SOLUTIONS TO THE JANUARY 7, 2013 UNIVERSITY OF MICHIGAN QUALIFYING EXAM IN TOPOLOGY

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1. Problem 1

(1) Show that if X and Y are connected topological spaces, then $X \times Y$ is also connected.

Proof. First we will need a quick lemma. Suppose $X = U \cup V$, U and V are connected, and $U \cap V = \emptyset$. Then it follows that X is connected as well. To prove this we note that $X = \coprod X_i$ where the X_i are the connected components of X. Since U and V are each connected and have nonempty intersection, it follows that U and V belong to the same connected component. Since their union is X, it follows that X consists of exactly one connected component, and thus one may conclude that $X = U \cup V$ is connected. Now, let $x \in X$ and $y \in Y$. We now have that $X \times \{y\}$ is homeomorphic to X, and that $\{x\} \times Y$ is homeomorphic to Y. Since connectedness is a topological property, we can conclude that $X \times \{y\}$ and $\{x\} \times Y$ are connected. Since their intersection in nonempty we conclude from the opening remarks that $X \times Y$ is connected.

□

(2) Recall that $\mathbb{R}^\omega$ denotes the product of countably many copies of R with the product topology, and that $\mathbb{R}^\infty$ denotes the subspace of $\mathbb{R}^\omega$ which consists of points that have only finitely many non-zero coordinates. Show that $\mathbb{R}^\infty$ is connected.

Proof. To prove this, we note that it is actually path-connected. To see this, just take any two points in $\mathbb{R}^\infty$ and start at one and then go along each coordinate until you get to one that differs between the two points. Now take the straight-line path connecting these two points and repeat. This will only take finitely many steps, and one thus has a path between the two points. Since path-connected implies connected, we are done. □

Date: Sunday, February 24, 2013.

2. Problem 2

Let $\sim$ be the smallest equivalence relation on $\mathbb{R}^n$ such that $x \sim -x$ for every $x \in \mathbb{R}^n$. For what values $n = 1, 2, 3, ...$ is $X = \mathbb{R}^n / \sim$ (with the qoutient topology) a topological manifold?

Proof. For $x \in \mathbb{R}^n$, let $[x]$ denote its equivalence class in X. Now, note that $\mathbb{R}^n \backslash \{0\}$ deformation retracts to S^{n-1}. Since $S^{n-1} \backslash \sim$ is homeomorphic to $\mathbb{RP}^{n-1}$, we have that $X \backslash \{[0]\}$ deformation retracts to $\mathbb{RP}^{n-1}$. Since for $\mathbb{R}^n \backslash \{0\}$ we have that $H_{n-1}(\mathbb{R}^n \backslash \{0\}) = \mathbb{Z}$ for $n > 2$ and $H_{n-1}(X \backslash \{[0]\} = \mathbb{Z}_2$ for $n > 2$, we have that no neighborhood of $[0]$ can be homeomorphic to euclidean space if $n > 2$. Now we consider the remaining two cases separately. The case $n = 1$ is easy for then X is homeomorphic to a ray and is thus not a topological manifold. The case $n = 2$ is best done via pictures.

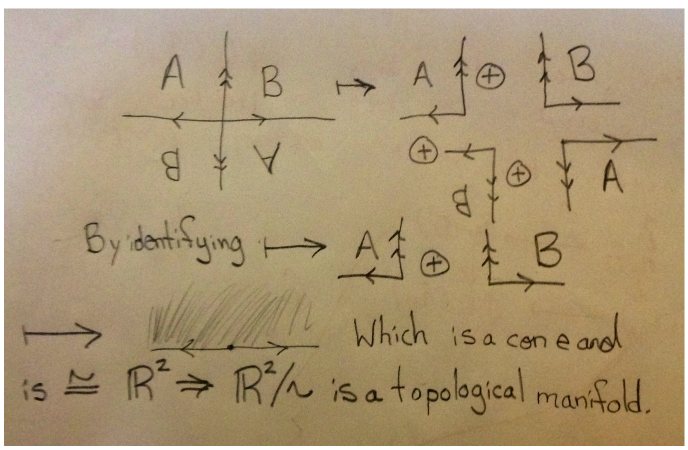

Figure 1. The case $n = 2$

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3. Problem 3

For which values of c is the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + 3y^2 - 6y + z^2 + 4z = c\}$$

a nonempty smooth 2-dimensional submanifold of $\mathbb{R}^3$? Explain your answer carefully for all values of c.

