

Distributed Multi-task Formation Control under Parametric Communication Uncertainties

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Abstract—Formation control is a key problem in the coordination of multiple agents. It arises new challenges to traditional formation control strategy when the communication among agents is affected by uncertainties. This paper considers the robust multi-task formation control problem of multiple non-point agents whose communications are disturbed by uncertain parameters. The control objectives include 1. achieving the desired configuration; 2. avoiding collisions; 3. preserving the connectedness of uncertain topology. To achieve these objectives, first, a condition of Linear Matrix Inequalities (LMIs) is proposed for checking the connectedness of uncertain topologies. Then, by preserving the initial topological connectedness, a gradient-based distributed controller is designed via Lyapunov-like barrier functions. Two numerical examples illustrate the effectiveness of the proposed method.

I. INTRODUCTION

Formation control is a key problem in the coordination of multi-agent systems. We focus on whether the communication network of multiple agents is disturbed or not, rather than an exhaustive review from all aspects (excellent surveys of formation control can be found in [1]). For the case of communication network without disturbances, efficient methods are proposed by using algebraic graph theory, Lyapunov stability theorem, LaSalle invariance principle, behavior-based approaches, etc [2], [3]. However, in real world, the communication among agents is usually disturbed by sorts of uncertainties, under which the methods developed for disturbance-free cases are usually not applicable [4].

In order to improve the robustness to the uncertainties in communication, different uncertain models and useful methods have been proposed recently [5]–[7]. According to the types of uncertainties, these models can be roughly divided into two categories: *Additive norm-bounded* uncertainties and *additive stochastic* uncertainties. In the first category, it is assumed that an uncertainty is added to the control input for each agent, where the l_2 -norm or l_∞ -norm of this additive uncertainty is bounded, and Lyapunov stability theorem is employed for controller design [5], [6]. In the second category, Gaussian noises or white noises are added to the measurement or to the control input, reflecting the unreliable information exchange [4], [7]. However, additive uncertainties cannot meet all practical demands, e.g., (as a

motivating example for this work) in an aerospace multi-robot network, the communication among robots are disturbed by parameters like temperature, radiation, magnetic field intensity, such that the value of information exchange can be modeled as a polynomial function of these parameters.

Besides additive uncertainty, the existing literature usually assumes that the topology has a spanning tree, or it is connected. Though this condition is essential, the network cannot be always connected if there is no control input guaranteeing the connectedness. Thus, researchers start focusing on the connectedness maintenance in multi-agent networks [3], [8]–[13]. In [12], for uncertainty-free case, a decentralized navigation function is introduced for connectedness maintenance. In [9], a robust rendezvous problem is studied with unknown dynamics and bounded disturbances, but without considering collision avoidance.

Motivated by aforementioned results, based on our previous work [14], this paper considers a robust formation control problem of uncertain networks, whose communication weights are generic polynomial functions of uncertain parameters. The main contributions of this paper are:

- Rather than calculating the eigenvalues of Laplacian matrix, a necessary and sufficient condition is provided for checking the connectedness of uncertain networks. Based on the real Positivstellensatz, a relaxed condition is given via sum-of-squares programming. Then, by using the square matrix representation, a solvable condition of LMIs is obtained for checking the connectedness of uncertain network (Section III.A).
- Different from [10] and [15], a new Lyapunov-like barrier function is introduced, which guarantees a bounded control input. Then, a gradient-based controller is designed for solving the robust formation problem, with considering both connectedness maintenance and collision avoidance (Section III.B).

II. PRELIMINARIES

A. Model Formulation

In this paper, we consider a second-order system of multiple agents described by:

$$\begin{aligned} \dot{x}_i(t) &= \rho_i(t) \\ \dot{\rho}_i(t) &= u_i(t), \quad i \in \mathcal{N}, \end{aligned} \quad (1)$$

where $\mathcal{N} = \{1, \dots, N\}$, $x_i(t) \in \mathbb{R}^n$ denotes the position state, $\rho_i(t) \in \mathbb{R}^n$ denotes the velocity state, and $u_i(t) \in \mathbb{R}^n$ denotes the control input on i -th agent. In the sequel, we will omit the arguments t and x of functions whenever possible for the brevity of notations.

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This work is supported by the Automotive Research Center (ARC) in accordance with Cooperative Agreement W56HZV-14-2-0001 U.S. Army TARDEC in Warren, MI, USA, and an Early Career Faculty Grant from NASA's Space Technology Research Grants Program.

