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Analysis QR May 10, 2013

1. Let $F(x) = \int_{\infty}^{x} f(t)\,dt$

Since f is integrable, we have that

$\lim\limits_{x \mapsto \infty} F(x) = 0$. Given $\varepsilon > 0$, $\exists$

$\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $G_\delta(x) = F(x + \delta/2) - F(x)$.

Then, $\lim\limits_{x \mapsto \infty} G_\delta(x) = 0$

Also, $G_\delta(x) = \delta/2 \, f(\xi_x)$, $\xi_x \in [x, x+\delta/2]$

By the Mean Value Theorem.

We have $|\xi_x - x| < \delta \Rightarrow |f(\xi_x) - f(x)| < \varepsilon$

and, $\lim\limits_{x \mapsto \infty} \delta/2 \, f(\xi_x) = 0 \Rightarrow f(\xi_x) \mapsto 0$ as $x \mapsto \infty$

$\Rightarrow$ For x large, $|f(\xi_x) - f(x)| \approx |f(x)|$

$\leq \cancel{|f(\xi_x)| + |f(x)|}$

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In Particular for x large enough,

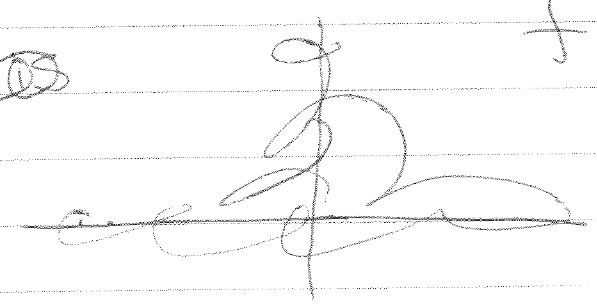
$|f(\xi_x)| < \varepsilon \Rightarrow |f(x)| < 2\varepsilon.$

So, for x large, $f(x) \in B_{2\varepsilon}(0)$

Since ε is arbitrary, we have that

$f(x) \longmapsto 0$ as $x \longmapsto \infty$. //

(b) Consider the function $f(x)$

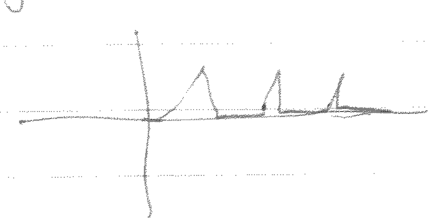


Defined by $f(n) = 1$, $n \in \mathbb{Z}^{>0}$

$f(x) =$ on $\mathbb{R} \setminus \bigcup_{n=1}^{\infty} (n-2^{-n}, n+2^{-n})$

and the rest of the image is constructed

by "connecting the dots"



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Then, $\|f\|_1 = \sum_{n=1}^{\infty} 2^{-n} = 1$,

But, $\lim_{x \mapsto \infty} f(x) \neq 0$. //

2. We have that $f(x) = \dfrac{g(x)}{x^p + x^{-p}}$.

So, ~~f is clearly~~ Now,

$\|f\|_1 = \left\| \dfrac{g(x)}{x^p + x^{-p}} \right\|_1 \leq \|g\|_2 \, \|(x^p + x^{-p})^{-1}\|_2$

But, $\displaystyle\int_0^\infty \left(\frac{1}{x^p + x^{-p}} \right)^2 dx \leq$

$\displaystyle\int_0^\infty \frac{1}{x^{2p} + x^{-2p}} dx \leq \int_0^1 x^{2p} dx + \int_1^\infty x^{-2p} dx$

$\displaystyle= \frac{x^{2p+1}}{2p+1} \Big|_0^1 + \frac{x^{1-2p}}{1-2p} \Big|_1^\infty$

$\displaystyle= \frac{1}{2p+1} + \frac{1}{2p-1} < \infty$

$\Rightarrow \|g\|_2 \, \|(x^p + x^{-p})^{-1}\|_2 < \infty \Rightarrow \|f\|_1 < \infty$