Proof. Note that

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2x + 3y^2 - 6y + z^2 + 4z = c\} =$$

$$\{(x, y, z) \in \mathbb{R}^3 : (x+1)^2 + 3(y-1)^2 + (z+2)^2 = 8 + c\}$$

.

Now, consider the function $f : \mathbb{R}^3 \to \mathbb{R}$. The Jacobian of this function evaluated at a point (a_1, a_2, a_3) is $(2(a_1+1), 6(a_2-1), 2(z+2))$. This $=$ $(0, 0, 0)$ iff $(a_1, a_2, a_3) = (-1, 1, -2)$. Therefore by the regular value theorem, we have that $f^{-1}(a)$ is a 2-dimensional manifold if $c > -8$. If $c = -8$ then $a = 0$, and $f^{-1}(0) =$ finite number of points, which is a 1-dimensional manifold and not a 2-dimensional manifold. Thus, we conclude that the described set is a 2-dimensional manifold iff $c > -8$.

□

4. Problem 4

Let $T = S^1 \times S^1$ and let $x, y \in T$ be two distinct points. Let Y be the qoutient space obtained from $T \times \{1, 2\}$ by identifying the points $(x, 1)$ and $(x, 2)$ into a single point $\bar{x}$, and identifying the points $(x, 1)$ and $(x, 2)$ into a single point $\bar{x}$, and identifying the points $(y, 1)$ and $(y, 2)$ into a single point $\bar{y}$, distinct from $\bar{x}$. Compute $\pi_1(Y, \bar{x})$.

Proof. Note that the space in question is just the disjoint union of two tori with two points on each tori identified in such a way that the space deformation retracts to $S^1 \wedge T \wedge T$. The fundamental group of this space is given by $F_1 * F_2 * F_2$. □

5. Problem 5

Let X be a Hausdorff space and let $f : X \to \mathbb{R}^n$ be a proper injective continuous function. Show that f is a homeomorphism of X onto $f(X)$. (Recall that a continuous map is proper if the pre-image of any compact set is compact.)

Proof. I will solve the problem by first showing that f is a closed map. Let $K \subset X$. I wish to show that $f(K)$ is closed. I will do this by showing that $f(K)$ contains all of its limit points. Let y be a limit point of $f(K)$. Then, y belongs to some some closed compact neighborhood $U \subset \mathbb{R}^n$. Furthermore, y is a limit point of $f(K)$ iff y is a limit point of $f(K) \cap U$. since U is compact, it follows that $f^{-1}(U)$ is also compact since f is a proper map. Therefore, $K \cap f^{-1}(U)$ is compact. By continuity, it follows that $f(K \cap f^{-1}(U))$ must also be compact, and hence closed since it lives inside of $\mathbb{R}^n$. Note that $f(K \cap f^{-1}(U)) = f(K) \cup U$ is closed. Therefore, $y \in f(K) \cup U \subset f(K)$, which implies that $f(K)$ contains all of its limit points and is therefore closed. Now, note that a bijective continuous closed mapping is a homeomorphism. □

6. Problem 6

Let T denote the torus $S^1 \times S^1$ and let M denote the Moebius band. Let C be a simple closed curve in T which bounds a 2-disk. Form a space X by identifying the boundary of M with C by a homeomorphism. Compute all the homology groups of X.

Proof. First we will produce a CW-complex structure on the space X. Here it is.