Based on algebraic graph theory [16], a network of multiple agent can be described by a *weighted undirected dynamic graph* $\mathcal{G}(t) = (\mathcal{A}, \mathcal{E}(t), G)$ with the set of nodes $\mathcal{A} = \{A_1, \dots, A_N\}$, the set of undirected edges $\mathcal{E}(t) = \{(A_i, A_j) \mid A_i, A_j \in \mathcal{A}\}$, and the *weighted adjacency matrix* $G = (G_{ij})_{N \times N}$. Th set of edges $\mathcal{E}(t)$ is constructed similar to [9] as shown in Fig. 1. A graph $\mathcal{G}(t)$ is *connected* at time

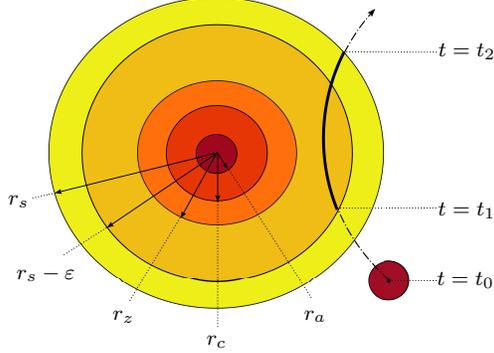


Fig. 1. The model of a single agent and the switching strategy of edges between two agents: r_a is the radius of each agent; r_z is the radius of collision avoidance region; r_z is the radius of region that the control with collision avoidance objective is active; r_s is the radius of sensing range; constant $\epsilon \in [0, r_s - r_z]$ is a distance parameter for the hysteresis in adding new edges. The solid line for $t \in (t_1, t_2)$ depicts the part of trajectory when there is an edge between the two agents.

t if there is a path between any pair of distinct nodes A_i and A_j in $\mathcal{G}(t)$. The *Laplacian matrix* is given as

$$L(t) = \Delta(t) - G(t) \quad (2)$$

where $\Delta(t) = \text{diag}(\sum_{j=1}^N G_{ij}(t))$. A relationship between $L(t)$ and the connectedness of $\mathcal{G}(t)$ is given in [16], [17]:

Lemma 1: Let $\lambda_1(L(t)) \leq \lambda_2(L(t)) \leq \dots \leq \lambda_N(L(t))$ be the ordered eigenvalues of $L(t)$. Then, 1_N is an eigenvector of $L(t)$ with the corresponding eigenvalue $\lambda_1(L(t)) = 0$. Moreover, $\lambda_2(L(t)) > 0$ if and only if $\mathcal{G}(t)$ is connected. \square

B. Problem Formulation

In this paper, the parametric uncertainties are considered in communications among agents. Specifically, it is supposed that the weighted adjacency matrix G is affected by uncertain parameters. We denote such a matrix as $G(t, \theta)$ where $\theta \in \mathbb{R}^r$ is an uncertain vector constrained as

$$\theta \in \Omega, \quad \Omega = \{\theta \in \mathbb{R}^r : s_i(\theta) \geq 0 \quad \forall i = 1, \dots, h\} \quad (3)$$

for some functions $s_1, \dots, s_h : \mathbb{R}^r \rightarrow \mathbb{R}$. In the sequel, we will assume that the entries of $G(t, \theta)$ and $s_1(\theta), \dots, s_h(\theta)$ are polynomials.

For a decentralized controller of agent i , the control law relies on the local information of agent i . Specifically,

$$u_i = \sum_{j \in \mathcal{N}_i^s(t)} f(x_i(t) - x_j(t), \rho_i(t) - \rho_j(t), G_{ij}(t, \theta)) \quad (4)$$

where $\mathcal{N}_i^s(t) = \{j \mid (A_i, A_j) \in \mathcal{E}(t)\}$ is the neighborhood set of agent i (in the sensing range of agent i). Different

with flocking and rendezvous problems, some useful sets are defined for formation control: \mathcal{N}_i^f is the neighborhood set to agent i in the desired formation configuration, i.e., $\mathcal{N}_i^f = \{j \mid (A_i, A_j) \in \mathcal{E}^f, x_i - \tau_i - (x_j - \tau_j) = 0\}$, where τ_i is the ideal displacement of agent i in the desired formation configuration, whose edge set is \mathcal{E}^f ; We also define sets $\mathcal{N}_i^{sf}(t) = \{j \mid j \in \mathcal{N}_i^s(t), j \in \mathcal{N}_i^f\}$ and $\mathcal{N}_i^{sz}(t) = \{j \mid j \in \mathcal{N}_i^s(t), \|x_i - x_j\| < r_z\}$, which will be used in Section III.

Now, let us propose the *robust multi-objective formation control problem* we are concerned with:

Problem 1: Under the dynamics (1) and the effect of θ given in (3), find a controller u_i in the form of (4) such that

- 1) $\lim_{t \rightarrow \infty} \|(x_i(t) - \tau_i) - (x_j(t) - \tau_j)\| = 0$, and $\lim_{t \rightarrow \infty} \|\rho_i - \rho_j\| = 0$, for all $i \in \mathcal{N}$ and $j \in \mathcal{N}_i^f$.
- 2) $\mathcal{G}(t)$ is connected, for all $t > t_0$, where t_0 is the initial time.
- 3) $\|x_i(t) - x_j(t)\| > d_s$, for all $t > t_0$, where d_s denotes a user-defined safety distance for collision avoidance. \square

For this problem, we assume that:

- Assumption 1: The desired configuration given by τ_i is achievable, i.e., $r_z \leq \|\tau_i - \tau_j\| \leq r_s - \epsilon$, for all $i \in \mathcal{N}$, $j \in \mathcal{N}_i^f$. In other words, the desired distance between agent i and agent $j \in \mathcal{N}_i^f$ is always between $r_s - \epsilon$ and r_z .
- Assumption 2: The neighbor set of agent i at time t_0 satisfies $\mathcal{N}_i^f \subseteq \mathcal{N}_i^s(t_0)$, which means that the desired topology is contained in the initial graph.
- Assumption 3: To achieve both objectives of collision avoidance and connectedness maintenance, we require $r_s - \|\tau_{ij}\| > d_s + \|\tau_{ij}\|$, for all $i, j \in \mathcal{N}$.

III. MAIN RESULTS

In this section, a condition for checking the connectedness of uncertain networks is provided, and a distributed controller is designed to solve Problem 1.

A. Checking the Connectedness of Uncertain networks

The connectedness of uncertainty-free network can be established by checking whether $\lambda_2(L)$ is positive from Lemma 1. However, it is not easy to calculate $\lambda_2(L(t^*, \theta))$ at a chosen time t^* under the disturbance of θ given in (3). In this subsection, we propose a method of Linear Matrix Inequalities (LMIs) to check the connectedness of an uncertain graph $\mathcal{G}(t^*)$ at a fixed time t^* . We omit the argument t in this subsection.

First, a necessary and sufficient condition is proposed for establishing the connectedness of uncertain networks:

Lemma 2: Consider a matrix $M \in \mathbb{R}^{N \times N-1}$ satisfying $\text{img}(M) = \ker(1_N^T)$, then, a new matrix is constructed based on the Laplacian matrix as:

$$\widehat{L}(\theta) = M^T L(\theta) M \quad (5)$$

where $\text{img}(A)$ denotes the image of matrix A , $\ker(A)$ denotes the null space of A , 1_N denotes a N dimensional column vector with all the entries equal to 1. The graph \mathcal{G}

is connected under the effect of parametric uncertainties θ if and only if there exists a symmetric definite matrix function $P(\theta) : \mathbb{R}^r \rightarrow \mathbb{R}^{N-1 \times N-1}$ such that

$$P(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P(\theta) > 0, \quad \forall \theta \in \Omega. \quad (6)$$

Proof: See [18], Lemma 2. \square

Note that the matrix polynomial inequalities (6) is not easy to be established, thus a tractable condition is provided based on the Real Positivstellensatz as follows:

Lemma 3: Condition (6) holds if there exist matrix polynomials $P(\theta)$, $R_i(\theta)$, for $i = 1, \dots, h$, and positive scalars $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{cases} R_i(\theta) \in \mathcal{P}_{\text{SOS}}^{N-1 \times N-1} \\ P(\theta) - c_1 I_{N-1} \in \mathcal{P}_{\text{SOS}}^{N-1 \times N-1} \\ H(\theta) - c_2 I_{N-1} \in \mathcal{P}_{\text{SOS}}^{N-1 \times N-1} \end{cases} \quad (7)$$

where $\mathcal{P}_{\text{SOS}}^{N-1 \times N-1}$ is the set of SOS matrix polynomials introduced in Section II,

$$H(\theta) = P(\theta)\widehat{L}(\theta) + \widehat{L}(\theta)^T P(\theta) - \sum_{i=1}^h R_i(\theta)s_i(\theta), \quad (8)$$

and θ , s_i are defined in (3).

Proof: See [18], Lemma 3. \square

Note that (7) requires to find appropriate c_1 , c_2 , $P(\theta)$ and $H(\theta)$ at the same time which is difficult and computationally demanding. Next, we propose an efficient method by using the Square Matrix Representation (SMR). Specifically, let $M(\theta) \in \mathcal{P}_{\text{SOS}}^{s \times s}$ be a symmetric matrix polynomial of size $s \times s$ of degree $\text{Deg}(M)$ in $\theta \in \mathbb{R}^r$ (this means that the highest degree of all the entries of $M(\theta)$ is $\text{Deg}(M)$ in θ), i.e., $\text{Deg}(M) = \max(\text{deg}(M_{ij}))$, $d_M = \lceil \frac{\text{Deg}(M)}{2} \rceil$. Then, $M(\theta)$ can be written as

$$M(\theta) = \Phi(\bar{M} + D(\delta), d_M, s) \quad (9)$$

where $\Phi(\bar{M} + D(\delta), m, s) = (*)^T (\bar{M} + D(\delta)) (\phi(r, d_M) \otimes I_s)$, \bar{M} is a symmetric matrix, and $D(\delta)$ is a linear parametrization of the linear subspace $\mathcal{D} = \{D(\delta) \in \mathbb{R}^{l(r, d_M) \times s \times l(r, d_M) \times s} : D(\delta) = D^T(\delta), (*)^T D(\delta) (\phi(r, d_M) \otimes I_s) = 0\}$. Now, we can propose the condition of LMIs for checking the connectedness of uncertain networks:

Theorem 1: Condition (6) holds if $c^* > 0$, where c^* is the solution of the convex optimization problem

$$\begin{aligned} c^* &= \sup_{c, \bar{R}_i, \bar{P}, \delta} c \\ \text{s.t.} \quad &\begin{cases} \bar{R}_i \geq 0, \bar{P} \geq 0, \text{trace}(\bar{P}) = 1 \\ \bar{H} + D(\delta) - cI_{(N-1) \cdot l(r, d_H)} - \sum_{i=1}^h \bar{\Upsilon}_i(\bar{R}_i) \geq 0. \end{cases} \end{aligned} \quad (10)$$

The matrices involved in this problem are defined by $R_i(\theta) = \Phi(\bar{R}_i, d_{R_i}, N-1)$, $R_i(\theta)s_i(\theta) = \Phi(\bar{\Upsilon}_i(\bar{R}_i), d_{R_s}, N-1)$, $P(\theta) = \Phi(\bar{P}, d_P, N-1)$, $H(\theta) = \Phi(\bar{H} + D(\delta), d_H, N-1)$ in which $2d_{R_i}$, $2d_P$

and $2d_H$ are the degrees of $R_i(\theta)$, $P(\theta)$, and $H(\theta) - cI$, respectively.

Proof: See [18], Theorem 1. \square

B. Multi-Objective Controller Design

To achieve collision avoidance and connectedness maintenance, Lyapunov-like barrier functions are employed. For the brevity of expressions, let $y_i = x_i - \tau_i$, $y_{ij} = y_i - y_j$, $x_{ij} = x_i - x_j$, $\tau_{ij} = \tau_i - \tau_j$ and $\rho_{ij} = \rho_i - \rho_j$.

For connectedness maintenance, based on Assumption 2, the desired topology is contained in the initial graph. The main idea is to preserve the desired topology $\mathcal{E}^f \subseteq \mathcal{E}(t)$ such that the network is always connected for $t \geq t_0$. To do this, we would like to make the following condition satisfied: $\|x_{ij}\| < r_s$, for all $i \in \mathcal{N}$ and $j \in \mathcal{N}^{\text{sf}}(t)$ which holds if $r_s - \|\tau_{ij}\| - \|y_{ij}\| > 0$. Thus, the following function is used:

$$\Psi_{ij}^e = \frac{\|y_{ij}\|^2}{\hat{r}_s - \|y_{ij}\| + \frac{\hat{r}_s^2}{\mu_1}}, \quad i \in \mathcal{N}, j \in \mathcal{N}_i^{\text{sf}}(t), \quad (11)$$

where $\hat{r}_s = r_s - \|\tau_{ij}\|$, $\mathcal{N}_i^{\text{sf}}(t) = \{j \mid j \in \mathcal{N}_i^s(t), j \in \mathcal{N}_i^f(t)\}$ defined in Section II, μ_1 is a positive scalar such that Ψ_{ij}^e is bounded when $\|y_{ij}\|$ tends to \hat{r}_s .

For collision avoidance, the basic idea is to keep the distance between any two agents i and j greater than a minimum user-defined safety distance $d_s > 2r_c$. In other words, the condition is required that $\|x_{ij}\| > d_s$, which holds if and only if $\|y_{ij} + \tau_{ij}\| - d_s > 0$. Thus, we select the following barrier function:

$$\Psi_{ij}^c = \frac{(\|y_{ij} + \tau_{ij}\| - \|\tau_{ij}\|)^2}{\|y_{ij} + \tau_{ij}\| - d_s + \frac{(d_s - \|\tau_{ij}\|)^2}{\mu_2}}, \quad i \in \mathcal{N}, j \in \mathcal{N}_i^{\text{sz}}(t), \quad (12)$$

where $\mathcal{N}_i^{\text{sz}}(t) = \{j \mid j \in \mathcal{N}_i^s(t), \|x_{ij}\| < r_z\}$ introduced in Section II. μ_1 is a positive scalar such that Ψ_{ij}^e is bounded when $\|y_{ij}\|$ tends to d_s .

Remark 1: We assume μ_1 and μ_2 satisfying $\mu_1 > \mu_{\max}$ and $\mu_2 > \mu_{\max}$ with $\mu_{\max} := \frac{1}{2} \sum_{i=1}^N (\sum_{j \in \mathcal{N}_i^f} \Psi_{ij}^e (\|\hat{r}_s - \hat{\varepsilon}\|) + y_i(t_0)^T \sum_{j=1}^N G_{ij}(t_0) y_{ij}(t_0) + \rho_i(t_0)^T \rho_i(t_0)) + (N-1)N\Psi_{ij}^c (\|d_s - \hat{\varepsilon}\|)$, where $0 < \hat{\varepsilon} < \min\{\frac{1}{2}d_s - r_c, \varepsilon\}$. Barrier function in this paper is different with the existing work in the sense that in [3], [8]–[10], the objective of collision avoidance is not considered; in [10], [15], the control input is not bounded. \square