$\Rightarrow f \in L_1(0, \infty)$. //

3. $\overset{A}{\overline{F_t \setminus G_t}} = \{x \in X : f(x) > t \geq g(x)\}$

Similarly, we have that

$\underset{B}{\underline{G_t \setminus F_t}} = \{x \in X : g(x) > t \geq f(x)\}$

So, we have, $\int_{-\infty}^{\infty} \mu(F_t \Delta G_t)\,dt$

$$= \underbrace{\int_{-\infty}^{\infty} \mu(A)\,dt}_{(1)} + \underbrace{\int_{-\infty}^{\infty} \mu(B)\,dt}_{(2)}$$

So, we have for (1),

$(1) = \int_{-\infty}^{\infty}\int_X \mathbb{1}_A \,d\mu\,dt$. Since $\mathbb{1}_A$ is

a non-negative measurable function on

$\mathbb{R} \times X$ we can invoke Fubini. We have

$(1) = \int_X \int_{-\infty}^{\infty} \mathbb{1}_A \,dt\,d\mu =$

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$$\int_X \int_{g(x)}^{f(x)} \mathbb{1}_{f(x)>g(x)} \, dt \, d\mu$$

$$= \int_X [f(x) - g(x)] \mathbb{1}_{f(x)>g(x)} \, d\mu$$

Now, for (2) We follow the same steps and we get,

$$(2) = \int_X [g(x) - f(x)] \mathbb{1}_{g(x) \geq f(x)} \, d\mu$$

Combining, we get $(1)+(2) =$

$\int_X |f-g|$ Proving the Claim, //

4. We have that $\limsup f_n = f^+$ is measurable and $\liminf f_n = f^-$ is measurable, $f^+ : X \mapsto \overline{\mathbb{R}}$, $f^- : X \mapsto \overline{\mathbb{R}}$. Let $A = (f^+)^{-1}(\mathbb{R})$ + $B = (f^-)^{-1}(\mathbb{R})$ which are both measurable sets $\Rightarrow A \cap B$ is $\Rightarrow (f^+ - f^-)_{|A \cap B} : X \mapsto \mathbb{R}$ is measurable. So, then we have $(f^+ - f^-)^{-1}(0)$ is measurable. But, this is equal to set of pts x s.t. $f_n(x)$ converges as $n \to \infty$. //

5. Note, on $[0,\infty)$ we have

$$e^{-x} > f_n(x) = \frac{x^n}{(1+x^n)e^x} \quad \forall n$$

Also, $\int_0^\infty e^{-x}dx = 1 \Rightarrow e^{-x}$

dominates f_n's $\Rightarrow$ We can invoke the Dominated Convergence Thm.

Thus, $\lim_{n\to\infty} \int_0^\infty \frac{x^n}{(1+x^n)e^x}\,dx$

$$= \int_0^\infty \lim_{n\to\infty} \frac{x^n}{1+x^n e^x}\,dx$$

$$= \int_1^\infty \frac{1}{e^x}\,dx = e^{-1}. //$$

Now, for a_0 we have

$$a_0 = \frac{1}{2\pi i}\int_{\gamma} \frac{dz}{(e^z-1)z}$$

$$\operatorname{Res}(f,0) = \lim_{z\mapsto 0} \frac{d}{dz}\left(\frac{z}{e^z-1}\right)$$

$$= \lim_{z\mapsto 0} \frac{(e^z-1) - ze^z}{-(e^z-1)^2}$$

$$= \lim_{z\mapsto 0} \frac{\cancel{e^z} - \cancel{e^z} - ze^z}{-2(e^z-1)}$$

$$= \lim_{z\mapsto 0} \frac{-e^z - ze^z}{-2e^z} = \frac{1}{2}$$

Ect...

(a)

(2) Consider $f(z) = \frac{1}{10} z^3 (z + 1/2)$

Then $|f(z)| \leq \frac{1}{10}(1)(1 + 1/2) < 1$

on $\mathbb{D} \Rightarrow f: \mathbb{D} \mapsto \mathbb{D}$, has a

triple zero at the origin + is clearly

analytic. Also, $f(-1/2) = 0$

which is as small as possible $\Rightarrow$

0 is Answer //

(b) $f(z) = z^3 g(z)$ where $g(0) \neq 0$

and $g(z)$ analytic on $\mathbb{D}$.