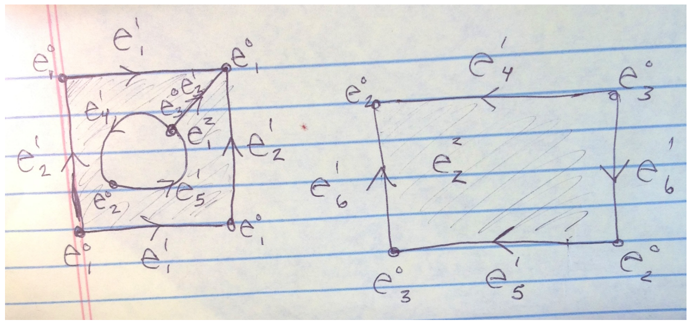

FIGURE 2. CW-complex structure on X

Furthermore, we have that $\partial_2(e_1^2) = e_4^1 + e_5^1$, $\partial_2(e_2^2) = 2e_6^1 - e_4^1 + e_5^1$. From this we can conclude that $\ker \partial_2 = 0$. Therefore, $H_2(X) = 0$. We also have that $\partial_1(e_1^1) = \partial_1(e_2^1) = 0$. Also, $\partial_1(e_3^1) = e_1^0 - e_3^0$ and $\partial_1(e_6^1) = \partial_1(e_4^1) = e_2^0 - e_3^0 = -\partial(e_5^1)$. Therefore, $\ker \partial_1 =< e_1^1, e_2^1, e_6^1 - e_4^1, e_4^1 + e_5^1 >$. We conclude that $H_1(X) = \ker \partial_1 / \mathrm{Im}\ \partial_2 =< e_1^1, e_2^1, e_6^1 - e_4^1, e_4^1 + e_5^1 > / < e_4^1 + e_5^1, 2e_6^1 - e_4^1 + e_5^1 >=< e_1^1, e_2^1, e_6^1 - e_4^1, e_4^1 + e_5^1 > / < e_4^1 + e_5^1, 2e_6^1 - 2e_4^1 >= \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$. Since X is connected, we have that $H_0(X) = \mathbb{Z}$. □

7. PROBLEM 7

Let $f : X \to Y$ be a continuous map of spaces where X is compact Hausdorff, Y is Hausdorff and both spaces are path-connected and locally path-connected. Suppose that for every $x \in X$ there exists an open neighborhood U such that $f(U)$ is open in Y and $f|U : U \to f(U)$ is a homeomorphism. Is f necessarily a covering?

Yes it is.

Proof. First we will show that f is surjective. X is compact. By continuity, $f(X)$ must also be compact in Y. But $Y$is Hausdorff, so compactness in Y implies closed in Y. Now, since f is a local homeomorphism, it implies that f is an open mapping, which further implies that $f(X)$ is open in Y. Since Y is connected and $f(X)$ is open, closed, and nonempty in Y, it follows that $f(X) = Y$ which implies that f is surjective. We will now show that if $y \in Y$, then $f^{-1}(y)$ is finite. Suppose that $f(x) = y$. Then since f is a local homeomorphism, we have that there exists some open set $U_x \subset X$ such that $U_x \cong f(U_x)$. It follows that $U_x \cap f^{-1}(y) = \{x\}$. Therefore, $f^{-1}(y)$ must be a discrete set, and since it is also a closed subset of a compact space it follows that $f^{-1}(y)$ is a finite set. Now we will show that f is an even covering. Let $f^{-1}(y) = \{x_1, x_2, ..., x_n\}$. Since f is a local homeomorphism, and since X is Hausdorff, we can find pairwise disjoint open sets $U_{x_1}, ..., U_{x_n}$ in X around $x_1, .., x_n$ respectively that are homeomorphic onto their image. Let $V = \bigcap_{i=1}^{n} f(U_{x_i})$. Then, let $V_{x_i} = U_{x_i} \cap f^{-1}(V)$. Then, $V_{x_i} \cong f(V_{x_i}) = V$. This shows that f is an even covering. Thus f is a covering map. $\square$

8. Problem 8

Which of the following spaces nontrivially cover themselves?

(1) S^3.

The question is essentially asking us if there exists a k-sheeted covering map $(k > 1)$ $\rho : S^3 \to S^3$. If ρ is a covering map, then it has $[\pi_1(S^3) : \rho(\pi_1(S^3))]$ sheets. But, S^3 is simply connected, so that $\pi_1(S^3) = 0$. Therefore, it follows that S^3 cannot nontrivially cover itself.

(2) $S^1 \times S^1$.

S^1 covers itself by the map $\rho : S^1 \times S^1$ given by $\rho(z) = z^2$. This gives us a 2-sheeted covering. Therefore, $S^1 \times S^1$ nontrivially covers itself via the map $\rho \times \rho$.

(3) $S^1 \times S^2$.

Let $Id : S^2 \times S^2$ be the identity map. Then letting ρ be as before, we have that $S^1 \times S^2$ nontrivially covers itself via the map $\rho \times Id$.

(4) The closed orientable surface of genus 2.

First, recall that if $\rho : X \to Y$ is a k-sheeted covering map, then $\chi(X) = k \cdot \chi(Y)$. The closed orientable surface of genus 2 is just $T\#T$. The torus has a CW-complex structure consisting of one 0-cell, two 1-cells, and one 2-cell. Therefore $\chi(T) = 1 - 2 + 1 = 0$. It follows that $\chi(T\#T) = 0+0-2 = -2$. Therefore if $\rho : T\#T \to T\#T$ is a covering map, we have that $-2 = k \cdot -2$, which implies that $k = 1$. In conclusion, there does not exist a nontrivial covering.