For the brevity of notations, let us introduce $\Psi_i^e = \sum_{j \in \mathcal{N}_i^{\text{sf}}} \Psi_{ij}^e$, $\Psi_i^c = \sum_{j \in \mathcal{N}_i^{\text{sz}}} \Psi_{ij}^c$, $x = (x_1^T, x_2^T, \dots, x_N^T)^T$, $\rho = (\rho_1^T, \rho_2^T, \dots, \rho_N^T)^T$. A distributed parameter-dependent controller is provided as follows:

$$\begin{aligned} u_i &= - \underbrace{\sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \nabla_{y_i} \Psi_{ij}^e (\|y_{ij}\|)}_{\alpha^e} - \underbrace{\sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \nabla_{y_i} \Psi_{ij}^c (\|y_{ij}\|)}_{\alpha^c} \\ &\quad - \underbrace{\sum_{j \in \mathcal{N}_i^s(t)} G_{ij}(t, \theta) y_{ij}}_{\beta^y} - \underbrace{\sum_{j \in \mathcal{N}_i^s(t)} G_{ij}(t, \theta) \rho_{ij}}_{\beta^p}, \end{aligned} \quad (13)$$

where θ is introduced in (3), G_{ij} is the ij -th entry of weighted adjacency matrix. The gradient-based terms α^e and α^c are designed to achieve connectedness maintenance and collision avoidance, respectively; the parameter-dependent terms β^y and β^ρ are designed to achieve the convergence of y_{ij} and ρ_{ij} , respectively. The following result shows that under the proposed controller (13), the connectedness of uncertain networks can be ensured.

Theorem 2: Under Assumption 1-3, and provided that (10) is satisfied for $\mathcal{G}(t_0, \theta)$, the uncertain graph $\mathcal{G}(t, \theta)$ is connected, and collision avoidance is guaranteed for all $t \geq t_0$ and for all $\theta \in \Omega$ by using the controller (13).

Proof: This proof aims to show the concerned set is in a positively invariant set, which implies the connectedness and collision avoidance in the uncertain network. Specifically, we assume that the edge set $\mathcal{E}(t)$ changes at $t_l, l = 0, 1, 2, \dots$. For each $[t_l, t_{l+1})$, \mathcal{G} is fixed. Let us introduce a parameter-dependent Lyapunov-like function

$$W = \frac{1}{2} \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \Psi_{ij}^e(\|y_{ij}\|) + \sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \Psi_{ij}^c(\|y_{ij}\|) + y_i \sum_{j=1}^N G_{ij}(t, \theta) y_{ij} + \rho_i^T \rho_i \right). \quad (14)$$

Consider the time interval $[t_0, t_1)$, one has $\Psi_{ij}^e > 0$ from (11), Ψ_{ij}^c from (12), and $\rho_i^T \rho_i \geq 0$. In addition, $\sum_{j=1}^N y_i \sum_{j=1}^N G_{ij}(t, \theta) y_{ij} = \sum_{j=1}^N y_i \sum_{j=1}^N L_{ij}(t, \theta) y_j = y^T (L(t, \theta) \otimes I_n) y \geq 0$ on account of the fact that $L(t, \theta) = L(t_0, \theta)$, and (10) is satisfied for $\mathcal{G}(t_0, \theta)$. Thus, one has that $W_0 = W(t_0, \theta) > 0$ for all $\theta \in \Omega$. Moreover, for $t \in [t_0, t_1)$, $G_{ij}(t, \theta)$ is fixed, one has

$$\begin{aligned} \dot{W} &= \frac{1}{2} \sum_{i=1}^N \left(\sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \dot{\Psi}_{ij}^e(\|y_{ij}\|) + \sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \dot{\Psi}_{ij}^c(\|y_{ij}\|) \right) \\ &\quad + \sum_{i=1}^N \dot{y}_i \sum_{j=1}^N L_{ij} y_j + \sum_{i=1}^N \rho_i^T \dot{\rho}_i \\ &= \sum_{i=1}^N \sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \dot{y}_i^T \nabla_{y_i} \Psi_{ij}^e(\|y_{ij}\|) + \sum_{i=1}^N \dot{y}_i \sum_{j=1}^N L_{ij} y_j \\ &\quad + \sum_{i=1}^N \sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \dot{y}_i^T \nabla_{y_i} \Psi_{ij}^c(\|y_{ij}\|) + \sum_{i=1}^N \rho_i^T \dot{\rho}_i \\ &= -\rho^T (L(t_0, \theta) \otimes I_n) \rho. \end{aligned} \quad (15)$$

