S \) implies that \( \mathcal{U}_N \) is also uniformly compact near all \( \bar{x} \in S \). We will denote \( \bar{u}_N \in \mathcal{U}_N \) as the vector containing only normal agents’ control inputs; i.e. \( \bar{u}_N = \left[ u_1^T, \ldots, u_{|N|}^T \right]^T \), \( \{i_1, \ldots, i_{|N|}\} \in N \).

Our ultimate aim is to demonstrate that for all \( t \geq 0 \),
\[
\dot{h}(\bar{x}(t)) + \alpha(h(\bar{x}(t))) \leq 0. \tag{11}
\]

The dependence of \( \dot{x}(t) \) on \( t \) will be omitted for brevity. Prior results have typically focused on designing continuous \( u(\cdot) \) functions that guarantee that (11) is satisfied. Satisfying (11) in sampled-data systems for all intermediate times \( t \in [t_k, t_{k+1})\), \( k \in \mathbb{Z}_{\geq 0} \) is more challenging since \( u(\cdot) \) is constant on each interval \( t \in [k\Gamma, (k + 1)\Gamma) \). Inspired by [9], this challenge will be addressed as follows: given the sampled state \( \bar{x}(t^k) \) and the state \( \bar{x}(t) \), \( t \in [t^k, t^{k+1}) \), define the error term
\[
e(t, t^k) = \left( \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) \right) + \alpha(h(\bar{x})) - \alpha(h(\bar{x}^k)) \right). \]

From the LHS of (11) we obtain
\[
\dot{h}(\bar{x}) + \alpha(h(\bar{x})) = \dot{h}(\bar{x}^k) + \alpha(h(\bar{x}^k)) + \alpha(h(\bar{x})) - \alpha(h(\bar{x}^k)) \leq \dot{h}(\bar{x}^k) + \alpha(h(\bar{x}^k)) + e(t, t^k) \leq \dot{h}(\bar{x}^k) + \alpha(h(\bar{x}^k)) + \sup_{t \in [t^k, t^{k+1})} \|e(t, t^k)\|.
\]

By defining a function \( \eta(\cdot) \) such that \( \|\eta(\Gamma)\| \geq \sup_{t \in [t^k, t^{k+1})} \|e(t, t^k)\| \), the inequality condition in (11) is therefore satisfied for all times on the interval \( t \in [t^k, t^{k+1}) \) if for every \( t^k \in T \) the following condition holds:
\[
\dot{h}(\bar{x}^k) + \alpha(h(\bar{x}^k)) + \eta(\Gamma) \leq 0. \tag{12}
\]

Satisfaction of (12) implies that \( \dot{h}(\bar{x}) + \alpha(h(\bar{x})) \leq \dot{h}(\bar{x}^k) + \alpha(h(\bar{x}^k)) + \eta(\Gamma) \leq 0 \) for all \( t \in [t^k, t^{k+1}) \). To define such a function \( \eta(\cdot) \), the following Lemma will be used.

**Lemma 3.** Consider the system (1). There exist constants \( c_f, c_g, c_a, c_{\gamma}, c_h \in \mathbb{R} \) such that for all \( \bar{x}^1, \bar{x}^2 \in S \), all of the following inequalities hold:
\[
\sum_{i \in N} \| L_{f_i}^x \bar{x}^i(\bar{x}) - L_{f_i}^x \bar{x}^i(\bar{x}^2) \| \leq c_f \| \bar{x}^1 - \bar{x}^2 \|, \tag{13}
\]
\[
\sum_{i \in N} \| L_{g_i}^x \bar{x}^i(\bar{x}) - L_{g_i}^x \bar{x}^i(\bar{x}^2) \| \leq c_g \| \bar{x}^1 - \bar{x}^2 \|, \tag{14}
\]
\[
\| \alpha(h(\bar{x})) - \alpha(h(\bar{x}^2)) \| \leq c_a \| \bar{x}^1 - \bar{x}^2 \|, \tag{15}
\]
\[
\sum_{j \in A} \left\| \gamma_j^{\max}(\bar{x}) - \gamma_j^{\max}(\bar{x}^2) \right\| \leq c_{\gamma} \| \bar{x}^1 - \bar{x}^2 \|, \tag{16}
\]
\[
\sum_{i \in V} \left\| L_{\phi_i}^x \bar{x}^i(\bar{x}^2) \right\| \leq c_h \sum_{i \in V} \phi_i^{\max}. \tag{17}
\]

