 Analysis QR September 7, 2013

(1) Note that μ^* defines an outer measure.

Thus, we have that $\mu^*(X) =$

$\mu^*(X \setminus A^c \cup A^c) \leq \mu^*(X \setminus A^c) + \mu^*(A^c)$

$= \mu^*(A) + \mu^*(A^c) \Rightarrow$

$\mu^*(X) - \mu^*(A^c) \leq \mu^*(A)$

But, $\mu^*(X) = \mu(X)$ so we have

$\cancel{\mu_*(A) \leq \mu_*(}$ $\mu_*(A) \leq \mu^*(A)$. //

(2) Consider the map $x \mapsto x_n$. This map is unambiguous for all numbers which ~~cannot~~ do not have $x_n = 9$ for all but finitely many n. I.e., $0.100000\ldots0\ldots = 0.09\overline{9}$. Therefore, we can choose to always write numbers like 0.1 as $0.0\overline{9}$. Then $x \mapsto x_n$ is ~~was~~ well-defined. (i.e. in $0.0102\ldots$ $x_2 = 1$) Also, We only ~~do~~ had to make a choice on a set of measure zero Since the set of all point eventually ending in all zeros or all 9s is a countable union of finite sets $\Rightarrow$ countable.

> Since proj[illegible] continuous, the product of continuous fns is continuous + continuous $\Rightarrow$ measurable

Therefore, the map given by $f_n(x,y) = x_n y_n$ is measurable, So $f_n^{-1}(5)$ is measurable. We have that $A = \bigcap_{n=1}^{\infty} f_n^{-1}(5)$ which is a Countable intersection of measurable sets $\Rightarrow$ measurable. Now, $\mathbb{1}_A(x,y)$, the indicator function on A is measurable since A is, so we can apply Fubinis Thm for non-negative measurable fns. We have

$$\int_{I\times I} \mathbb{1}_A(x,y)\, d(x\times y) = m(A) = \int_I \int_I \mathbb{1}_A(x,y)\, dx\, dy.$$

But, $x_n y_n = 5 \ \forall n \in \mathbb{N} \Rightarrow$ if $(x,y) \in A$ and we Know x, then we Know y and vice versa. I.E. x determines y + vice versa.

Thus, for fixed $y \in I$, $\mathbb{1}_A(x,y) = 0$ for all but at most one $x \Rightarrow \mathbb{1}_A(x,y) = 0$ a.e. for fixed $y \Rightarrow \int_I \int_I \mathbb{1}_A(x,y)\,dx\,dy = \int_I 0\,dy = 0 = m(A)$. So, A is measurable and has measure 0. //

(3) Prove $\lim_{n \to \infty} \int_{\mathbb{R}} \frac{\cos^n(\pi x)}{(x-n)^2+1} \, dx$ ~~exists~~ exists + find it.

We use the change of variables $x = y+n$

$\Rightarrow dx = dy$. So,

$$\int_{\mathbb{R}} \frac{\cos^n(\pi x)}{(x-n)^2+1} \, dx = \int_{\mathbb{R}} \frac{\cos^n(\pi(y+n))}{y^2+1} \, dy$$

Let $f_n(y) = \frac{\cos^n(\pi(y+n))}{y^2+1}$. Then,

$f_n(y) \leq \frac{1}{y^2+1} = f(y) \quad \forall n \Rightarrow$

Also, f measurable + $\int_{\mathbb{R}} f(y) \, dy$

$= \tan^{-1}(y) \Big|_{-\infty}^{\infty} = \pi \Rightarrow f$ integrable,

So f dominates the f_n and so by the Dominated Convergence Theorem, we have that

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(y) \, dy = \int_{\mathbb{R}} \lim_{n \to \infty} f_n(y) \, dy$$

But, $\cancel{\cos^n(\pi(y+n))}$ $|\cos(\pi(y+n))| < 1$ for almost all $y \Rightarrow |\cos^n(\pi(y+n))| \mapsto 0$ as $n \mapsto \infty \Rightarrow \lim_{n \mapsto \infty} f_n(y) = 0$

and so $\displaystyle \lim_{n \mapsto \infty} \int_{\mathbb{R}} \frac{\cos^n(\pi x)}{(x-n)^2+1}\, dx = 0.$ //