Taking into account that (10) is satisfied for $\mathcal{G}(t_0, \theta)$, one has $L(t_0, \theta) \geq 0$ for all $\theta \in \Omega$, which implies that $\dot{W} \leq 0$. Thus, $W(t, \theta) \leq W(t_0, \theta) \leq \mu_{\max}$, for $t \in [t_0, t_1)$. From (11), (12) and Remark 2, one has that $\Psi_{ij}^e(\hat{r}_s) = \mu_1 > \mu_{\max}$, and $\Psi_{ij}^c(\hat{d}_s) = \mu_2 > \mu_{\max}$, which implies that no collision occurs during $[t_0, t_1)$, and no agent j has left the set $\mathcal{N}_i^{\text{sf}}$ for agent i . Hence, the network $\mathcal{G}(t, \theta)$ is still connected. Let us consider $t = t_1$, we assume that the number of new agents added in the set $\mathcal{N}_i^{\text{sz}}$ is k_i for agent i . One has that $\sum_{i=1}^N k_i + \sum_{i=1}^N \text{num}_i(\mathcal{N}_i^{\text{sz}}) \leq N(N-1)$, and $\text{num}_i(\mathcal{N}_i^{\text{sz}})$

is the number of agents in $\mathcal{N}_i^{\text{sz}}$. It yields that

$$\begin{aligned} W(t_1, \theta) &\leq W(t_1^-, \theta) + \sum_{i=1}^N k_i \tilde{\Psi} \\ &\leq W(t_0, \theta) + \sum_{i=1}^N k_i \tilde{\Psi} \\ &< \mu_{\max}, \end{aligned} \quad (16)$$

where $\tilde{\Psi} = \frac{1}{2} \sum_{j \in \mathcal{N}_i^{\text{sz}}} \Psi_{ij}^c(\|\hat{d}_s - \hat{\varepsilon}\|)$.

The above argument can be applied to time intervals $[t_l, t_{l+1})$. The condition still holds that $\dot{W}(t, \theta) \leq 0$, and one has

$$W(t, \theta) \leq W(t_l) \leq \mu_{\max}, \quad (17)$$

which implies that no collision occurs during $[t_l, t_{l+1})$, and no agent j has left the set $\mathcal{N}_i^{\text{sf}}$ for agent i . Hence, the graph $\mathcal{G}(t, \theta)$ is connected for $t \in [t_l, t_{l+1})$. \square

The above result achieves the objectives of last two items in Problem 1. Next, we will show that the objective of the first item in Problem 1 can also be achieved.

Theorem 3: If Assumption 1-3 hold and (10) is satisfied for $\mathcal{G}(t_0, \theta)$, then, under the controller (13), the following conditions hold for all $\theta \in \Omega, i \in \mathcal{N}$:

- 1) $\lim_{t \rightarrow \infty} \|\rho_i - \rho_j\| = 0$, for $j \in \mathcal{N}$;
- 2) $\lim_{t \rightarrow \infty} \|x_i(t) - \tau_i - (x_j(t) - \tau_j)\| = 0$, for $j \in \mathcal{N}_i^f$.

Proof: 1) For the first statement, we assume that the edge set $\mathcal{E}(t)$ changes at $t_l, l = 0, 1, 2, \dots$, and there exists a time \hat{t}_l such that the topology of \mathcal{G} is fixed. For $t \in [\hat{t}_l, \infty)$, from the construction of W , one has that

$$\frac{1}{2} \sum_{i=1}^N y_i \sum_{j=1}^N G_{ij}(t, \theta) y_{ij} \leq \mu_{\max}, \quad \frac{1}{2} \sum_{i=1}^N \rho_i^T \rho_i \leq \mu_{\max}.$$

Since the topology of \mathcal{G} is fixed, we have that G_{ij} is also fixed for $t \in [\hat{t}_l, \infty)$. On account of the symmetry of G , let λ_{\max} be the largest eigenvalue of G , one has that

$$\frac{1}{2} y^T (L(\hat{t}_l, \theta) \otimes I_n) y \leq \frac{1}{2} \lambda_{\max} \|y\|^2 \leq \mu_{\max},$$

which implies that $\|y\| \leq \sqrt{\frac{2\mu_{\max}}{\lambda_{\max}}}$. Via similar arguments, we obtain that $\|\rho\| \leq \sqrt{2\mu_{\max}}$. Let us consider the set $\Xi = \{y \in \mathbb{R}^{Nn}, \rho \in \mathbb{R}^{Nn} \mid W(y, \rho) \leq \mu_{\max}, \|y\| \leq \sqrt{\frac{2\mu_{\max}}{\lambda_{\max}}}, \|\rho\| \leq \sqrt{2\mu_{\max}}\}$, which is closed and bounded, and thus a compact set. Now, let us investigate the largest invariant set in $\mathcal{I} = \{y \in \mathbb{R}^{Nn}, \rho \in \mathbb{R}^{Nn} \mid \dot{W} = 0\}$.

Based on (15), one has

$$\dot{W} = -\rho(L(\theta) \otimes I_n)\rho = \frac{1}{2} \sum_{i \in \mathcal{N}, j \in \mathcal{N}_i^s} G_{ij} \|\rho_i - \rho_j\|^2,$$

which implies that $\dot{W} = 0$ if and only if $\rho_1 = \dots = \rho_N$. By using LaSalle's invariance principle, one has that all the trajectories started in the set Ξ will converge to set \mathcal{I} , i.e., $\rho_1 = \dots = \rho_N$.