The proof is omitted due to space constraints. In addition to the inequalities in Lemma 3, observe that each set \( \mathcal{U}_i \) being uniformly compact implies that there exist a constant \( u_{\text{max}} \geq 0 \) such that \( \| u_i^k \| \leq u_{\text{max}} \) for all \( i \in N, k \geq 0 \). Using this definition of \( u_{\text{max}} \), the constants defined in Lemma 3, and the function \( \epsilon(\cdot) \) in (10) we define the function \( \eta : \mathbb{R}_{\geq 0} \to \mathbb{R} \) as follows:
\[
\eta(\Gamma) = (c_f + c_g u_{\text{max}} + c_a + c_{\gamma}) \epsilon(\Gamma) + c_h \sum_{i \in V} \phi_i^{\max}. \tag{18}
\]

The proof that \( \eta(\Gamma) \geq \sup_{t \in [t^k, t^{k+1})} \|c(t, t^k)\| \) will be given in Theorem 1. This definition of \( \eta(\cdot) \) is used to define the following safety-preserving controls set for the normal agents in \( N \):
\[
K(\bar{x}) = \{ \bar{u}_N \in \mathcal{U}_N : \| L_{f_i}^x \bar{x}^i(\bar{x}) + L_{g_i}^x \bar{x}^i(\bar{x}^2) u_i^k \| + \sum_{j \in A} \gamma_j^{\max}(\bar{x}) + \alpha(h(\bar{x}^k)) + \eta(\Gamma) \leq 0 \}. \tag{19}
\]

Using this definition of \( K(\cdot) \), the following Theorem presents conditions under which the set \( S \) can be rendered forward invariant for the system (1) with synchronous sampling times despite the actions of the adversarial agents.

**Theorem 1.** Consider the system (1) with synchronous sampling times. If \( \bar{x}^k \in S \) for \( k \geq 0 \), then for any control input \( \bar{u}^k \in K(\bar{x}^k) \) the trajectory \( \bar{x}(t) \) satisfies \( \bar{x}(t) \in S \) for all \( t \in [k\Gamma, (k + 1)\Gamma) \).

**Proof.** First, denote \( \dot{h}(\bar{x}^k) = \dot{h}(\bar{x}^k) - \sum_{i \in V} L_{\phi_i}^x \bar{x}^i(\bar{x}^k) \). In words, \( \dot{h}(\bar{x}^k) \) is equal to \( \dot{h}(\bar{x}^k) \) with all disturbance-related Lie derivatives subtracted out. Observe that
\[
\dot{h}(\bar{x}^k) + \left( \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) \right) = \dot{h}(\bar{x}^k) + \sum_{i \in V} L_{\phi_i}^x \bar{x}^i(\bar{x}^k) + \left( \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) - \sum_{i \in V} L_{\phi_i}^x \bar{x}^i(\bar{x}^k) \right),
\]
\[
= \dot{h}(\bar{x}^k) + \left( \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) \right),
\]
\[
= \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) + \sum_{i \in V} L_{\phi_i}^x \bar{x}^i(\bar{x}).
\]

From (1) and the definition of adversarial agents in (4), define the error term
\[
e'(t, t^k) = \left( \dot{h}(\bar{x}) - \dot{h}(\bar{x}^k) \right) + \sum_{i \in V} L_{\phi_i}^x \bar{x}^i(\bar{x}) + \left( \alpha(h(\bar{x})) - \alpha(h(\bar{x}^k)) \right),
\]
\[
= \left( \sum_{i \in V} L_{f_i}^x \bar{x}^i(\bar{x}) - L_{f_i}^x \bar{x}^i(\bar{x}^k) \right) + \sum_{i \in V} L_{g_i}^x \bar{x}^i(\bar{x}) + \left( \sum_{i \in V} L_{g_i}^x \bar{x}^i(\bar{x}) - L_{g_i}^x \bar{x}^i(\bar{x}^k) u_i^k \right) + \sum_{j \in A} \gamma_j^{\max}(\bar{x}) + \left( \gamma_j^{\max}(\bar{x}) - \gamma_j^{\max}(\bar{x}^k) \right) + \left( \alpha(h(\bar{x})) - \alpha(h(\bar{x}^k)) \right).
\]