(4) $(X, \mathcal{A}, \mu)$ is σ-finite and $\mu(X) = +\infty$

$$\Rightarrow X = \bigcup_{n=1}^{\infty} A_n, \ \mu(A_n) < +\infty \ \forall n$$

Now define $B_n = A_n \setminus \bigcup_{m=1}^{n-1} A_m$. Then $X = \bigcup_{n=1}^{\infty} B_n$ so we may assume the A_n are disjoint. Also, if an A_n is such that $m(A_n) = 0$ then we can get rid of it so we still have $\mu(X) = \mu(\bigcup_{n=1}^{\infty} A_n) = +\infty$ where $\mu(A_n) > 0$ and $< +\infty$. So, we have constructed a sequence of sets $A_1, A_2, \ldots$ such that $\bigcup_{n=1}^{\infty} A_n$ has measure $+\infty$, each A_n has measure $\in (0, +\infty)$ and A_n are disjoint. Now, define $F: X \mapsto \mathbb{R}$ by

$$F(x) = \sum_{n=1}^{\infty} n^{-p} [\mu(A_n)]^{-1} \mathbb{1}_{A_n}(x)$$

Then, we have that

$$\left[\int_X |F(x)|^p \, d\mu\right]^{1/p} =$$

$$\left[\sum_{n=1}^{\infty} n^{-p}\right]^{1/p} < +\infty \iff$$

$$\sum_{n=1}^{\infty} n^{-p} < +\infty \iff p > 1$$ by Calc II

p-test. //

QR Part II

(1) No, such a function does not exist. First off, there is no polynomial $p(z)$ satisfying $p(e^n)=n$. Suppose there was, $p(z)=\sum_{m=0}^{N} a_m z^m$ where $N\neq 0$. Then for n large, $p(e^n)\approx a_N e^{nN}=n$

$\Rightarrow a_N>0$, $N\neq 0$ since then p is constant

$\Rightarrow \Rightarrow\Leftarrow$ since Ae^{Bx} grows faster than x for $A,B>0$. Therefore, f is not a polynomial $\Rightarrow f$ has an essential singularity at ∞.

By Picard's Thm $\exists$ a sequence $z_n \mapsto \infty$ with $f(z_n)\longmapsto 0$ since

(5) WLOG, we may assume f_n are nonnegative. Then, we have that each $f_n = \mu_n(x) = \mu_n([0,x])$ for some nonnegative measure μ_n. Similarly, we have that $\cancel{\mu(x)} = f(x) = \mu([0,x])$.

We can write $d\mu_n = f_n' d\lambda + d\nu_n$ Where λ is Lebesgue Measure and we have that ν_n is singular. Similarly we have that $d\mu = f' d\lambda + d\nu$. Let N be a Borel null set such that $\nu_n([0,1] \setminus N) = 0 \; \forall n$, $\nu([0,1] \setminus N) = 0$. Then, we have that $\sum_n \mu_n([0,x]) = \mu([0,x]) \; \forall x$. It follows that $\sum_n \mu_n(E) = \mu(E)$ for Borel measurable sets.

So, if E is borel measurable such that

$$E \cap N = \emptyset \Rightarrow \int_E \sum_n f_n' \, d\lambda =$$

$$\sum_n \int_E f_n' \, d\lambda$$ By Monotone Convergence Thm

$$= \sum_n \mu_n(E) = \int_E f' \, d\lambda$$ which shows

$$\sum_{n=0}^{\infty} f_n' = f' \text{ a.e.}$$ //

every nbhd of ∞ is dense in $\mathbb{C}$

$\Rightarrow$ we can choose $z_n \in B^n(\infty) \setminus \{\infty\}$,

$B^n(\infty)$ some nbhd of ∞, $B^{n+1}(\infty) \subset B^n(\infty)$

and $z_n \in B^n(\infty)$, $z_n \notin B^{n+1}(\infty)$. with

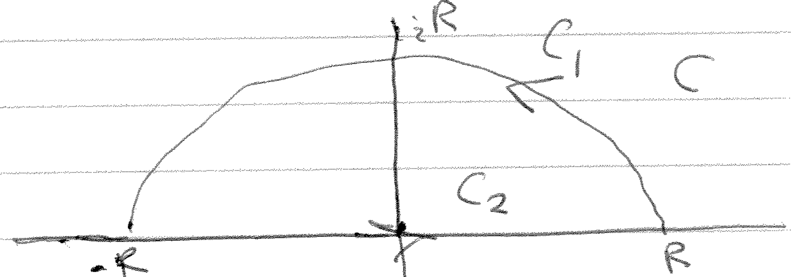
$f(z_n) \in B_{1/n}(0)$. Then, $z_n \mapsto \infty$ and