2) For the second statement, consider the case of $t \geq \hat{t}_l$, one has $\rho_i - \rho_j = 0$ for all $i, j \in \mathcal{N}$. Then, (13) can be rewritten as

$$\begin{aligned}
u_i &= - \sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \nabla_{y_i} \Psi_{ij}^e(\|y_{ij}\|) - \sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \nabla_{y_i} \Psi_{ij}^c(\|y_{ij}\|) \\
&\quad - \sum_{j \in \mathcal{N}_i^{\text{s}}(t)} G_{ij}(t, \theta) y_{ij}, \\
&= - \sum_{j \in \mathcal{N}_i^{\text{sf}}(t)} \frac{\partial \Psi_{ij}^e(\|y_{ij}\|)}{\partial \|y_{ij}\|} \cdot \frac{1}{\|y_{ij}\|} y_{ij} - \sum_{j \in \mathcal{N}_i^{\text{s}}(t)} G_{ij}(t, \theta) y_{ij} \\
&\quad - \sum_{j \in \mathcal{N}_i^{\text{sz}}(t)} \frac{\partial \Psi_{ij}^c(\|y_{ij}\|)}{\partial \|y_{ij}\|} \cdot \frac{1}{\|y_{ij}\|} y_{ij}.
\end{aligned} \tag{18}$$

Observe that $\frac{\partial \Psi_{ij}^e(\|y_{ij}\|)}{\partial \|y_{ij}\|} \cdot \frac{1}{\|y_{ij}\|}$ is positive and bounded as $\|y_{ij}\| \rightarrow 0$, one has that $u_i = -(\tilde{L}(t) \otimes I_n + L(\theta, t) \otimes I_n)y$ with $\tilde{L}(t) \geq 0$ and $L(\theta, t) \geq 0$. From algebraic graph theory [16], it yields that $\lim_{t \rightarrow \infty} y = \text{span}(1_{Nn})$, i.e., $y_i - y_j = 0$, for all $i, j \in \mathcal{N}$. \square

IV. EXAMPLES

Two examples are provided for illustration. The MATLAB toolbox SeDuMi is used for solving LMI problems.

A. Example 3: A 6-agent case

With an initial topology shown in Fig. 2 (a), an uncertain 6-agent system is considered. The model of agents is set up with the parameters: $r_a = 0.75$, $r_s = 8$, $r_z = 2.5$, $r_c = 1.25r_a$, $d_s = 2r_c$, and $\epsilon = 0.1$.

It is assumed that the network is affected by two uncertain parameters, i.e. θ_1 and θ_2 . Specifically, the uncertain matrix $G(t_0, \theta)$ is given by

$$\begin{bmatrix}
0 & 3 + 2\theta_1 & 0 & 0 & 0 & 0 \\
3 + 2\theta_1 & 0 & 4 + \theta_1 & 0 & 0 & 0 \\
0 & 4 + \theta_1 & 0 & 3 - \theta_2 & 2\theta_1 + \theta_2 & 0 \\
0 & 0 & 3 - \theta_2 & 0 & 0 & 0 \\
0 & 0 & 2\theta_1 + \theta_2 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 0
\end{bmatrix}.$$

where $\theta \in \mathbb{R}^2$ is constrained in the set Ω chosen as $\Omega = \{\theta : \|\theta\| \leq 1\}$. Hence, we have $n = 4$ and $r = 2$. For the newly added edge (i, j) for $t > t_0$, we assume $G(i, j) = G(j, i) = 1$. Moreover, Ω can be described as in (3) with

$$s_1(\theta) = 1 - \theta_1^2 - \theta_2^2.$$

Let us use (6) to investigate the connectedness of network. We look for a constant matrix function $P(\theta)$ satisfying (6), and by solving (10) we find $c^* = 0.589$. Therefore, $\mathcal{G}(t_0, \theta)$ is connected.

Next, we apply the proposed controller (13), and the results are shown in Fig. 3. Let us observe that $\min(x_{ij})$ is always larger than d_s in Fig. 4. Thus, the collision avoidance is achieved. For the connectedness maintenance, by only preserving connections of $j \in \mathcal{N}_i^{\text{sf}}$, the controller allows breaks of edges when system evolves. Demonstrated by Fig. 2-3, our approach guarantees the connectedness of the uncertain network, and the robust formation is achieved.