Since \( t^{k+1} - t^k = (k + 1)\Gamma - h\Gamma = \Gamma \) for all \( k \geq 0 \), by Lemma 2 we have \( \| \bar{x} - \bar{x}^k \| \leq \epsilon(\Gamma) \) for all \( t \in [t^k, t^{k+1}). \)
Using Lemma 3 and the definition of $\eta(\cdot)$ in (18) yields the following upper bound on $\|e'(t, t^k)\|$: 
\[
\sup_{t \in \{t^k, t^{k+1}\}} \|e'(t, t^k)\| \leq (c_f + c_d u_{\text{max}} + c_\alpha + c_\gamma) \epsilon(\Gamma) + c_h \sum_{i \in V} \phi^\text{max}_i,
\]
\[
\implies \sup_{t \in \{t^k, t^{k+1}\}} \|e'(t, t^k)\| \leq \eta(\Gamma).
\]
Therefore for all $t \in [t^k, t^{k+1})$, it holds that 
\[
\dot{h}(\vec{x}) + \alpha(h(\vec{x})) = \dot{h}'(\vec{x}) + \left(\dot{h}'(\vec{x}) - \dot{h}'(\vec{x}^k)\right) + \sum_{i \in V} L_{fi} h^x_i(\vec{x}) + \alpha(h(\vec{x})) + (\alpha(h(\vec{x})) - \alpha(h(\vec{x}^k)) \right) + \sum_{i \in V} L_{fi} h^x_i(\vec{x}) + \alpha(h(\vec{x}^k)) + \sum_{i \in V} \gamma^\text{max}_j(x_i) + \alpha(h(x_i)) + \eta(\Gamma),
\]
\[
\sum_{i \in \mathcal{N}} L_{fi} h^x_i(\vec{x}) + \alpha(h(\vec{x})) + \sum_{i \in \mathcal{N}} \gamma^\text{max}_j(x_i) + \alpha(h(x_i)) + \eta(\Gamma) > 0,
\]
\[
\implies \dot{\bar{h}}(\vec{x}) + \alpha(h(\vec{x})) \leq 0.
\]
Therefore any $\vec{u}_{\text{nom}}^k \in K(\vec{x})$, observe from (19) that 
\[
\sum_{i \in \mathcal{N}} L_{fi} h^x_i(\vec{x}) + \sum_{j \in \mathcal{A}} \gamma^\text{max}_j(x_i) + \alpha(h(\vec{x})) + \eta(\Gamma) \leq 0.
\]

C. Asynchronous Sampling Times

The assumption of identical, synchronous sampling times typically does not hold in practice. In addition, a distributed system may not have access to a centralized entity to solve the QP in (22) to obtain $\vec{u}_{\text{nom}}^k$. This subsection will therefore consider asynchronous sampling times and a distributed method for computing local control inputs. Each agent $i \in \mathcal{V}$ is assumed to have a nominal sampling period $\Gamma_i \in \mathbb{R}_{>0}$ and the perturbed sequence of sampling times 
\[
\mathcal{T}_i = \{t^0_i, t^1_i, \ldots\} \text{ s.t. } t^k_i - t^k_i = \Gamma_i + \delta_i(k), \forall k \in \mathbb{Z}_{\geq 0},
\]
where $\delta_i(k)$ is a disturbance satisfying $\|\delta_i(k)\| \leq \delta^\text{max}_i$. The function $\delta_i$ can be used to model time delays due to disturbances such as clock asynchrony or packet drops in the communication network. We denote $\Gamma^\text{max} = \max_{i \in \mathcal{V}} \Gamma_i$ and $\delta^\text{max}_i = \max_{i \in \mathcal{V}} \delta^\text{max}_i$. Recall from Section II-A that we denote $\vec{x}_i^k = \vec{x}(t^k_i)$ and $u_i^k = u_i(t^k_i)$.

Each agent $i \in \mathcal{N}$ updates its control input $u_i^k$ at sampling times $t^k_i$ and also broadcasts $u_i^k$ to all other agents in the network. Each agent $i$ stores the values of the most recently received inputs from its normal in-neighbors $l \in \mathcal{N}$. The notation $\vec{u}_{\text{nom}}^k$ denotes the most recently received input value by an agent $i$ from agent $l$ at time $t^k_i$.