$f(z_n) \mapsto 0$ $\Rightarrow$ No fn exists. //

(2) $$\int_{\mathbb{R}} \frac{\cos x}{1+x^2} e^{-ixt}\, dx = \frac{1}{2}\int_{\mathbb{R}} \frac{e^{ix(1-t)} + e^{-ix(1+t)}}{1+x^2}\, dx$$

We will compute $\displaystyle\int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\, dx$

Case I, $\alpha > 0$, Consider the contour



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Then,

$$\int_C \frac{e^{\alpha i z}}{1+z^2}\, dz = \int_{C_1} \frac{e^{\alpha i z}}{1+z^2}\, dz + \int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\, dx$$

$$= 2\pi i \left[\frac{e^{-\alpha}}{2i}\right] = \pi e^{-\alpha} = \int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\, dx$$

(5)
$$f(e^{1/z}) = \frac{1}{z}$$

for $\frac{1}{n} \mapsto 0$

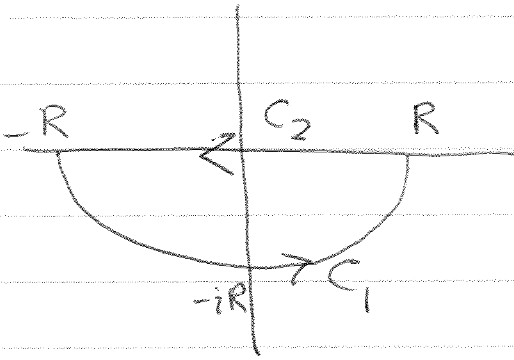
e^{n} ~~$\frac{1}{n}$~~ n

$$-\mathrm{Log}(e^{-n}) = f(e^{-n})$$

$$-\mathrm{Log} = -\mathrm{Log}(e^{-n}) = f(e^{n})$$

$-\cancel{h} = g = f$ on e^{-n}

Case 2, $\alpha < 0$, then use the contour C



Then, $$\int_C \frac{e^{\alpha i z}}{1+z^2}\,dz = \int_{C_1} \frac{e^{\alpha i z}}{1+z^2}\,dz - \int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\,dx$$

$$\Rightarrow 2\pi i\left[\frac{e^{\alpha}}{\cancel{-}2i}\right] = \cancel{-}\int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\,dx$$

$$\Rightarrow e^{\alpha}\pi = \int_{\mathbb{R}} \frac{e^{\alpha i x}}{1+x^2}\,dx$$

Now, for $t \in (0,1)$ we have that

$$\int_{\mathbb{R}} \frac{\cos x}{1+x^2} e^{-ixt}\, dx =$$

$$\frac{\pi}{2}\left[e^{-(1-t)} + e^{-(1+t)}\right] = \cancel{\frac{e^{t}\pi}{2}\left[e^{1}+e^{1}\right]}$$

$$\cancel{= e^{t}\pi\cosh(1).//}$$

~~for $t \in (1,\infty)$ we have~~

$$\cancel{= \frac{\pi}{2}\left[e^{(1-t)} + e\right]} \quad = \frac{\pi}{2e}\left[e^{t} + e^{-t}\right]$$

$$= \frac{\pi}{e}\cosh(t).// \qquad \text{For } t \in (1,\infty)$$

$$= \frac{\pi}{2}\left[e^{(1-t)} + e^{-(1+t)}\right] = \frac{\pi}{e^{t}}\cosh(1).//$$

$$\text{For } t = 1 \quad = \frac{1}{2}\tan^{-1}(x)\Big|_{-\infty}^{\infty} + \frac{\pi}{2}e^{-(1+t)}$$

$$= \frac{\pi}{2}\left[1 + e^{-(1+t)}\right].//$$

By Min Modulus Principle

(3) (a) Suppose $f(0) \neq 0$, then $f(z) \neq 0$ on $t\mathbb{D}$, $t \in (0,1)$. So, we have that f attains it ~~max~~ min on $t\overline{\mathbb{D}}$ on $\partial t\overline{\mathbb{D}}$. Call this pt z_0. Then, we have that $|f(z_0^2)| \leq |f(z_0)| \Rightarrow f(z)$ is constant $\Rightarrow \rightarrow\leftarrow \Rightarrow f(0)=0$.

