Analysis Qualifying Review May 8, 2014

(1) Consider $f(z) = e^{-2\pi} e^{2\pi i z}$.

$f(z)$ is obviously an entire function. We also have that $|f(z)| \leq e^{-2\pi} e^{2\pi|z|} < 1$ on $\mathbb{D}$

$\Rightarrow f : \mathbb{D} \mapsto \mathbb{D}$. Clearly, we have

$f(-1/2) = e^{-2\pi} e^{2\pi i(-1/2)} = e^{-2\pi} e^{2\pi i(-1/2+1)}$

$= f(1/2)$. Also, $f'(z) =$

$e^{-2\pi}(2\pi i) e^{2\pi i z} \neq 0$ on $\mathbb{D}$. //

(2) f is analytic on $0.9\mathbb{D}$ and continuous on its boundary $\Rightarrow$ By the Maximum Modulus Principle, On $0.9\overline{\mathbb{D}}$, f attains its max on $\partial 0.9\overline{\mathbb{D}}$. $\Rightarrow$ on $0.9\mathbb{D}$,

$|f(z)| \leq 1-(0.9)^2+(0.9)^{1000} =$

$0.19+(0.9)^{1000} < 0.2 \Rightarrow |f(0)| \leq 0.2$ //

(4) By the Reimann Mapping Thm, we have that Ω_2 is conformally eqivalent to $\mathbb{D}$. Thus, $\exists\ f: \Omega_1 \mapsto \mathbb{D}$ iff $\exists\ f: \Omega_1 \mapsto \Omega_2$. Now, Suppose $\exists$ $f: \Omega_1 \mapsto \mathbb{D}$. Then f is bounded $\Rightarrow$ All singularities are removable $\Rightarrow$ f can be extended to a function $f: \mathbb{C} \mapsto \mathbb{D}$. However, Louivilles Thm says that any bounded entire function is constant $\Rightarrow$ Such a map does not exist. //

(a)

(5) We have that $\sqrt{z^2-1} = z\left(1-\left(\frac{1}{z}\right)^2\right)^{1/2}$

$$= (z) \sum_{n=0}^{\infty} \binom{1/2}{n} \left(-1/z^2\right)^n$$

$$= z\left[1 + \frac{1}{2}\left(-\frac{1}{z^2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{1}{z^2}\right)^2 + \cdots\right]$$

$$= z - 1/2\, z^{-1} - 1/4\, z^{-3} + \cdots$$

$$\Rightarrow \alpha = 1,\ \beta = 0,\ \gamma = -1/2,\ \delta = 0,\ \varepsilon = -1/4$$

(b) $$\int_{|z|=2} (5+6z+7z^2) f(z)\, dz$$

$$= \int_{|z|=2} (5+6z+7z^2)\left[z - 1/2\, z^{-1} - 1/4\, z^{-3} + \cdots\right] dz$$

$$= \int_{|z|=2} \left[-\frac{5}{2} z^{-1} - 7/4\, z^{-1} + \cdots\right] dz$$

$$= 2\pi i \left[-\frac{17}{4}\right].$$ //

Part II

(1) First we extend f to a continuous function on $\mathbb{R}$ by defining $f(x) = f(a)$, $x \leq a$, $f(x) = f(b)$, $x \geq b$.

Now, define $f_n(x) = n\left[f(x+1/n) - f(x)\right]$

Then each f_n is measurable so,

$f_n \xmapsto{a.e} f^+$, $f^+(x)$ is ~~an a.e.~~ measurable function, $f^+ : \mathbb{R} \mapsto \overline{\mathbb{R}}$. Now, define

$g_n(x) = -n\left[f(x-1/n) - f(x)\right]$,

$g_n \xmapsto{a.e} \cancel{g}\, f^-$ ~~a.e.~~ measurable, $\cancel{g(x)}\, f^-(x) : \mathbb{R} \mapsto \overline{\mathbb{R}}$

Let $A = f^{+\,-1}(-\infty, \infty)$, $B = f^{-\,-1}(-\infty, \infty)$.

Since f^+ and f^- are measurable functions, we have that A & B are measurable

$\Rightarrow A \cap B$ is

The set of pts for which f is differentiable is equal to $(f^+ - f^-)^{-1}_{|A \cap B}(0)$ which is the inverse image of a closed set of the difference of two measurable fns (w range $\mathbb{R}$ since restricted to $A \cap B$ so makes sense) $\Rightarrow$ it is measurable. //

(2) $f_n \mapsto f$ is measure $\Longrightarrow$ $\exists$ a subsequence

$\{f_{n_j}\}$ such that $f_{n_j} \mapsto f$ a.e.

But, $\cancel{\sup f_{n_j}}$

$f = \sup f_{n_j} \overset{a.e.}{\leq} g \Longrightarrow f \overset{a.e.}{\leq} g.$ //

4. (a) $\|Tf\|_1 = \int_{-\infty}^{\infty} \left| \int_{x-1}^{x+1} f(y)\,dy \right| dx$

$$\leq \int_{-\infty}^{\infty} \int_{x-1}^{x+1} |f(y)|\,dy\,dx.$$

Since $\|Tf\|_1 = \|4f\|_1 = 4\|f\|_1 < +\infty$

We can apply Fubini's Theorem,

and so we get,

$$\int_{-\infty}^{\infty} \int_{x-1}^{x+1} |f(y)|\,dy\,dx = \int_{-\infty}^{\infty} \int_{y-1}^{y+1} |f(y)|\,dx\,dy$$

$$= 2 \int_{-\infty}^{\infty} |f(y)|\,dy = 2\|f\|_1$$

So, $4\|f\|_1 = \alpha \|f\|_1$, $\alpha \leq 2$

$\Rightarrow \|f\|_1 = 0 \Rightarrow f = 0$ a.e.

4. (b) $\int_{x-1}^{x+1} e^{ry}\,dy = 4e^{rx}$

$\Rightarrow \frac{e^{ry}}{r}\Big|_{x-1}^{x+1} = 4e^{rx}$

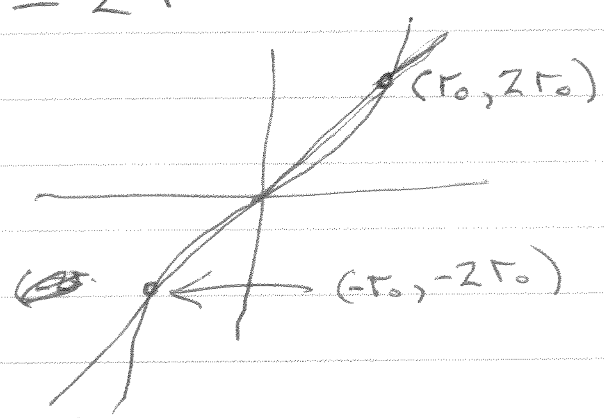
$\Rightarrow \frac{e^{r(x+1)}}{r} - \frac{e^{r(x-1)}}{r} = 4e^{rx}$

$\Rightarrow \frac{e^{rx}}{r}\left[e^{r} - e^{-r}\right] = 4e^{rx}$

$\Rightarrow 2\sinh(r) = 4r$

$\Rightarrow \sinh(r) = 2r$

Picture is



$\Rightarrow$ intersects at $r=0$, r_0, $-r_0$.

Now, $r=0$ doesn't work (& actually we multiplied through by r so division by 0 going on) But, r_0 & $-r_0$ works.

So, we have that $Ae^{r_0 x} +$
$Ae^{-r_0 x}$ works, in particular
$\cosh(r_0 x)$ and $\sinh(r_0 x)$ are
good fns. //