9. Problem 9

Let M be a connected topological manifold. Prove rigorously that for two points $x, y \in M$, the spaces $M \backslash \{x\}$ and $M \backslash \{y\}$ are homeomorphic.

Proof. First off, we note that if $\mathbb{D}^n$ is the closed unit disk in $\mathbb{R}^n$, and $x, y \in \mathbb{D}^n$, then there exists a homeomorphism $h : \mathbb{D}^n \to \mathbb{D}^n$ which is the identity on $\partial\mathbb{D}^n$ and such that $h(x) = y$. Now, let $\gamma : [0, 1] \to M$ be a path in M such that $\gamma(0) = \gamma(1)$. Now, consider open sets $U \subset M$ such that U admits a chart $\phi : U \to \mathbb{R}^n$ such that $\phi(U)$ is the open disk in $\mathbb{R}^n$. Now consider the sets $\gamma^{-1}(U)$. These form an open cover for $[0, 1]$. Choose a finite subcover $\{\gamma^{-1}(U_1), ..., \gamma^{-1}(U_N)\}$. We can choose $t_0 = 0, ..., t_N = 1$ such that $\gamma(t_i)$ and $\gamma(t_{i+1})$ both lie in the same chart U_i. This produces a sequence of homomorphisms

$$M \backslash \{\gamma(t_0)\} \to ... \to M \backslash \{\gamma(t_N)\}$$

□

10. Problem 10

The cone CW on a topological space X is defined to be a quotient space of $X \times [0, 1]$ obtained by identifying $X \times \{0\}$ to a point. Prove or provide counterexamples to each of the following two statements.

(a) X is Hausdorff if and only if CX is Hausdorff.

Proof. We will first deal with the forward direction. Suppose that X is indeed Hausdorff. Now let $[(x_1, y_1)]$ and $[(x_2, y_2)]$ be two points in X. First assume that neither y_1 or y_2 is 0. Since $[(x_1, y_1)]$ and $[(x_2, y_2)]$ are distinct points. We have that either $x_1 \neq x_2$ or $y_1 \neq y_2$. Suppose that $x_1 \neq x_2$. Then since x is a Hausdorff space, we can find disjoint open neighborhoods U_{x_1} and U_{y_2} around x_1 and x_2 respectively. Then the sets $U_{x_1} \times (0, 1)$ and $U_{x_2} \times (0, 1)$ are disjoint open sets in CX. Similarly, if instead $y_1 \neq y_2$, then since $(0, 1)$ is Hausdorff, we can find disjoint open sets U_{y_1} and U_{y_2} contained in $(0, 1)$ around y_1 and y_2 respectively. Then we have that $X \times U_{y_1}$ and $X \times U_{y_2}$ are disjoint open sets in CX. Now, suppose without loss of generality that y_1 is 0. This automatically means that y_2 is not 0. Then we can find disjoint open sets U_{y_1} and U_{y_2} in $[0, 1]$. Then the sets $X \times U_{y_1}$ and $X \times U_{y_2}$ are disjoint open sets in CX, and so we conclude that CX is Hausdorff. Now, suppose that CX is Hausdorff. We wish to conclude that X is Hausdorff as well. Let $x_1 \neq x_2$ and consider the points $[x_1, 1]$ and $[x_2, 1]$ in CX. Then since CX is Hausdorff, we can find disjoint open sets U_{x_1} and U_{x_2} around $[x_1, 1]$ and $[x_2, 1]$ respectively in CX. Now consider the open set $V_{x_1} = U_{x_1} \cap X \times (\frac{1}{2}, 1]$ and $V_{x_2} = U_{x_2} \cap X \times (\frac{1}{2}, 1]$ around $[x_1, 1]$ and $[x_2, 1]$ respectively. These are 2 open sets in $CX \subset X \times (\frac{1}{2}, 1]$ and it is easy to see that these sets being open in CX is equivalent to them being open in $X \times [0, 1]$. Therefore (after a little work this is clear, I skipped a couple of obvious details), $(x_1, 1)$ and $(x_2, 1)$ are contained in disjoint open sets $U_{x_1} \times V$ and $U_{x_2} \times V$ where U_{x_1}, U_{x_2} are open in X and V is open in $[0, 1]$. This shows that X is Hausdorff. □

(b) X is connected if and only if CX is connected.

This is false. Just consider the space X, where X is a disjoint union of two copies of $\mathbb{R}$.