Now, By Schwartz, we have that

$|z^3 g(z)| \leq |z| \Rightarrow |z^2 g(z)| \leq 1$

But, $|z^2 g(z)| \neq 1$ since not

constant

$\Rightarrow |z^2 g(z)| < 1$, so By Schwartz $|z^2 g(z)| \leq |z|$

$\Rightarrow |z g(z)| \leq 1$. Same argument Repeated, we get that

$|g(z)| \leq 1 \Rightarrow$

$|f(z)| = |z^3 g(z)| \leq |z^3|$

$\Rightarrow \max f(-1/2) = \frac{1}{8}$ attained

By using $f(z) = z^3$. //

(3) $|f(z)| \leq 5|z|^{-\sqrt{2}}$

$\Rightarrow |z^2 f(z)| \leq 5|z|^{2-\sqrt{2}}$

So $z^2 f(z): \mathbb{C} \mapsto \mathbb{C}$ is entire

Now, $z^2 f(z)$ is either a polynomial or it has an essential singularity at ∞. Let $g(z) = z^{-2} f(z^{-1})$.

Then $|g(z)| \leq 5|z|^{\sqrt{2}-2}$

and $\lim_{z \mapsto 0} |z^2 g(z)| = 0$

$\Rightarrow$ 0 pole at worst $\Rightarrow$ Not essential singularity $\Rightarrow z^2 f(z)$ is a polynomial.

Moreover, $|z^2 f(z)| \le 5|z|^{2-\sqrt{2}}$

$\Rightarrow z^2 f(z) = \text{constant}$ since

$0 < 2-\sqrt{2} < 1 \Rightarrow z^2 f(z) = A$

$\Rightarrow f(z) = \dfrac{A}{z^2}$.

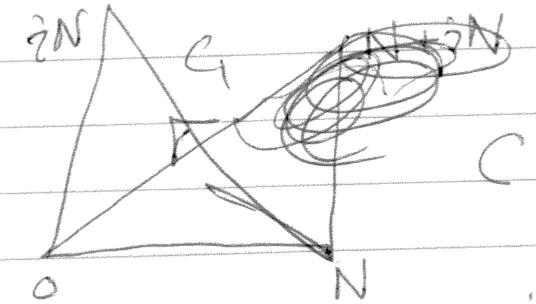
But, the only A for which

$\left|\dfrac{A}{z^2}\right| \le 5|z|^{-\sqrt{2}} \quad \forall z \in \mathbb{C} \setminus 0$

is $A = 0 \Rightarrow f(z) = 0$

$\Rightarrow \sup |f(1)| = 0$ //

4. (a) Consider the contour



$i\,dy = dx$

Then, $\displaystyle\int_C f(z)\,dz = 0$ $\quad iy = x$

$$= \lim_{N\to\infty}\int_0^N f(x)\,dx - \lim_{N\to\infty}\int_0^{iN} f(x)\,dx$$

$$+ \lim_{N\to\infty}\int_{C_1} f(z)\,dz$$

$N(1-t) + iNt$

$$= -\lim_{N\to\infty} i\int_0^N f(iy)\,dy$$

$$\lim_{N\to\infty}\int_0^1 f(N(1-t)+iNt)(-N+iN)\,dt$$

$$= \cancel{\lim_{N \to \infty}}$$

$$\le \lim_{N \to \infty} \int_0^1 |f(N(1-t) + iNt)| N\sqrt{2}\, dt$$

$$\le \lim_{N \to \infty} \int_0^1 10 e^{-Nt} N\sqrt{2}\, dt$$

$$= 0$$

$$\Rightarrow \lim_{N \to \infty} \int_0^N f(x)\,dx =$$

$$\lim_{N \to \infty} -i \int_0^\infty f(iy)\,dy$$

this is finite

by $|f(x+iy)| \le 10e^{-y}$. //