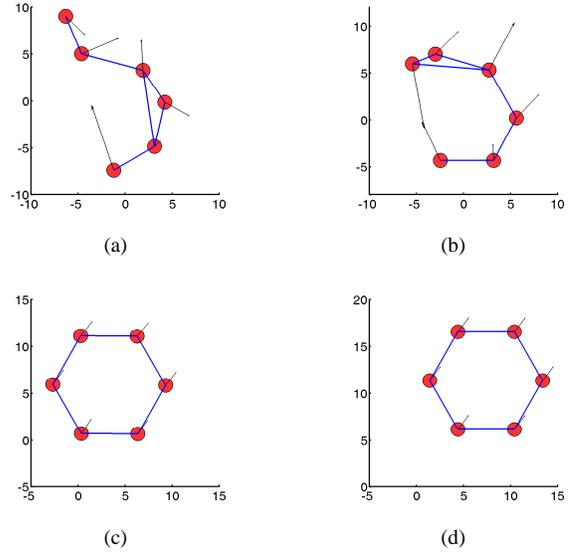


Fig. 2. Example 3: The motion of agents and the set of edges of network for $t = 0, 1, 3, 4$, respectively.

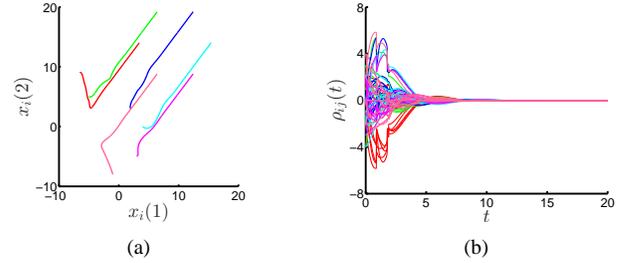


Fig. 3. Example 3: The trajectories of agents and the differences of velocities.

B. Example 4: A 50-agent case

In this scenario, a 50-agent system is considered with the parameters: $N = 50$, $r_a = 0.75$, $r_s = 5$, $r_z = 2.5$, $r_c = 1.25r_a$, $d_s = 2r_c$, and $\epsilon = 0.1$. The initial positions are selected as shown in Fig. 5, and the initial velocities are randomly chosen in $[-2, 2]$. The desired configuration is a circle of radius 30 on which all agents are symmetrically distributed. The communication are affected by three parameters. Due to limited space, we omit the expression of Ω and $G(\theta)$. By solving (10) we find $c^* = 0.642$, which ensures that $\mathcal{G}(t_0, \theta)$ is connected. The controller (13) guarantees the collision avoidance, and the robust formation is achieved as shown in Fig. 6-7. Finally, it is worth noting that the proposed method is scalable for a large number of agents.

V. CONCLUSION AND DISCUSSION

This paper addresses the robust formation control problem of multiple agents under parametric communication uncertainties. A distributed controller is proposed by introducing a parameter-dependent Lyapunov-like barrier function, which encodes control objectives like connectedness maintenance, collision avoidance and convergence to the desired configuration. First, a necessary and sufficient condition is proposed

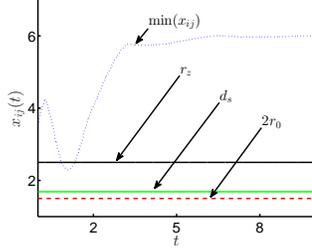


Fig. 4. Example 3: The minimal distance between agents.

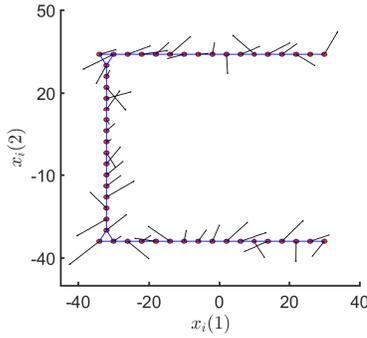


Fig. 5. Example 4: The initial topology and states.

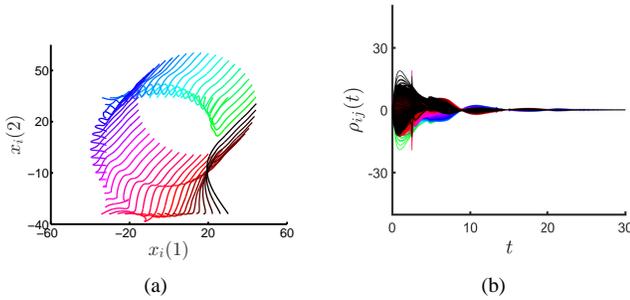


Fig. 6. Example 4: The trajectories of agents and the differences of velocities.

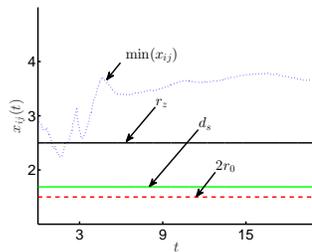


Fig. 7. Example 4: The minimal distance between agents.

for checking the connectedness of uncertain networks under communication uncertainties. Based on this condition, a solvable condition consisting of LMIs is given by employing the square matrix representation. Second, a parameter-dependent Lyapunov-like barrier function is introduced, and a gradient-based controller with bounded input is designed such that the collision avoidance and connectedness maintenance can be achieved. Finally, this paper also shows that the systems converges to the desired formation configuration by using LaSalle invariance principle.

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