Using the definition of $\eta(\cdot)$ from (18), the following safety-preserving control set is defined for each $i \in \mathcal{N}$: 
\[
K_i(\vec{x}_i^k) = \left\{ u_i \in \mathcal{U}_i : L_{fi} h^x_i(\vec{x}_i^k) + L_{gi} h^x_i(\vec{x}_i^k) u_i + \sum_{i \in \mathcal{N} \setminus \{i\}} [L_{fi} h^x_i(\vec{x}_i^k) + L_{gi} h^x_i(\vec{x}_i^k) u_i] + \sum_{j \in \mathcal{A}} \gamma^\text{max}_j(x_i^k) + \alpha(h(x_i^k)) + \eta(\Gamma_i + \delta^\text{max}_i) \leq 0 \right\}
\]

Theorem 2 presents conditions under which forward invariance of the set $\mathcal{S}$ can be guaranteed for the distributed, asynchronous system described in this subsection.

Theorem 2. Consider the system (1) with sampling times described by (23). If at sampling time $t^k_i$ for $k \geq 0$, $i \in \mathcal{N}$ it holds that $\vec{x}_i^k \in \mathcal{S}$, then for any $u_i^k \in K_i(\vec{x}_i^k)$ the trajectory $\vec{x}(t)$ satisfies $\vec{x}(t) \in \mathcal{S}$ for all $t \in [t^k_i, t^{k+1}_i]$. The proof is omitted due to space constraints, but follows similar logic as in Theorem 1. Under the communication protocol described previously, each agent can use the most recently received inputs $\vec{u}_{\text{nom}}^k$ from other normal agents to calculate a control input $u_i^k \in K_i(\vec{x}_i^k)$. Such a $u_i^k$ can be computed by solving the following QP: 
\[
u_i^k = \arg \min_{u_i \in \mathcal{U}_i} \left\| u_i^k - \vec{u}_{\text{nom}}^k \right\|_2
\]
\[
\text{s.t.} \quad \left(L_{fi} h^x_i(\vec{x}_i^k) + L_{gi} h^x_i(\vec{x}_i^k) u_i + \sum_{i \in \mathcal{N} \setminus \{i\}} [L_{fi} h^x_i(\vec{x}_i^k) + L_{gi} h^x_i(\vec{x}_i^k) u_i] + \sum_{j \in \mathcal{A}} \gamma^\text{max}_j(x_i^k) + \alpha(h(x_i^k)) + \eta(\Gamma_i + \delta^\text{max}_i) \leq 0 \right\}
\]

Like the previous formulations, the values of $\gamma^\text{max}_j(\cdot)$ for $j \in \mathcal{A}$ can be calculated via solving a separate LP. By the results of Theorem 2, when each $K_i(\vec{x})$ is nonempty and each normal agent applies the controller defined by (24) the
multi-agent safe set is rendered forward invariant despite any collective worst-case behavior of the adversarial agents.

D. Maximum Safety-Preserving Control Action

One of the required conditions of the foregoing results is the nonemptiness of the safety-preserving sets $K(x)$ and $K_i(\bar{x})$, which is also closely related to the feasibility of the respective QPs (22), (24). Conditions under which such sets remain nonempty for general systems remains an open question. Guaranteeing both safety and the feasibility of the QP calculating the control input $u_i(z_k)\forall k$ has been a recent topic of study [23], [27], and can depend on the choice of extended class-$\mathcal{K}_\infty$ function $\alpha(\cdot)$.

In contrast, consider the sampled-data control law $u_i^{min}(\cdot)$ defined in (8). Intuitively speaking, (8) represents the strongest control effort agent $i \in N$ can apply towards minimizing the LHS of (3). This control input can be solved for by taking the arg min of the minimizing LP in (7):

$$ u_i^{min}(\bar{x}_k) = \text{arg min}_{u_i \in \mathbb{R}^m} L_g h^* (z_k)u_i $$

$$ \text{s.t.} \quad A_i(\bar{x}_k)u_i \leq b_i(z_k) \quad (25) $$

For any system satisfying Assumption 3, the set $\mathcal{U}_i(\bar{x}) = \{u : A_i(\bar{x})u \leq b_i(\bar{x})\}$ is nonempty for all $\bar{x} \in S$. This implies that (25) is always guaranteed to be feasible for $\bar{x} \in S$. However the question remains as to when the control action (8) can guarantee forward invariance of $S$. Towards this end, define the set

$$ \partial S_e = \left\{ x \in S : \min_{z \in \partial S} ||x - z|| \leq \epsilon \right\}, \quad \epsilon > 0. \quad (26) $$