(b) Suppose $z_0 \in 0<|z|<1$, $f(z_0)=0$. Then $f(z_0^{2^n})=0$ $\forall n \in \mathbb{Z}^{\geq 0}$ $\Rightarrow f(z)=0$ on some sequence converging to $z=0$ since $f(0)=0$ we can apply the Identity Thm to conclude $f \equiv 0$ on $\mathbb{D} \Rightarrow \rightarrow\leftarrow$. This proves Claim. //

(c) We have 0 is a pole of $f(z)$ so we have

$f(z) = z^n g(z)$ with $g(z) \neq 0 \; \forall z$

~~and $g(z)$~~ $\lim_{z \to 0} g(z) \neq 0, + < +\infty$

Now, define $h(z) = \dfrac{g(z^2)}{g(z)} = \dfrac{z^{-2n} \cancel{g} f(z^2)}{z^{-n} f(z)}$

$= \dfrac{f(z^2)}{f(z)} z^{-n} \Rightarrow |h(z)| \leq |z|^{-n}$.

on $t\overline{\mathbb{D}}$, $t \in (0,1)$, $h(z)$ attains its Max on

$\partial t\overline{\mathbb{D}} \Rightarrow |h(z)| \leq t^{-n} \; \forall t \in (0,1)$.

as $t \to 1$ we have $|h(z)| \leq 1$

$\Rightarrow |g(z^2)| \leq |g(z)|$. Since $g(0) \neq 0$

$\Rightarrow$ g is constant (By same argument in (a))

$\Rightarrow f(z) = az^n$, $a \in \mathbb{C}$, $n \in \mathbb{N}$. //

(4) There is a unique conformal map

$g: \Omega \mapsto \mathbb{D}$, $g(3) = \cancel{0}\,1$, $g'(3) > 0$.

Let $h(z) = \frac{\alpha - z}{1 - \bar{\alpha} z}$, $\alpha \in \mathbb{D}$.

Then, Let $f(z) = h(g(z))$.

$f'(z) = h'(g(z))g'(z)$

So, $f'(3) = \cancel{h'(0)g'(3)}$. $h'(1)g'(3)$

$$h'(z) = \frac{\alpha\bar{\alpha} - 1}{(1 - \bar{\alpha} z)^2} \Rightarrow h'(1) = \frac{|\alpha|^2 - 1}{(1 - \bar{\alpha})^2}$$

Let $\phi(t) = \frac{t^2 - 1}{(1-t)^2}$. We will use

L'Hopital to calc $\lim_{t \mapsto 1} \phi(t)$.

$$\lim_{t \mapsto 1} \phi(t) = \frac{-2t}{2(1-t)} = -\infty \text{ is}$$

we approach t from below.

So, let $h_n(z) = \dfrac{\left(\frac{n}{n+1}\right) - z}{\left(1 - \left(\frac{n}{n+1}\right)z\right)^2}$

Then, $\lim\limits_{n \mapsto \infty} h_n(1) = -\infty$

Let $f_n(z) = h_n(g(z))$.

Then, $f_n'(3) = h_n'(1)\, g'(3)$

and $g'(3) \neq 0 \Rightarrow$

$\lim\limits_{n \mapsto \infty} |f_n'(3)| = \lim\limits_{n \mapsto \infty} |h_n'(1)|\,|g'(3)| = +\infty$

Thus, $\sup = +\infty$ or you could say it never attains it because it can be arbitrarly large. But same thing.

(5) Each f_n misses the 2 pts 5, 6 so by Montels Thm we have that $\{f_n\}$ form a Normal Family, So every subsequence has a subsequence converging on $\frac{1}{2}\overline{\mathbb{D}}$ to some holomorphic function $f(z)$ with $f(z) = \lim_{n\to\infty} f_n(1/k)$ for $k = 2, 3, 4, 5, \ldots \Rightarrow$

$f(z)$ is defined on all ready on $\{1/2, 1/3, 1/4, 1/5, \ldots\} \cup \{0\} \Rightarrow$ By identity principle $f(z)$ is uniquely defined since it is on a sequence with limit point.

Therefore, ~~$\{f_n\}$~~ ever subsequence of $\{f_n\}$ has a subsequence $\mapsto f$ on $\frac{1}{2}\overline{\mathbb{D}} \Rightarrow$ on $\frac{1}{2}\mathbb{D}$, $\{f_n\} \mapsto f$. But $1/2$ was arbitrary, and ~~for~~ we could have chosen any α such that $0<\alpha<1$. So,

$$\{f_n\} \mapsto f \text{ on } \alpha\mathbb{D}, \ 0<\alpha<1 \Rightarrow$$

$$\{f_n\} \mapsto f \text{ on } \bigcup_{m=2}^{\infty} (1-1/m)\mathbb{D}$$

$$\Rightarrow \{f_n\} \mapsto f \text{ on } \mathbb{D}. //$$