In words, $\partial S_e$ is an “inner boundary region” of $S$ which includes all points in $S$ within distance $\epsilon$ of $\partial S$ with respect to a chosen norm. The following theorem presents a sufficient condition for when $u_i^{min}(\cdot)$ for each normal agent renders the set $S$ invariant in the presence of an adversarial set $A$.

Theorem 3. Let $\epsilon^* = \epsilon(\Gamma^{\max} + 2\delta^{\max})$ and define the sets $\partial S_\epsilon, \partial S_\epsilon'$ as per (26). Suppose that each normal agent $i \in N$ applies the control input $u_i^{min}(\bar{x}_k)$ from (8) for all sampled states $\bar{x}_k$, satisfying $\bar{x}_k \in \partial S_\epsilon$. Then $S$ is forward invariant if $\bar{x}(0) \in S \setminus \partial S_\epsilon'$, and the following condition holds:

$$ \max_{\bar{x} \in \partial S_\epsilon'} \left[ \sum_{i \in A} \max_{\bar{x} \in B(\bar{x}, \epsilon^*)} [\gamma_i^{min}(\bar{x})] \right] + $$

$$ \sum_{j \in A} \gamma_j^{max}(\bar{x}) + \alpha(h(\bar{x})) \leq -\eta(\Gamma^{\max} + 2\delta^{\max}). \quad (27) $$

Proof. The proof first demonstrates that the most recently sampled states of all agents always lie within a closed ball of radius $\epsilon^* = \epsilon(\Gamma^{\max} + 2\delta^{\max})$. Next, it shows that $\bar{x}(0) \in S \setminus \partial S_\epsilon$ implies that $\bar{x}(t)$ cannot leave $S$ without all agents sampling the state at least once within the region $\partial S_\epsilon$. Finally, it is shown that this fact combined with (27) implies that $S$ is forward invariant.

Choose any $i \in N$ and any sampling time $t_k^i$ for agent $i$. By the definition of $\Gamma^{\max}$ and $\delta^{\max}$, the next sampling time $t_{k+1}^i$ satisfies $t_{k+1}^i - t_k^i \leq t_k^i + \Gamma^{\max} + 2\delta^{\max}$. Since this holds for all $i \in N$, given any $i_1, i_2 \in N$ and interval $[t_i^k, t_{i+1}^k + \Gamma^{\max} + \delta^{\max}]$, there exists a sampling time for $i_2$ satisfying $t_{i_2}^k \geq t_i^k + \Gamma^{\max} + \delta^{\max}$. Using Lemma 2, this implies that the maximum normed difference between any two most recently sampled states $\bar{x}(t_{i_1}^k)$ and $\bar{x}(t_{i_2}^k)$ satisfies $\|\bar{x}(t_{i_1}^k) - \bar{x}(t_{i_2}^k)\| \leq \epsilon(\Gamma^{\max} + 2\delta^{\max}) = \epsilon^*$. Since this holds for all $i_1, i_2 \in N$ at any $t_k^i$, the most recently sampled states of all agents therefore always lie within a ball of radius $\epsilon^*$.

Next, consider any agent $i$ with sampling time $t_i^k$ such that $\bar{x}(t_i^k) \in S \setminus \partial S_e$. Then $\bar{x}(t_i^k) \in S \setminus \partial S_e$. Therefore, $\bar{x}(0) \in S \setminus \partial S_\epsilon$, which implies $\bar{x}$ cannot leave $S$ without each agent $i \in N$ having at least one sampling time $t_i^k$ such that $\bar{x}_k \in \partial S_\epsilon$. Note that $\bar{x}(0) \in S \setminus \partial S_\epsilon$, and we now have

Choose the first sampling time $t_{i_1}^k$ such that $t_{i_1}^k \geq \Gamma^{\max} + 2\delta^{\max}$ and $\bar{x}(\bar{x}_k) \in \partial S_\epsilon \subset \partial S_\epsilon'$. Since $\bar{x}(0) \in S \setminus \partial S_\epsilon$, by the Theorem statement, it can be shown using prior arguments that such a sampling time is guaranteed to exist. This choice of $t_{i_1}^k$ implies that all agents have sampled at least once at or before $t_{i_1}^k$. Let $t_{i_2}^k > t_{i_1}^k$ be the next normal agent sampling time strictly greater than $t_{i_1}^k$, with the associated agent denoted $i_2 \in N$. Let $\bar{x}_{k_{i_1}}, \ldots, \bar{x}_{k_{i_2}}$ denote the most recently sampled states of all normal agents. By prior arguments, $\bar{x}_{k_{i_1}} \in B(\bar{x}_{k_{i_1}} \epsilon^*)$ for all $l \in 1, \ldots, N$, and therefore by (27) at time $t_{i_1}^k$ we have

$$ \sum_{j \in A} \gamma_j^{max}(\bar{x}) + \alpha(h(\bar{x})) \leq \sum_{j \in A} \gamma_j^{max}(\bar{x}_{k_{i_1}}) + \alpha(h(\bar{x}_{k_{i_1}})) + \eta(\Gamma^{\max} + 2\delta^{\max}) \leq 0. $$

From this it holds that for all $t \in [t_{i_1}^k, t_{i_2}^k]$, we have

$$ \bar{x}(\bar{x}) \leq \sum_{j \in A} \gamma_j^{max}(\bar{x}_{k_{i_1}}) + \alpha(h(\bar{x}_{k_{i_1}})) + \eta(\Gamma^{\max} + 2\delta^{\max}) \leq 0. $$

It follows that $S$ is forward invariant on the interval $t \in [t_{i_1}^k, t_{i_2}^k]$. The preceding arguments can be repeated for any subsequent adjacent sampling times $t_{i_1}^k, t_{i_2}^k, t_{i_3}^k, \ldots, t_{i_p}^k$, to show that $S$ is forward invariant on $[t_{i_1}^k, t_{i_p}^k]$, which concludes the proof.
IV. Simulations

We simulate a network of $n = 5$ agents with unicycle dynamics in $\mathbb{R}^2$. Agents are nominally tasked with tracking time-varying trajectories defined by a Bezier curve, timing law, and local formation offsets. The agents must also avoid obstacles near the trajectory. Two agents misbehave by each pursuing the respective closest normal agent. The state of each unicycle $i \in \mathcal{V}$ is denoted $x_i = [x_{i,1}, x_{i,2}, x_{i,3}]^T$. Each unicycle is controlled via an input-output linearization method [28, Ch. 11] where each agent has the outputs $p_i = [p_{i,1}, p_{i,2}]^T$ defined as $p_{i,1} = x_{i,1} + b \cos(x_{i,3})$, $p_{i,2} = x_{i,2} + b \sin(x_{i,3})$, $b > 0$. The output $p_i$ is treated as having single integrator dynamics $\dot{p}_i = u_i = [u_{i,1}, u_{i,2}]^T$. Each agent $i$ is controlled by first computing the output control input $u_i$ and minimally modifying $u_i$ via the CBF-based QP method described previously. The final unicycle control inputs $[v_i, \omega_i]$ are then obtained via the transformation $[v_i, \omega_i] = \begin{bmatrix} -\cos(\theta_i) & \sin(\theta_i) \\ -\sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}[u_{i,1}, u_{i,2}]$. At any timestep where the QP is infeasible, each normal agent applies the best-effort safety preserving control (8) calculated via the LP (25).

Given control bounds $|u_i| \leq \nu_i^{\max}$ and $|\omega_i| \leq \omega_i^{\max}$, it can be shown that the corresponding linear control bounds on $u_{i,1}, u_{i,2}$ are $A_i(x_i) \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix} \leq b_i$, with

$$A_i(x_i) = \begin{bmatrix} \cos(\theta_i) & \sin(\theta_i) \\ -\cos(\theta_i) & -\sin(\theta_i) \\ -\sin(\theta_i)/b & \cos(\theta_i)/b \\ \sin(\theta_i)/b & -\cos(\theta_i)/b \end{bmatrix}, \quad b_i = \begin{bmatrix} \nu_i^{\max} \\ \nu_i^{\max} \\ \omega_i^{\max} \\ \omega_i^{\max} \end{bmatrix}$$ (28)

For strictly positive $\nu_i^{\max}$, $\omega_i^{\max}$, and $b$, the set $\mathcal{U}_i = \{u_i : A_i(x_i)u_i - b_i \leq 0\}$ satisfies the conditions of Assumption 3 for all $x_i \in \mathbb{R}^3$. In this simulation each normal agent has lower maximum linear and angular velocities than the normal agents with $\nu_j^{\max} = 2$, $\omega_j^{\max} = 1$, $j \in \mathcal{A}$. The safe set $S$ is defined using a boolean composition of pairwise collision-avoidance sets for normal-to-normal pairs, normal-to-adversarial pairs, and normal-to-obstacle pairs. More specifically, given $i, j \in \mathcal{N}$ each safe set $h_{i,j}(\vec{x})$ is defined with respect to the linearized outputs as $h_{i,j}(\vec{x}) = (R_i + 2b)^2 - \|p_i - p_j\|_2^2$, with partial derivative $\partial h_{i,j}/\partial p_i = -2(p_i - p_j)$. The normal-to-adversarial and normal-to-obstacle pairwise safe sets for $i \in \mathcal{N}$, $j \in \mathcal{A}$ are defined in a similar manner. The pairwise adversarial-to-adversarial and adversarial-to-obstacle safe sets are not considered (as per Remark 2), since the nominal control law by definition has no effect on adversarial agents. All pairwise safe sets are composed into a single CBF $h_{\text{tot}}$ via boolean AND operations using the log-sum-exp smooth approximation to the max(·) function:

$$h_{\text{tot}}(\vec{x}) = \text{LSE}([h_{1,1}, \ldots, h_p]) = \sigma + \frac{1}{\rho} \ln \left( \sum_{i=1}^p \rho^{h_i - \sigma} \right),$$

where $\rho \in \mathbb{R}_{>0}$, $\sigma \in \mathbb{R}$. The term $\sigma$ is used to ensure numerical stability. The term $\rho$ controls how tightly LSE(·) approximates max(·). The reader is referred to [29], [30, Eq (10)] for more details. Sampling times in this simulation are asynchronous; each agent has a nominal time period of $\Gamma = 0.01$ with a time-varying random disturbance satisfying $\delta_i^{\max} = .002$. For each agent $i \in \mathcal{V}$, the disturbance bound satisfies $\delta_i^{\max} = 1.73$, and the term $\eta$ is set as $\eta(\Gamma) = 8.0566$. Several frames from the simulation are shown in Figure 1. A plot of $h_{\text{tot}}$ is given in Figure 2. As shown by Figure 2, under the proposed resilient controller the safety bounds for normal agents are not violated for the duration of the simulation. This is achieved despite the actions of the adversarial agents.

For comparison, Figure 3 shows the result of a simulation run under the same parameters but with $\eta(\Gamma) = 0$ for all $t \geq 0$; i.e. sampling, disturbances, and time delays are not taken into account in the normal agents’ control actions. In this case Figure 3 shows that the safety of the normal agents is not preserved—the value of $h_{\text{tot}}$ is temporarily positive, indicating that one or more of the composed safe sets was not invariant for the duration of the simulation.

V. Conclusion

In this paper we presented a framework for normally behaving agents to render a safe set forward invariant in the presence of adversarial agents. The proposed method considers distributed sampled-data systems with heterogeneous, asynchronous control affine dynamics. Directions for future work include investigating methods for identification of adversarial agents, and studying how estimates of the forward reachable sets for each agent can be included in the analysis.

REFERENCES

Fig. 1. Still frames from the video of Simulation 1. Normal agents are represented by blue circles and adversarial agents are represented by red circles. The dotted red lines around the blue circles represent normal agents’ safety radii. The time-varying formation trajectory is represented by the dotted magenta line; the magenta diamond represents the center of formation. Black crosses represent agents’ nominal local time-varying formational points.

Fig. 2. The value of the composed function $h_{tot}$ representing the safe set $S$. Non-positive values represent safety of the normal agents.

Fig. 3. The value of the composed function $h_{tot}$ representing the safe set $S$ when $\eta(\Gamma) = 0$ for all normal agents; i.e. sampling times and disturbances are not accounted for in the control input calculations. The safety bound is violated for the normal agents.


